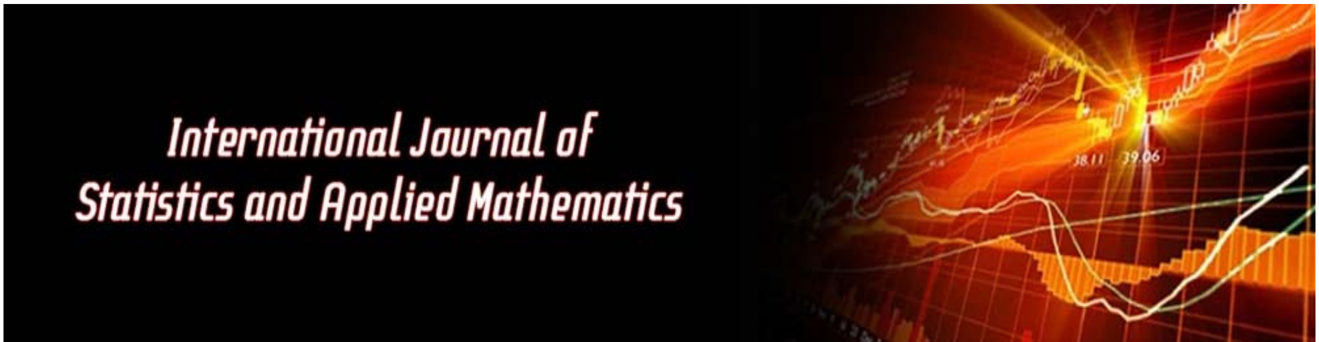

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Geometry of numbers and its applications

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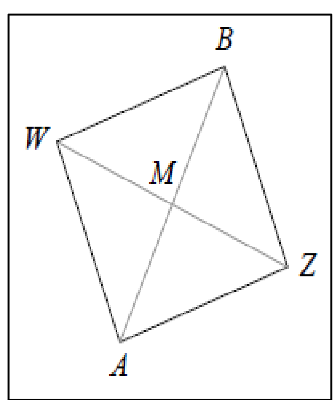
Abstract
Some geometrical results which are difficult to prove by purely Euclidean methods and approaches involving coordinate geometry or trigonometry are not always appropriate or may lead to messy algebraic manipulation. An alternative approach is to use complex numbers and for some results this may be the most convenient method of proof. Here we demonstrate the method by looking at a few results about polygons. That squares feature here is perhaps not surprising, since multiplication by i corresponds to a rotation through 90^0. Here we shall see, complex numbers may also help with results involving regular polygons.

Keywords: Euclidean methods, geometry or trigonometry

Introduction
The geometry of numbers studies convex bodies and integer vectors in n-dimensional space. The geometry of numbers was initiated by Hermann Minkowski (1910) [10].

The geometry of numbers has a close relationship with other fields of mathematics, especially functional analysis and Diophantine approximation, the problem of finding rational numbers that approximate an irrational quantity.

Results
1. Suppose M is the midpoint of AB. Then $\overrightarrow{AB} = 2\overrightarrow{AM}$ and so b-a = 2(m-a). Therefore $m = \frac{1}{2}(a + b)$.
2. The distance from Z to W, is the length ZW, is equal to $|w - z|$
3. If b - a = i(q - p), then AB and PQ are perpendicular and equal in length.
4. Suppose ABCD is a square (labelled anticlockwise). Then d - a = i(b - a).
5. Given two complex numbers a, b, what are the values of z, w so that AZBW is a square?

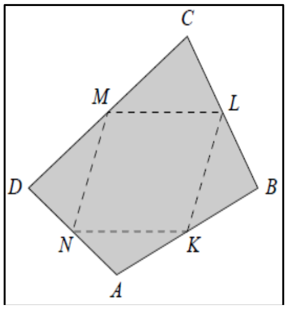


Let M be the midpoint of AB. Since MZ is perpendicular to BA and of half the length we have

$z - m = \frac{1}{2}i(a - b) \Rightarrow z = \frac{1}{2}(a + b) + \frac{1}{2}i(a - b)$

Similarly $w = \frac{1}{2}(a + b) - \frac{1}{2}i(a - b)$

Varignon's theorem: The midpoints of the sides of an arbitrary planar quadrilateral form a parallelogram.

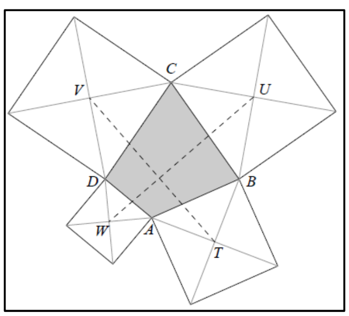


Let the midpoints be K, L, M and N, as shown. Then using the result we have $k = \frac{1}{2}(a + b)$, with similar results for l, m and n. Therefore $l - k = \frac{1}{2}(b+ c) - \frac{1}{2}(a + b) = \frac{1}{2}(c - a)$ and $m - n = \frac{1}{2}(c + d) - \frac{1}{2}(a + d) = \frac{1}{2}(c - a)$

Hence $l - k = m - n$, so, $\overrightarrow{KL} = \overrightarrow{NM}$ and therefore KLMN is a parallelogram.

The appearance of squares in the statement of the following theorem, and the nature of the conclusion, mean that the result lends itself readily to a proof by complex numbers.

Van Aubel's theorem: Given an arbitrary planar quadrilateral, place a square outwardly on each side. Then the two lines joining the centres of opposite squares are of equal length and perpendicular.



Let the quadrilateral be ABCD and the centres of the squares T, U, V, W, as shown.

Now A, T, B form three vertices of a square,

$t = \frac{1}{2}(a + b) + \frac{1}{2}i(a - b)$

Similarly,

$u = \frac{1}{2}(b + c) + \frac{1}{2}i(b - c)$

$v = \frac{1}{2}(c + d) + \frac{1}{2}i(c - d)$

$w = \frac{1}{2}(d + a) + \frac{1}{2}i(d - a)$

Hence $t - v = \frac{1}{2}(a+b-c-d) + \frac{1}{2}i(a-b-c+d)$

and $u - w = \frac{1}{2}(-a+b+c-d) + \frac{1}{2}i(a+b-c-d) = i(t - v)$

Therefore WU and VT are perpendicular and equal in length

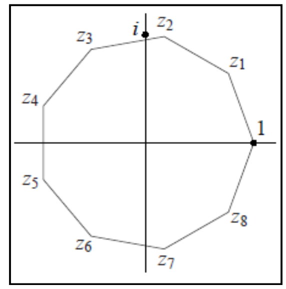
Th´ebault's first theorem: Given an arbitrary parallelogram, place a square outwardly on each side. Then the centres of these squares form a square.

Regular polygons

We conclude with a remarkable result about the diagonals of a regular polygon. The proof demonstrates the power of an approach by complex numbers, which brings into play a range of algebraic techniques.

Diagonals For any integer $n \geq 3$, suppose that a regular polygon with n sides is inscribed in a circle of radius 1. Then the product of the lengths of the diagonals of the polygon passing through a given vertex V is equal to n. (Here the edges are counted as diagonals.)

In order to keep the notation a little simpler, we present the proof for the case when n = 9, the nonagon. However, the proof is quite general.



Consider the nonagon in the complex plane with vertices at 1, $z_1, z_2, . . ., z_8$ on the unit circle centre 0, as shown in the figure below, and let V be the vertex at 1.

Now $1, z_1, z_2, . . ., z_8$ have modulus 1 and arguments $2\pi s$,

where $s = 0, \frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \ldots, \frac{8}{9}$, and so may be written as $e^{2\pi is}$

But $(e^{2\pi is})^9 = 1$

Since 9 is an integer. Hence $1, z_1, z_2, ..., z_8$ are the roots of $z^9 - 1 = 0$

Because $z^9 - 1 = (z-1)(z^8 + z^7 + \ldots + z + 1)$

Therefore $z_1, z_2, . ., z_8$ are the roots of $z^8 + z^7 + \ldots + z + 1 = 0$

Hence $z^8 + z^7 + \ldots + z + 1 = (z-z_1)(z-z_2)(z-z_3)\ldots\ldots(z-z_8)$

Putting z = 1 we get $(1-z_1)(1-z_2)(1-z_3)\ldots\ldots(1-z_8) = 9$

$|1-z_1||1-z_2||1-z_3|\ldots\ldots|1-z_8| = 9$

But $|1-z_k|$ is equal to the distance from zk to 1, that is, the length of one of the diagonals

through V, so that the result follows.

All pythagoran triangles whose areas equal to the perimeters

Let x,y,z be sides of a Pythagorean triangle

Then $x^2 + y^2 = z^2$

Now, area of Pythagorean triangle = Perimeter of the triangle

Therefore $\frac{1}{2}xy = x + y + z$

$Z = \frac{xy}{2} - x - y$

$z^2 = (\frac{xy}{2} - x - y)^2$

$x^2 + y^2 = \frac{x^2y^2}{4} + x^2 + y^2 - x^2y + 2xy - xy^2$

$x^2y + xy^2 = \frac{x^2y^2}{4} + 2xy$

$xy(x+y) = xy(\frac{xy}{4} + 2)$

$4x + 4y = xy + 8$

$xy - 4x - 4y = -8$

$(x-4)(y-4) = 8$

x-4=1,y-4=8; x-4=2,y-4=4; x-4=4,y-4=2; x-4=8,y-4=1

x=5,y=12; x=6,y=8; x=8,y=6; x=12,y=5

For x=5,y=12 or x=12,y=5 we have z =13

and For x=6,y=8 or x=8,y=6 we have z = 10

Hence the required Pythagorean triangles are those with sides 5,12,13 or 6,8,10

The radius of the inscribed circle of a Pythagorean triangle is always an integer.

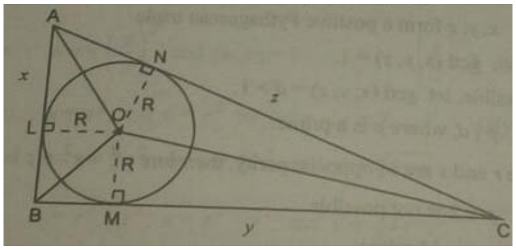
Let ABC be a Pythagorean triangle with sides x,y and z.

Then $x^2 + y^2 = z^2$

Therefore $x = k(r^2 - s^2)$, $y = 2krs$ and $z = k(r^2 + s^2)$ for some $r,s,k \in Z$ such that $r, s > 0$, $(r,s)=1$, $r \not\equiv s \pmod 2$

Let O be the centre of inscribed circle and R be the radius of incircle. Let the circle touches the sides at L,M and N

Then OL, OM and ON are perpendicular to sides AB, BC and AC respectively.



Now,

area of ΔABC = area of ΔOAB + area of ΔOBC + area of ΔOAC $\frac{1}{2}$AB.BC = $\frac{1}{2}$OL.AB + $\frac{1}{2}$OM.BC + $\frac{1}{2}$ON.AC

$xy = Rx + Ry + Rz$

$R = \frac{xy}{x+y+z} = \frac{k(r^2-s^2)2krs}{k(r^2-s^2)+2krs+k(r^2+s^2)} = ks(r-s) \in Z$

Hence radius of inscribed circle in a Pythagorean triangle is an integer.

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