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On certain subclasses of analytic multivalent functions involving generalized integral operator

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Abstract
In this paper several new subclasses of analytic functions which are defined by means of a generalized integral operator have been introduced. Next, inclusion properties for these subclasses are established. Many interesting applications are also discussed.

Keywords: analytic functions, univalent and multivalent functions, starlike functions, convex functions, differential subordination, hadamard product or convolution

Introduction

Let A_p denote the class of functions f (z) *normalized* by

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (p \in \mathbf{N}) \qquad \ldots(1.1)$$

which are analytic in the open unit disk U = {z: z ∈ C and | z | < 1 }. Also let $S_p^*(\alpha)$ and $K_p(\alpha)$ denote, respectively, the subclasses of A_p consisting of p-valent functions which are *starlike* and *convex* of order α in U with $0 \le \alpha < p$. In particular $S_p^*(0) = S_p^*$ and $K_p(0) = K_p$ are the well-known subclasses of p-valent *starlike* and p-valent convex functions in U, respectively.

Given two functions f and g, which are analytic in U with f (0) = g (0), the function f is said to be *subordinate* to g in U if there exists a function w, analytic in U, such that

w (0) = 0, | w (z) | < 1 (z ∈ U), and f (z) = g (w (z)) (z ∈ U).

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We denote this subordination by

$f(z) \prec g(z)$ in U

We also observe that

$f(z) \prec g(z)$ in U

iff $f(0) = g(0)$ and $f(U) \subset g(U)$

whenever g is *univalent* in U.

Let M be the class of analytic functions ϕ (z) in U normalized by ϕ (0) = 1, and let H be the subclass of M consisting of those functions ϕ which are univalent in U and for which ϕ (U) is convex and R {ϕ(z)} > 0 (z ∈ U).

We define the following subclasses $s_p^*(\phi)$ and $k_p(\phi)$ for $\phi \in H$ by

$$s_p^*(\phi) = \left\{ f : f \in A_p \text{ and } \frac{z f'(z)}{f(z)} \prec p\,\phi(z) \text{ in } U \right\}$$

$$k_p(\phi) = \left\{ f : f \in A_p \text{ and } 1 + \frac{z f''(z)}{f'(z)} \prec p\,\phi(z) \text{ in } U \right\},$$

Obviously

1. $s_p^*\left(\frac{1+z}{1-z}\right) = S_p^*$,

2. $k_p\left(\frac{1+z}{1-z}\right) = K_p$

3. $s_p^*\left(\frac{1+Az}{1+Bz}\right) = S_p^*[A,B] \quad (-1 \le B < A \le 1)$

4. $k_p\left(\frac{1+Az}{1+Bz}\right) = K_p[A,B] \quad (-1 \le B < A \le 1)$.

We also have $S_p^*[1,-1] = S_p^*$ and $K_p[1,-1] = K_p$. For p = 1, the above reduced classes S^* [A,B] and K [A,B] were investigated by Janowski [7] and Goel and Mehrok [4].

Clearly

$$f(z) \in k_p(\phi) \Leftrightarrow z f'(z) \in s_p^*(\phi) \; .$$

Further suppose that

$$h_p[(\alpha_q);(\beta_r);z] = z^p \; {}_qF_r(\alpha_1,...,\alpha_q;\beta_1,...,\beta_r;z)$$

$$= z^p + \sum_{n=p+1}^{\infty} B_p^{(\alpha_q),(\beta_r)}(n) z^n \qquad \text{...(1.2)}$$

$$(q \le r+1; \alpha_i \in R; \beta_j \in R \setminus Z_0^-; Z_0^- = \{0,-1,-2,...\};$$

i = 1,…, q; j = 1,…,r; z ∈ U)

where ${}_qF_r$ is the generalized hypergeometric function and

$$B_p^{(\alpha_q),(\beta_r)}(n) = \frac{(\alpha_1)_{n-p}(\alpha_2)_{n-p}...(\alpha_q)_{n-p}}{(\beta_1)_{n-p}(\beta_2)_{n-p}...(\beta_r)_{n-p}(n-p)!}. \qquad \text{...(1.3)}$$

Corresponding to the function $h_p[(\alpha_q);(\beta_r);z]$, Dziok and Srivastava [2, p.3, Eq.(3)] introduced a linear operator $\boldsymbol{H_{p,\alpha_q,\beta_r}}$ defined by the convolution

$$H_{P,\alpha_q,\beta_r} f(z) = h_p[(\alpha_q);(\beta_r);z] * f(z) \qquad \text{...(1.4)}$$

or equivalently by

$$H_{P,\alpha_q,\beta_r} f(z) = z^p + \sum_{n=p+1}^{\infty} B_p^{(\alpha_q),(\beta_r)}(n) a_n z^n \quad (z \in U). \qquad \text{...(1.5)}$$

Here $*$ stands for the convolution of two analytic multivalent functions f and g of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \; g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n \; (a_n, b_n \ge 0, \; p \in N)$$

and is defined by

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n. \qquad \text{...(1.6)}$$

The linear operator $H_{P,\alpha_q,\beta_r} f(z)$ includes various other linear operators considered earlier by Hohlov [6], Carlson-Shaffer [1], Goyal and Bhagtani [5], Ruscheweyh [9] etc.

Next by using the operator H_{P,α_q,β_r} we introduce the following classes of analytic functions for $\phi \in H$; $f \in A_p$; $\alpha_q > -1$ and $\beta_r \ge 1$

$$s^*_{p,\alpha_q,\beta_r}(\phi) = \{f : f \in A_p \text{ and } H_{p,\alpha_q,\beta_r} f(z) \in s_p^*(\phi)\}$$

$$k_{p,\alpha_q,\beta_r}(\phi) = \{f : f \in A_p \text{ and } H_{p,\alpha_q,\beta_r} f(z) \in k_p(\phi)\}$$

We also note that

$$f(z) \in k_{p,\alpha_q,\beta_r}(\phi) \Leftrightarrow z f'(z) \in s^*_{p,\alpha_q,\beta_r}(\phi) \quad \ldots(1.7)$$

In particular, we set

$$s^*_{p,r,s,t,2}\left(\frac{1+z}{1-z}\right) = S^*_{p,r,s,t},$$

$$s^*_{p,\lambda,\nu,\eta,\mu}\left(\frac{1+Az}{1+Bz}\right) = S^*_{p,\lambda,\nu,\eta,\mu}[A,B] \quad (-1 \le B < A \le 1),$$

$$k_{p,\lambda,\nu,\lambda,\mu}\left(\frac{1+Az}{1+Bz}\right) = K_{p,\lambda,\nu,\eta,\mu}[A,B] \quad (-1 \le B < A \le 1),$$

and for p = 1, we have

if r = s = n then $s^*_{n,2}\left(\frac{1+z}{1-z}\right) = S^*_n$,

$$\text{if } \nu = \lambda, \text{ then } s^*_{\lambda,\mu}\left(\frac{1+Az}{1+Bz}\right) = S^*_{\lambda,\mu}[A,B] \quad (-1 \le B < A \le 1)$$

And $$k_{\lambda,\mu}\left(\frac{1+Az}{1+Bz}\right) = K_{\lambda,\mu}[A,B] \quad (-1 \le B < A \le 1).$$

2. Inclusion properties involving H_{p,α_q,β_r}

The following result will be required in our investigation:

Lemma (Eenigenburg *et al.* [3]). *Let h be convex univalent in U with h (0) = 1 and*

$$R\{\beta h(z) + \gamma\} > 0 \qquad (\beta, \gamma \in C)$$

If p (z) is analytic in U with p (0) = 1, then

$$p(z) + \frac{z p'(z)}{\beta p(z) + \gamma} \prec h(z) \text{ in } U$$

implies that p (z) ≺ h (z) in U.

Our first inclusion theorem is stated as:

Theorem 1. *Let* $\alpha_q > -1$ and $\beta_r \ge 1$. *Then*

$$s^*_{p,\alpha_q,\beta_r+1}(\phi) \subset s^*_{p,\alpha_q,\beta_r}(\phi) \subset s^*_{p,\alpha_q+1,\beta_r}(\phi) \quad (\phi \in H)$$

Proof. First of all we show that

$$s^*_{p,\alpha_q,\beta_r+1}(\phi) \subset s^*_{p,\alpha_q,\beta_r}(\phi) \; (\phi \in H; \; \alpha_q > -1 \text{ and } \beta_r \ge 1)$$

Let f (z) $s^*_{p,\alpha_q,\beta_r+1}(\phi)$ and set

$$\frac{z\left(H_{p,\alpha_q,\beta_r} f(z)\right)'}{H_{p,\alpha_q,\beta_r} f(z)} = p\,\theta(z), \quad \ldots(2.1)$$

where $\theta(z) = 1 + c_1 z + c_2 z^2 + \ldots$.

Obviously θ (z) is analytic in U and θ (z) ≠ 0 for all z ∈ U.

Applying (1.21) in (2.1), we obtain

$$(\beta_r + p - 1)\frac{\left(H_{p,\alpha_q,\beta_r+1} f(z)\right)}{H_{p,\alpha_q,\beta_r} f(z)} = p\,\theta(z) + \beta_r - 1 \quad \ldots(2.2)$$

By using the logarithmic differentiation on both sides of (2.2) and multiplying with z, we have

$$\frac{z\left(H_{p,\alpha_q,\beta_r+1} f(z)\right)'}{H_{p,\alpha_q,\beta_r} f(z)} = p\,\theta(z) + \frac{p\,z\,\theta'(z)}{p\,\theta(z)+\beta_r - 1} \qquad \ldots(2.3)$$

Since $\beta_r \geq 1$, $\phi(z) \in H$ and $f(z) \in s^*_{p,\alpha_q,\beta_r}(\phi)$ from (2.3), we see that

$R\{p\,\phi(z) + \beta_r - 1\} > 0 \ (z \in U)$

and $p\,\theta(z) + \dfrac{p\,z\,\theta'(z)}{p\,\theta(z)+\beta_r - 1} \prec p\,\phi(z)$ in U

Thus by using the Lemma and (2.1) we observe that

$p\,\theta(z) \prec p\,\phi(z)$ in U

so that $f(z) \in s^*_{p,\alpha_q,\beta_r}(\phi)$

This implies that

$s^*_{p,\alpha_q,\beta_r+1}(\phi) \subset s^*_{p,\alpha_q,\beta_r}(\phi)$.

To prove the second part, let $f(z) \in s^*_{p,\alpha_q,\beta_r}(\phi)$ ($\alpha_q > -1$ and $\beta_r \geq 1$) and put

$$\frac{z\left(H_{p,\alpha_q+1,\beta_r} f(z)\right)'}{H_{p,\alpha_q+1,\beta_r} f(z)} = p\,\Psi(z)$$

where $\Psi(z) = 1 + d_1 z + d_2 z^2 + \ldots$ is analytic in U and $\Psi(z) \neq 0$ for all $z \in U$. Now by using arguments similar to those detailed above, it follows that

$p\,\Psi(z) \prec p\,\phi(z)$ in U,

which implies that $f(z) \in s^*_{p,\alpha_q+1,\beta_r}(\phi)$. Hence we conclude that

$$s^*_{p,\alpha_q,\beta_r+1}(\phi) \subset s^*_{p,\alpha_q,\beta_r}(\phi) \subset s^*_{p,\alpha_q+1,\beta_r}(\phi) \quad (\phi \in H)$$

which completes the proof of Theorem 1

Remark. By putting

$q = 2, r = 2; \alpha_1 = r, \alpha_2 = s, \beta_1 = t, \beta_2 = 2, (r,s,t \in N_0), \mu = 2$ and $\phi(z) = \dfrac{1+z}{1-z}$, $z \in U$ in Theorem 1, we obtain

$S^*_{p,r,s,t} \subset S^*_{p,r+1,s+1,t}$

Further if p = 1 and r = s = n then $S^*_n \subset S^*_{n+1}$ which was asserted earlier by Noor [8].

Theorem 2. *Let* $\alpha_q > -1$ and $\beta_r \geq 1$,

$$k_{p,\alpha_q,\beta_r+1}(\phi) \subset k_{p,\alpha_q,\beta_r}(\phi) \subset k_{p,\alpha_q+1,\beta_r}(\phi) \qquad (\phi \in H)$$

Proof. Applying (1.7) and Theorem 1, we observe that

$$f(z) \in k_{p,\alpha_q,\beta_r+1}(\phi) \Leftrightarrow H_{p,\alpha_q,\beta_r+1} + 1 f(z) \in k_p(\phi)$$

$$\Leftrightarrow z\left(H_{p,\alpha_q,\beta_r+1} f(z)\right)' \in s^*_p(\phi) \quad \Leftrightarrow \quad H_{p,\alpha_q,\beta_r+1}(z f'(z)) \in s^*_p(\phi)$$

$$\Leftrightarrow \ z f'(z) \in s^*_{p,\alpha_q,\beta_r+1}(\phi) \quad \Rightarrow z f'(z) \in s^*_{p,\alpha_q,\beta_r}(\phi) \Leftrightarrow H_{p,\alpha_q,\beta_r}(z f'(z)) \in s^*_p(\phi)$$

$$\Leftrightarrow z\left(H_{p,\alpha_q,\beta_r} f(z)\right)' \in s^*_p(\phi) \quad \Leftrightarrow \quad H_{p,\alpha_q,\beta_r} f(z) \in k_p(\phi) \quad \Leftrightarrow f(z) \in k_{p,\alpha_q,\beta_r}(\phi)$$

and

$$f(z) \in k_{p,\alpha_q,\beta_r}(\phi) \Leftrightarrow z f'(z) \in s^*_{p,\alpha_q,\beta_r}(\phi)$$

$$\Rightarrow z f'(z) \in s^*_{p,\alpha_q+1,\beta_r}(\phi) \Leftrightarrow z(H_{p,\alpha_q+1,\beta_r} f(z))' \in s^*_p(\phi)$$

$$\Leftrightarrow H_{p,\alpha_q+1,\beta_r} f(z) \in k_p(\phi) \Leftrightarrow f(z) \in k_{p,\alpha_q+1,\beta_r}(\phi)$$

which evidently proves Theorem 2.
Taking

$$\phi(z) = \frac{1+Az}{1+Bz} \qquad (-1 \le B < A \le 1 ; z \in U)$$

in Theorems 1 and 2, we have
Corollary. *Let* $\alpha_q > -1$ and $\beta_r \ge 1$ *and* $-1 \le B < A \le 1$. *Then*

$$S^*_{p,\alpha_q,\beta_r+1}[A,B] \subset S^*_{p,\alpha_q,\beta_r}[A,B] \subset S^*_{p,\alpha_q+1,\beta_r}[A,B]$$

and

$$K_{p,\alpha_q,\beta_r+1}[A,B] \subset K_{p,\alpha_q,\beta_r}[A,B] \subset K_{p,\alpha_q+1,\beta_r}[A,B]$$

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