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Neeta Mahesh Patil
Ph. D. Scholar AISECT University, Madhya Pradesh, India

Dr. Basant Singh
Guide, AISECT University, Bhopal, Madhya Pradesh, India

Correspondence
Neeta Mahesh Patil
Ph. D. Scholar AISECT University, Madhya Pradesh, India

Borel measures on polar sets

Neeta Mahesh Patil and Dr. Basant Singh

Abstract
We define the cofine topology in the present setting, which shows that it satisfies the conditions introduced by J. B. Walsh. We modify such a topology on a left polar set and it must have a cofine topology. The definition introduced here is canonical. We combine the exceptional sets N, N_1, and N into one set, which we denote it by N.

Keywords: Borel, polar sets

Introduction
The terminology *Borel measure* is used by different authors with different meanings:
(A) Some authors use it for measures μ on the σ-algebra B of Borel subsets of a given topological space X, i.e. functions $\mu: B \to [0, \infty]$ which are countably additive.

(B) Some authors use it for measures μ on the σ-algebra of Borel sets of a locally compact topological space satisfying the additional property $that\ \mu(K) < \infty$ for every compact set K.

(C) Some authors use it for outer measures μ on a topological space X for which the Borel sets are μ-measurable (hence the difference between acception (A) and (B) is small, however when coming to the terminology *Borel regular* we will see more important discrepancies).

Borel Regular Measures
In these three different contexts *Borel regular measures* are then defined as follows:
(A) Borel measures μ for which
$sup\{\mu(C): C \subset E\ is\ closed\}$
$= \mu(E)$ any Borel set E. $sup\{\mu(C): C \subset E\ is\ closed\}$
$= \mu(E)$ for any Borel set E.

(B) Borel measures μ such that
$sup\{\mu(C): C \subset E\ is\ compact\}$
$= \mu(E) for\ any\ Borel\ set\ Esup\{\mu(C): C \subset E\ is\ compact\}$
$= \mu(E) for\ any\ Borel\ set\ E$
and
$inf\{\mu(U): U \supset E\ is\ open\}$
$= \mu(E) for\ any\ Borel\ set\ Einf\{\mu(U): U \supset E\ is\ open\}$
$= \mu(E)\ for\ any\ Borel\ set\ E$

(C) Borel (outer) measures such that for any $A \subset XA \subset X$ there is a Borel set BB with $\mu(A) = \mu(B)\mu(A) = \mu(B)$

Terminology: Using terminology (A) call tight the measures called *Borel regular* by authors using terminology (B). Moreover they call τ-*smooth* those Borel measures for which $\mu(Fk) \to 0\mu(Fk) \to 0$ for any sequence (or, more in general, net) of closed sets with $Fk \downarrow \emptyset Fk \downarrow \emptyset$.

The study of Borel measures is often connected with that of Baire measures, which differ from Borel measures only in their domain of definition: they are defined on the smallest σσ-algebra β_0 for which continuous functions are β_0 measurable.
In particular observe that the two concepts coincide on topological spaces such that for any open set U there is a continuous function f with $f^{-1}(]0,\infty[) = Uf^{-1}(]0,\infty[) = U$.
The concept of tightness and τ-smoothness can be extended to Baire measures as well (and in fact using terminology (B) call *Baire regular* the Baire measures which are tight;
If X is a completely regular space, then any τ-smooth (tight) finite Baire measure can be extended to a regular τ-smooth (tight) finite Borel measure

Polar Set
The polar set of analytic function $f(z)$ of the complex variable
$z = (z_1, \dots, z_n), n \geq 1$ is the set P of points in some domain D of the complex space C_n for which:
a) $f(z)$ is holomorphic everywhere in $D \setminus P$;
b) $f(z)$ cannot be analytically continued to any point of P; and
c) for every point $a \in P$ there are a neighbourhood U_a and a function $q_a(z) \not\equiv 0$, holomorphic in U_a, for which the function $p_a(z) = q_a(z)f(z)$, which is holomorphic in $D \cap \{U_a \setminus P\}$, can be holomorphically continued to U_a. At every point $a \in p_a \in P$ one has $q_a(a) = 0$. The polar set P consists of the poles (cf. Pole (of a function)) $a \in p_a \in P$ of $f(z)$, for which $p_a(a) \neq 0$, and the points $a \in p_a \in P$ of indeterminacy of $f(z)$, for which $p_a(a) = 0$ (it is assumed that $p_a(z)$ and $q_a(z)$ have no common factors that are holomorphic and vanish at a). Every polar set is a complex analytic variety (by which we mean the set of common zeros of a finite set of holomorphic functions) of complex dimension $n - 1$.
A polar set in potential theory is a set E of points of the Euclidean space R^n $n \geq 2$, for which there exists a potential $U\mu(x)$, $x \in R^n$, for some Borel measure μ, that takes the value $+\infty$ at the points of E and only at those points.
In the case of the logarithmic potential for $n = 2$ and the Newton potential for $n \geq 3$, for a bounded set E to be a polar set it is necessary and sufficient that E is a set of type G_δ and has zero outer capacity. Here, in the definition of a polar set, one can replace "potential" by "superharmonic function". The main properties of polar sets in this case are: a) the set $\{a\}$ which consists of a single point $a \in R^n$ is a polar set; b) a countable union of polar sets is a polar set; c) any polar set has Lebesgue measure zero in R^n; and d) under a conformal mapping a polar set goes to a polar set.
A set E as described under 2) is usually called a complete polar set. A (not necessarily complete) polar set is defined as a subset of a complete polar set. A bounded set is polar if and only if it has zero outer capacity.
The sets described under 1) are also called pole sets, or sets of poles, which avoids confusion, and Meromorphic function.
In parabolic potential theory, a set A is polar if and only if there exists an open covering W of A and, for any $V \in W$, a positive supercaloric function u_v on V such that $u_v = \infty$ on $A \cap V$. Again, points are polar and a countable union of polar sets is polar. Any polar set is totally thin but, in contrast with classical potential theory, not every totally thin set is polar. A similar theory of polarity holds in harmonic spaces, or in the more general case of balayage spaces. In probabilistic potential theory, a Borel set is polar if its first hitting time T_A satisfies $T_A = \infty$ a.s.

1. Definition
A Borel set A is elusive at a points $y \in E - N$ if $yP_A(x) < 1$ for some $x \in E_g$. We say A is elusive at a points $y \in N$ if y is not in the ordinary closure of A. The definition of elusiveness at points of the exceptional set N may seem arbitrary, but N is left polar. We observe that if $y \in A - N$ then A is not elusive at y (for $y^P\{ \tau < \infty, X_\tau = y \in A \} = 1$ for all $x \in E_g$). Hence, we determine that the idea of elusiveness is of interest only when $y \notin A$.

2. Proposition
Let $y \in E - N$ and let $A \in \beta$ such that $y \in A$. Then A is elusive at y *iff* $y^P\{ L_A < \rho \} = 1$, all $x \in E_g$. ….1.

Proof
We consider that $S_A = \infty$ if $L_A = 0$, so for $\in E_g$,
$1 - {}_yP_A(x) = y^P \{ S_A = \infty \} = y^{P^X} \{ L_A = 0 \} \leq y^{P^X} \{ L_A < \tau \}$ ….2.

Now $\{L_A < \tau \}$ is invariant, so the right-hand side of (2) is either zero or one.
If A is elusive, the left-hand side is strictly positive for some x, which implies that (1) holds. Conversely if (1) holds, there exists t such that

$0 < y^{P^X} \{ L_A \leq t < \rho \} = y^{P^X}\{ S_A \circ \theta_t = \infty, \rho > 0 \} = y^{P^X}\{ 1 - {}_yP_A (X_t) \}$. …3

Thus ${}_yP_A \not\in 1$, and A is elusive at y.
Hence the assumption is valid.
Hence, if A is elusive at y, then X_{t^-} is in $E - A$ for all t sufficiently close to ρ.

3. Definition
A Borel set $V \subset E$ is a cofine neighborhood of $y \in E$ is $y \in V$ and if $E - V$ is elusive at y.
It is evident from (1) that the intersection of two cofine neighborhoods is a cofine neighborhood. Thus the cofine neighborhood from a neighborhood base for a topology on E, called the cofine topology. We extend this to $E \cup \delta$, by making δ an isolated point.

We find that any ordinary neighborhood of y is a cofine neighborhood. Thus the cofine topology is finer than the ordinary topology.

4. Theorem

Let A be Borel. Then the cofine closure of A is also Borel.

Proof

Let μ_o be a reference measure. Let L be an 0-measurable time which is ${}_yP^{\mu_0}$- a.s. equal to L_A, for each y. In view of assumptions of (2.) cofine closure of A is

$A \cup (N \cap A)\{ y \in E - N: {}_yP^{\mu_0}\{ \mathrm{L} = \tau \} > 0.$

This is Borel structure.

We now prove that the name "cofine topology" is justified.

5. Theorem

The cofine topology defined in (2) is a cofine topology, i.e., it satisfies conditions of (1).

Proof

Let L be a bounded Borel function which is cofine continuous, except possibly at a left polar set. We show $t \to f(X_t)$ is left continuous for it is sufficient to show it is left continuous on (0, L_K),where K is any compact set. Let X be the right continuous process $\{ \overline{X}_t = X_{L_{K-t}-}, 0 \le t \le L_K \}$. It is then sufficient to show $f(X_t)$ is right continuous, and even sufficient to show $f(X_t)$ is right continuous at a given stopping time T. But $(L_K - T) \vee 0$ is co-optional for the original process. Hence, it is sufficient to show that for any co-optional time L,

$1\mathrm{im}\, f(X_t) = f(X_L)\, P^x$ a. s., all x. …..4.

Let $y \in E - N$ be such that f is cofine continuous at y.

The set $j_n = \{ x: | f(x) - f(y) | > 1/n \}$ is then elusive at y;

hence, $y^{P^X}\{L_{J_n} < \rho \} = 1, x \in E_g$.

If follows that $| f(X_s) - f(y) | < 1/n$ for s sufficiently close to ρ. As n is arbitrary $y^{P^X}\left\{ \lim_{s\uparrow p} f(\mathrm{x}_s) = f(y)\right\} = 1$ …..5.

If L is a finite co-optional time, the process killed at L is an h −transform,

with $h(x) = P^x\,(L > 0)$. Since X_L – exists, implies that h is a Green's potential, say $h = G_\mu$.

We find that,

$y^{P^X}\{ X_{\rho^-} = y \} = 1$, we have, if $A = \lim_{s \to L} f(X_s) = f(X_L)$

$P^x \{ \wedge / L > 0 \} = \frac{1}{h(x)} \int \mathrm{g}_y (\mathrm{x})\, y^{P^X} \lim_{s\uparrow p} f(X_s) = f(y)\}\, \mu(dy)$

$= \frac{1}{h(x)} \int \mathrm{g}_y (\mathrm{x})\, \mu(dy) = 1$ ……6.

where the penultimate equality follows from the equation (5).

We further suppose f is a bounded Borel function for which $s \to f(X_S)$ has left limits P^x – a. s. for all x. If y is not isolated in the cofine topology, we define $f(x) = \underset{x - y, x \neq y}{cofine\ lim} \sup f(x)$ and

$f(x) = \underset{x - y, x \neq y}{cofine\ lim} \inf f(\mathrm{y})$ and, if y is isolated, put $\bar{f}(\mathrm{y}) = f(\mathrm{y})$. Both f and $\bar{f}$ are Borel functions. The set on which f is not cofine continuous is contained in $A = \{ y: \overline{f}(y) > f(y) \}$. If K is a compact subset of A, P_K is a Green's potential, say $P_K = G_\mu$, and sup(μ) ⊂ K. The process killed at L_K being a P_K- transform, if $x \in \mathrm{E}_P$

$P^x \{ \lim_{s\uparrow L_x} f(X_s-) > \lim f(X_s-) / L_K > 0 \} = P^{K^{P^X}} \lim f(X_s-) > \lim_{s\uparrow\rho} f(X_s-)$

$= \frac{1}{P_K(x)} \int_K \mathrm{g}_y(\mathrm{x})\, {}_yP^x \{ \lim f(X_s-) > \overline{\lim} f(X_s-)\} > \mu(dy).$ …..7.

But for $y \in K \cap E - N$ and $\varepsilon > 0$ both $\{ x: f(x) > \overline{f}(y) - \varepsilon \}$

and $\{ x: f(x) < f(y) + \varepsilon \}$ are not elusive at y; theg_y- transform enter both arbitrarily near to ρ. In other words, we get an inequality

${}_yP^x\{ \underset{s \uparrow \rho}{lim} f(X_s-) > \bar{f}(\mathrm{y}) - \varepsilon, \lim_{s\uparrow\rho} f(X_s-) < f(y) + \varepsilon \} = 1.$ …….8.

We combine equations (7) and (8) which equal = $\frac{1}{P_K(x)}\int_k g_y(x)\ \mu(dy) = 1$.

But now $s \rightarrow f(X_s-)$ has left limits P^x – a. s., so $P^x\{ 0 < L_K < \infty \}\ \overline{\lim_{s\uparrow\rho}} f(X_s-) > \underline{\lim_{s\uparrow\rho}} f(X_s-)$. This contradicts (7) unless $E_{P^K} = \emptyset$, i.e., unless K is left polar. K is arbitrary, so it follows that A is left polar. Thus the set at which f fails to have a cofine limit is left polar.

Now the set $\{: \overline{f(x)} < f(x) - \varepsilon\}$ can have no cofine accumulation points, from which it follows that f is cofine contradicts except at A. By (CFI), which we have verified,

both $s - f(X_s-)$ and s - X_s- are left continuous.

We consider the two left continuous processes $Y_t = \lim_{s\uparrow t} f(X_s-)$ and $Z_t = f(X_s-)$. We show that they are indistinguishable. It is sufficient to show equality at an arbitrary finite co-optional time L. Let $h = G_\mu$ be the Green's potential $P^x\ \{L > 0\}$. If $= \lim_{s\uparrow\rho} f(X_s-) = f(X_\rho-)$, then, because the process killed at L is an h –transform,

$P\ \{ Y_L = Z_L / L > 0 \} = {}_hP^X\{\wedge\} = \frac{1}{h(x)}\int g_y(x)\ y^{P^X}\ \{\wedge\}\ \mu(dy).$ …9.

Since the set A $\{$ x: $\bar{f}$(x) > f(x) $\}$ is left polar, we may assume that

$\bar{f}(y) = \text{confine} \lim_{x\rightarrow y, x\neq y} f(x)$, which means that the set $\{ x \neq y: |f(x) - f(y)| \} > \varepsilon$ is elusive at y, for any $\varepsilon > 0$.

It follows that $y^{P^X}\{\lim_{\tau\subset X_{E\neq y}} f(X_s-) = \bar{f}(y)\} = 1$ ……..10.

But we know that $\lim_{s\uparrow\rho} f(X_s-)$ exists, so if y is not in a left polar set, $y^{P^X}\{\wedge\} = 1$, x ∈ E_g, ; hence the last term in (9) is one. Hence, the theorem is proved.

There are different topologies associated with the topology on sets and Getoor's d –topology, and extend some of the results.

We define a topology as follows: Let us consider a Borel set V is a neighborhood of $y \in E - N$ in the τ - topology if $y \in V$ and if V^c is co-thin at y, and V is a neighborhood of $y \in N$ if it is an ordinary neighborhood of y. These neighborhoods from a base for a topology which we denote it by τ - topology.

We compare the τ - topology with the cofine topology, with the observations that it M_A and L_A are respectively the last exits of X_t and X_t- from A, then V is an τ - neighborhood of y *iff*

$y \in V$ and ${}_yP^X\{ M_{V^c} < \tau, \tau > 0 \}, = 0$ ……11.

while V is a cofine neighborhood *iff* $y \in V$ and ${}_yP^X\{ L_{V^c} < \rho, \rho > 0 \} = 0$. …….12.

The difference between the two only involves interchanging X_t and X_t- but it is interesting feature that τ - topology is not always a cofine topology. However, the two topologies coincide in the case X is continuous and in fact they coincide in the case of standard processes in duality, as we will see.

We know that X is said to satisfy Runt's hypothesis (B) if, for each semi polar B, $\{ P^x \exists t: X_t \in B$ and $X_t- \neq X_t = 0, all\ x \in E\}$.

6. Theorem

Suppose that X is quasi-left continuous, and satisfies Runt's hypothesis. Then the τ - topology and the cofine topology coincide.
Before proving the theorem (6), we consider another theorem whose assertions are suitably applied. We generalize here the results due to J. B. Walsh.

7. Theorem

Let $A \in \beta$. If x is quasi-left continuous, $P^x\{ T_A \leq S_S \} = 1$. If X satisfies hypothesis, $\{ S_A \leq T_A \} = 1$.

Proof

We consider the case where A is compact. Suppose X is quasi-left continuous and left $f(x) = E^X\{e^{-S_A}\}$. S_A is predictable on $X_{S_A} \notin A$, hence $X_{S_A} = X_{S_A} \in A$, while on the set $X_{S_A} \in A$, we must have $f(X_{S_A}) = 1$ by the strong Markov property. But $\{x: f(x) = 1\} \subset A$. Thus in either case $T_A \leq S_A$.

Suppose X satisfies Runt's Hypothesis and let U_A be the terminal time

$U_A = \inf\{ : X_t \in A$ and $X_t- = X_t \}$.

And let $h(x) = E^X\{ e^{-U_A} \}$, and let $A' = \{x: h(x) = 1\}$.

Suppose we further show that $A - A'$ is semi polar. Since A is compact, $X_{T_A} \in A'$, then, since

$S_A \le U_A$, $S_A \circ \theta_T = 0$ a.s., so $S_A \le T_A$. On the other hand, P^X { $X_T \in$ A – A′, $X_T \neq X_T$} = 0, and again $S_A \le T_A$.
We further show that $A - A'$ is semi polar. Consider the set $B = A \cap \{x: f(x) < 1 - \varepsilon\}$, and let $U^1 = U_B$, $U^{n+1} = U^n + U_B$ o θ_{U^n}, A is compact and B is fine closed, hence $X_{U^n} \in B$ if $U^n < \infty$.
Since $B \subseteq A$, $U_B \ge U_A$, so on { $U^n < \infty$ }, $E\{e^{-(U^{n+1} - U^n)}/\} \le 1 - \varepsilon.\tau_{U^n}\} \le 1 - \varepsilon$
It thus follows that 1im $U^n = \infty$. But we consider the interval (U^n, U^{n+1}). There cannot be at in this interval for which $X_t \in B$ and $X_{t^-} = X_t$. Since X_t is continuous except on a countable set, X_t can be in B for at most countably many t ∈ (U^n, U^{n+1}), and hence for at most contably many $t \ge 0$. According to Dellacherie B is semipolar. Then so is $A - A'$, which is a countable union of such sets. Hence, the theorem is proved.

Proof of Theorem (6)
Let $V \in \beta$ contain y. since $y \notin V^c$,
yP^X\{ $S_{V^c} = \rho$ or T_{V^c} = μ\} = 0. It follows from this and the lemma that yP^X\{ $L_{V^c} = M_{V^c}$\} = 1 for $X \in E_{g_y}$, since

L_A = sup { t ∈ Q: S_{V^c} o $\theta_t < \infty$} and M_A = sup { $t \in Q, T_{V^c}$ o $\theta_t < \infty$} ……13.

Thus conditions (11) and (12) reduce to the same condition. Hence, the theorem is proved.
An important topology, called the d-topology, was introduced by Getoor in order to study regular excessive functions. We examine that it also coincide with the cofine topology. The d-open set is a cofine neighborhood of quasi every one of its points.

8. Definition
A Borel set D is a d-set if, for each initial measure μ and each increasing sequence $\{T_n\}$ of stopping times with the property that $X_{T_n} \in D$ P^μ- a.s. on { $T_n < \infty$}, { $X_T \in D$ } where $T = lim\ T_n$. The complements of d –sets form a base for a topology called the d –topology.

9. Theorem
Suppose X is a Runt process satisfying hypothesis of theorem (1). Then the d –topology is finer than the cofine topology. Conversely, if A is Borel measurable and open in the d –topology, it is a cofine neighborhood of all but a left polar set of $y \in A$.

Proof
Let $D \in \beta$ and let K be compact in $E - D$. Let M_D(t) = sup { $s < t$: $X_s \in D$} be the last exit before t, and define a predictable set $\wedge_{D,K}$ by $\wedge_{D,K}$ = { (t, w): t > 0, $X_{t^-} \in K$ and M_D(t) = t }. We show that D is a d-set *iff* $\wedge_{D,K}$ is evanescent for all compacts ⊂ D^c, and all initial distributions. For if $\wedge_{D,K}$ is not P^x- evanescent, there exists a predictable time T with $[T] \subset \wedge_{D,K}'$ and P^x { < ∞ } > 0. Let (T_n) announce T. Now $M_D(T) = T$ on { T < ∞}, so the process hits D infinitely often on the interval (T_n, T). We can find a stopping time T_n such that $X_{T_n} \subset D$ a. s. on { < ∞ } and $T_n \le T_n{}' \le T$. Since T is predictable, $X_T = X_T$ K. But $X_{T_n}{}' \in D$ *and* $X_T \in D^c$ means that D is not a d –set. Conversely, if D is not a d –set, there exists x ∈ E, stopping times $T_n \uparrow T$, and a compact $K \subset E - D$ such that P^x { $X_{T_n} \in D$, ∀n and $X_T \in K$ } > 0. By quassi-left continuity,

1im $X_{T_n} = X_T \in K$, so $P^x\{ X_{T^-} \in K, M_D(T) = T \} > 0$, ……..14.

and $\wedge_{D,K}$ is not evanescent.
We further suppose A is cofine open and let $K \subset A$ be compact.
Let $(w) = sup\{ t: (t,w) \in \wedge_{A^c,K} \}$ be the end of $\wedge_{A^c,K}$. Notice that L is co-optional. By an argument often used above, the process killed at L in an h-transform, with $h(x) = P^x$ { $L > 0$ } = G_μ, where supp (μ) ⊂ K. Thus if $x \in E_h$,

P^x { $M_{A^c} = L > 0$ } = $\int_K g_y(x)$ yP^x { $M_{A^c} = \rho$ } μ$(dy) = 0$ ……..15.

Since A^c is elusive, hence co-thin, at all $y \in K$.
We suppose A^c is d –set and let K be a compact subset of those $y \in A$ for which A^c is not elusive (and hence not co-thin). The process killed at L_K is a P_K –transform, and $P_K = G_\mu$ with supp (μ) ⊂ K, so

P^x { $M_{A^c}(L_K) = L_K > 0$ } = $\int_K g_y(x)$ yP^X { $M_{A^c} = \rho$ } μ(dy)
= $\int_K g_y$(x) μ(dy) = P_K (x), …….16.

Since A is not a co-fine neighborhood of any $y \in K$. Thus if $L_K > 0$, then $X_{L_K} \in K$ and $M_{A^c}(L_K) = L_K$ i.e., [L] ⊂ $\wedge_{A^c,K}$. Therefore $\wedge_{A^c,K}$ is not evanescent unless K is left polar.
It is interesting feature that the d-topology is strictly finer then the co-fine topology at every point, even in the best of cases. We illustrate it by considering Brownian motion in the plane. If x is any point and D a line passing through x, D, being closed, is a d-set. Since points are polar, $D' = D - \{x\}$ is still a d –set. Thus $R^2 - D'$ is a neighborhood of x in the d-topology, but it is not a co-fine neighborhood. Hence, the fine and co-fine topology coincide in this case, and it is known that Brownian motion from x will encounter D'.
We discuss here elementary properties of Green's function in the context of results derived by K. Janssen on the existence of harmonic space.

Let δ be the cone of excessive functions. By a theorem of Mokobodzki δ is a union of caps C_f of the form $C_f = \{ v \in g: < v, f > \leq 1 \} \ldots$ ………..17.

Where *f* is Borel and strictly positive of E and

$< v, f > = \int_E v(x) f(x) \mu_0(dx)$.

Also there exists a function g such that $g(x) = 0$ *iff* f(x) = 0, and such that if C_f is given the topology induced by $L_1(g \,.\, \mu_0)$, is compact and metrizable. Let $\mathfrak{S}$ be the Borel field of C_f.

10. Lemma

If $K(v, x; dy)$ is a positive e kernel from $C_f \times E$ to E, then $(v, x) \rightarrow Kv(x)$ is $\mathfrak{S} \times \beta$ – measurable.

Proof

If K is of the form $K(v, x; dy) = p(v)\, q(x)\, r(y)\, \mu_0(dy)$, where p, q and r are bounded and measurable,
we obtain $Kv(x) = p(v)\, q(x) < v, r >$, …………18.
and $v \rightarrow < v, r >$ is a continuous linear functional. This extends to K of the form
$K(v, x; dy) = K(v, x, y)\, \mu_0(dy)$ by the monotone class theorem.

In general, let us consider $K_\lambda(v, x; .) = \lambda \int K(v, x; dz)\, R_\lambda(z, .)$, …….19.

Where R_λ is the resolvent. Since R_λ (z,.) << μ_0 (μ_0 is the reference measure), K_λ is of the form, so $(v, x) \rightarrow Kv(x)$ is $\mathfrak{S} \times \beta$ – measurable. As v is excessive, $Kv(x) = 1im < K\, v(x) >$.
It follows that

$(v, x) \rightarrow Pt\, v(x)$ and $(v, x) \rightarrow v(x)$ are $\mathfrak{S} \times \beta$ – measurable $K(x, .) = \delta_x(.)$, as is
$(v, x) \rightarrow {}_vP_t f(x)$ for a positive Borel *f*.

Let Ω be the space of functions from [0, ∞] to $E \cup \delta$ which admit a lifetime ρ are right continuous on [0, ∞), and have left limits except possibly at. Let X be defined canonically on Ω, and let τ^0 and ${\tau^0}_t$ be the natural fields.

11. Theorem

If $\wedge \in \tau^0$, then $(v, x) \rightarrow vP^x \{ \wedge \}$ is $\mathfrak{S} \times \beta$ – measurable.

Proof

This is true if $\wedge = \{ X_t \in A \}$ for some Borel set A, and for $\wedge$ of the form
$\{X_{t_1} \in A_1, \ldots, X_t \in A_n\}$ by induction. It extends to all $\wedge\, \tau^0$ by the monotone class theorem. We state a lemma before proving the main theorem.

12. Lemma

For $y \in E, (v, x, y) \rightarrow vP^x\{ X_\rho = y\}$ is $\beta \times \beta$ – measurable.

Proof

Let $\{A_{nj}\}_{j=1}^{\infty}$ be a partition of E into Borel sets of diameter bounded by 1/n for each j, $(v, x) \rightarrow vP^x \{ X_{\rho}- \in A_{nj} \}$ is ×β – measurable by (11).
Let us put
$u_n\,(v, x, y) = \sum_j I_{A_{nj}}$ (y) $vP^x \{ X_{\rho}- \in A_{nj}\}$. Then u_n is $\mathfrak{S} \times \beta \times \beta$ measurable, and converges to $vP^x \{X_\rho = y\}$ as $n \rightarrow \infty$. Hence, the lemma is proved.
We consider the set $\mathfrak{S}_f = \{ v \in C_f: < v, f > = 1 \}$. This is a compact, convex, metrizable set; hence the ∂C_f of its extreme points is a G_δ. These extreme points are minimal excessive functions.
We define a mapping p: v → pole of v, from those v ∈ ∂C_f which have poles, to E. We know that N_1 is the Borel set of hypothesis (G), so if $y \in E - N_1$, these exists at most one v ∈ ∂C_f with $P(v) = y$.

13. Theorem

P is $\mathfrak{S}$ - measurable, and the restriction of ρ^{-1} to $E - N_1$ is one-to-one and β –measurable.

Proof

Let $h(v, y) = \int_E vP^x \{ X_\rho = y \}\, \mu_0(dy) = {}_vP^{\mu_0} \{ X_\rho = y \}$. We combine the assertions of the theorem (12) and, h is $\mathfrak{S} \times \beta$ - measurable, $vP^x \{ X_{\rho}- = y \}$ is constant on E_v, being one if $y = \rho(v)$ and zero otherwise. Hence, either $E_v = \emptyset$ or
$\mu_0(E_v) > 0$. If $B \in \beta$ and $K \in \mathfrak{S}$, $K \subset \partial C_f$, let

$A_{BK} = \{ (v, y) : v \in K, y \in B \text{ and } h(v, y) > 0 \}$
$= \{ (v, y) : v \in K, y \in B \text{ and } y = \rho(v)\}$. ……….20.

Now $\boldsymbol{\rho}^{-1}(B)$ is the projection of $A_{B,}\partial c_f$ on ∂c_f. $A_{B.}\partial c_f$ is a Borel subset of a polish space and hence is a Lusin space; the projection is a continuous map, and it is one-to-one since each $v \in \partial c_f$ has at most one pole. Also $\boldsymbol{\rho}^{-1}(B) \in \mathfrak{S}$. Thus ρ is measurable. If we restrict ρ^{-1} to $E - N_1$, it becomes one-to-one, and we can project $A_{E-N_{1,K}}$ on E to conclude that ρ^{-1} is also measurable. Hence the theorem is proved.

14. Theorem

$(x,y) \rightarrow g(x,y)$ is $\beta \times \beta$ measurable.

Proof

The set $\Gamma\lambda = \{(x,y) \in E \subset N^c : g(x,y) < \lambda \}$ is the projection on $E \times E$ of the $\mathfrak{S} \times \beta \times \beta$ measurable set

$$\{ (v,x,y) \in \partial^{C}{}_{f} \times E \times N^c : y = \rho(v) \; and \; v(x) < \lambda \} \quad \text{.........21.}$$

For any$(x,y) \in E \times N^c$, there is at most one v $\in \partial^{C}{}_{f}$ for which ρ(v) = y.
Hence by Lusin's theorem, $\Gamma\lambda$ is $\beta \times \beta$ – measurable.

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