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Maximum and maximal roman free functions

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Abstract
In this paper we consider Maximum Roman Free Functions and Maximal Roman Free Function with minimum cardinality. We consider the change in the Roman Free Number and an rf-number of a graph when a vertex is removed from the graph. We prove a necessary and sufficient condition under which the Roman Free Number of a graph increases. Also we prove a necessary and sufficient conditions under which the rf-number of a graph increases or decreases.

Keywords: Roman Free Function, Maximal Roman Free Function, Maximum Roman Free Function, rf-function, Roman Free Number, rf- number.

1. Introduction

The concept of Roman Domination was introduced in [5] by Ernie J. Cockayne, T.W. Haynes, and others. A Dominating Function gives rise to a Roman Dominating Function. The concept of independence can be extended to a new concept which involves a function whose range is $\{0,1,2\}$. In [2] we define the concept of a Roman Free Function. We observed that every Independent set gives rise to a Roman Free Function. It is natural to define the concepts like Maximal Roman Free Function and Maximum Roman Free Function. This concepts have been studied in [2]. In this paper we bring Maximal Roman Free Functions to play some other role. We may note that a Maximal Independent Set in a graph is a Minimal Dominating Set. Therefore we have Independent Domination Number for graphs. Here we consider Maximal Roman Free Functions with minimum weight which are called rf-functions. The weight of such a function is called the rf-number of the graph. We state and prove necessary and sufficient conditions under which the rf-number increases or decreases when a vertex is removed from the graph. We also prove a necessary and sufficient condition under which the Roman Free Number of a graph decreases when a vertex is removed from the graph.

2. Preliminaries and Notations

In this paper we consider only those graphs which are simple and finite. If G is a graph, $V(G)$ will denote the vertex set of graph G and $E(G)$ will denote the edge set of graph G. If G is a graph and $v \in V(G)$ then $G - v$ will denote the subgraph obtained by removing the vertex v from G. The Roman Domination Number of the graph G is denoted as $\gamma_R(G)$, whereas the Domination Number of the graph G is denoted as $\gamma(G)$. If $f: V(G) \to \{0,1,2\}$ is a function then we write,

$\boldsymbol{V_2(f)} = \{v \in V(G) \,/\, f(v) = 2\}$
$\boldsymbol{V_1(f)} = \{v \in V(G) \,/\, f(v) = 1\}$
$\boldsymbol{V_0(f)} = \{v \in V(G) \,/\, f(v) = 0\}$

Obviously the above sets are mutually disjoint and their union is the vertex set V(G). The weight of this function $f = \sum_{v \in V(G)} f(v)$. This number is denoted as $w(f)$.

Definition 2.1 [2]**:** Let G be a graph. A function $f: V(G) \to \{0,1,2\}$ is said to be a *Roman Free Function* if for every edge $e = uv, f^*(e) = f(u) + f(v) \leq 2$.

Definition 2.2 [2]**:** A Roman Free Function with maximum weight is called a *Maximum Roman Free Function* and its weight is called *Roman Free Number* of a graph and it is denoted as $\beta_{rf}(G)$..

Definition 2.3 [2]**:** A Roman Free Function f is said to a *Maximal Roman Free Function* if whenever $g: V(G) \to \{0,1,2\}$ is any function such that $g > f$ then g is not a Roman Free Function.

Definition 2.4 [2]**:** A Maximal Roman Free Function with minimum weight is called *rf-function* and its weight is called the *rf-number* of a graph and it is denoted as $rf(G)$.

Maximum Roman Free Functions

Proposition 3.1: Let G be a graph and $v \in V(G)$ then $\beta_{rf}(G - v) \leq \beta_{rf}(G)$.

Proof: Let g' be a Maximum Roman Free Function defined on $G - v$.

Now define $g: V(G) \to \{0,1,2\}$ as follows:

$$g(v) = 0 \text{ and}$$
$$g(w) = g'(w); if\ w \neq v$$

Obviously g is a Roman Free Function on G.

Therefore $\beta_{rf}(G) \geq w(g) = w(g') = \beta_{rf}(G - v)$.

i.e. $\beta_{rf}(G - v) \leq \beta_{rf}(G)$. ◻

Now we state and prove the necessary and sufficient condition under which the Roman Free Number increases when a vertex is removed from the graph.

Theorem 3.2: Let G be a graph and $v \in V(G)$ then $\beta_{rf}(G - v) < \beta_{rf}(G)$ if and only if for every Maximum Roman Free Function f defined on $V(G)$ $f(v) \neq 0$.

Proof: First suppose $\beta_{rf}(G - v) < \beta_{rf}(G)$.

Let f be a Maximum Roman Free Function defined on $V(G)$ such that $f(v) = 0$. Now define $g: V(G - v) \to \{0,1,2\}$ as follows:

$$g(w) = f(w); \forall\ w \in V(G - v)$$

Obviously g is a Roman Free Function on $G - v$ and $f(v) = 0$.

Also $w(g) = w(f) = \beta_{rf}(G) > \beta_{rf}(G - v)$.

i.e. $w(g) > \beta_{rf}(G - v)$; which is a contradiction.

Therefore the condition is satisfied by every Maximum Roman Free Function on G.

Conversely suppose there is a Maximum Roman Free Function f defined on $V(G)$ such that $f(v) \neq 0$.

Suppose $\beta_{rf}(G - v) = \beta_{rf}(G)$.

Let g be a Maximum Roman Free Function defined on $V(G - v)$.

Define $g': V(G) \to \{0,1,2\}$ as follows:

$$g'(v) = 0 \text{ and}$$
$$g'(w) = g(w); if\ w \neq v$$

Then obviously g' is a Maximum Roman Free Function on G with $g'(v) = 0$; which contradicts the hypothesis.

Therefore $\beta_{rf}(G) > \beta_{rf}(G - v)$.

Example 3.3: Consider the following example in which the vertex set is $\{v_1, v_2, v_3, v_4\}$.

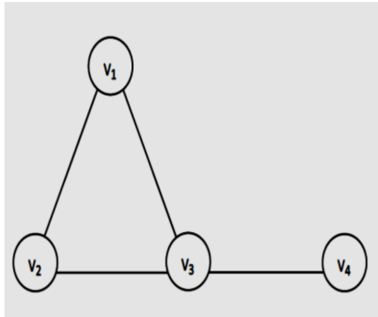

Fig 1: (GRAPH G)

Let $f: V(G) \to \{0,1,2\}$ be any function such that $f(v_1) = 2, f(v_2) = 0, f(v_3) = 0\ and\ f(v_4) = 2$

Then f is a Maximal Roman Free Function and $\beta_{rf}(G) = 4$.

Now consider the graph $G - v_3$.

The Roman Free Number of $V(G - v_3) = 4$ i.e.$\beta_{rf}(G - v_3) = 4$.

Thus we have $\beta_{rf}(G - v_3) = \beta_{rf}(G)$. ◻

Example 3.4: Consider the Cycle graph $G = C_5$ with vertices $\{v_1, v_2, v_3, v_4, v_5\}$.

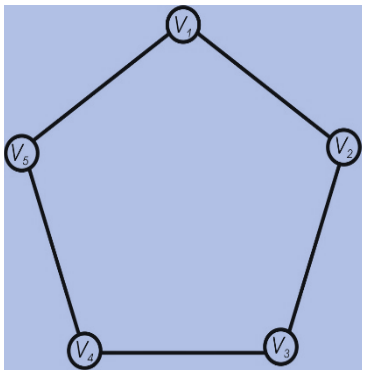

Fig 2: (GRAPH G)

Define $f: V(G) \to \{0,1,2\}$ as follows:

$$f(v_i) = 1, \forall i = 1,2,3,4,5$$

Obviously f is Maximal Roman Free Function and $\beta_{rf}(G) = 5$.

Now consider the graph $G - v_5$ which is a path graph P_4 with vertices $\{v_1, v_2, v_3, v_4\}$.

Then clearly $\beta_{rf}(G - v_5) = 4$.

Thus $\beta_{rf}(G - v_5) < \beta_{rf}(G)$. ◻

Maximal Roman Free Functions

We proved the following theorem in [2].

Theorem 4.1: Let G be a graph and $f: V(G) \to \{0,1,2\}$ be a Roman Free Function then f is a Maximal Roman Free Function if and only if whenever $f(v) < 2$ there is a vertex u adjacent to v such that $f(u) + f(v) = 2$. ◻

Now we consider Maximal Roman Free Functions with minimum cardinality which are defined as *rf*-functions. It is obvious that every *rf*-function is a Roman Dominating Function. Here we also consider the operation of removing the vertex from the graph on the *rf*-number of the graph.

Proposition 4.2: Let G be a graph and $v \in V(G)$. If there is an *rf*-function f on $G - v$ such that $f(w) = 2$ for some $w \in N(v)$ then $rf(G - v) \geq rf(G)$.
Proof: Let f be an *rf*-function on $G - v$ such that $f(w) = 2$ for some $w \in N(v)$.
Define $g: V(G) \to \{0,1,2\}$ as follows:
$$g(v) = 0 \text{ and}$$
$$g(w) = f(w); \forall\, w \neq v$$
Then g is a Maximal Roman Free Function.
Therefore $rf(G) \leq w(g) = w(f) = rf(G - v)$.
i.e. $rf(G) \leq rf(G - v)$. □

Proposition 4.3: Let G be a graph and v be an isolated vertex of G. If f is a Maximal Roman Free Function on G then $f(v) = 2$.
Proof: Consider f be a Maximal Roman Free Function with $f(v) = 0$ *or* 1.
Define $g: V(G) \to \{0,1,2\}$ as follows:
$$g(v) = 2 \text{ and}$$
$$g(w) = f(w); \forall\, w \neq v$$
Then g is a Roman Free Function on G and $f < g$; which contradicts the maximality of f.
Thus $f(v) = 2$. □
Now we state and prove the necessary and sufficient conditions under which the *rf*-number increases when a vertex is removed from the graph.

Theorem 4.4: Let G be a graph and $v \in V(G)$ then $rf(G - v) > rf(G)$ if and only if the following conditions are satisfied:
i) v is not an isolated vertex in G.
ii) $f(v) = 2$ for every *rf*- function f on G.
iii) There is no Maximal Roman Free Function g on $G - v$ such that $w(g) \leq rf(G)$ and $V_2(g)$ is a subset of $V(G) - N[v]$.

Proof: Suppose $rf(G - v) > rf(G)$
i) Suppose v is an isolated vertex, then for every Maximal Roman Free Function f on $G, f(v) = 2$.
Define $g: V(G - v) \to \{0,1,2\}$ as follows:
$$g(w) = f(w); \forall\, w \in V(G - v)$$
Then g is a Maximal Roman Free Function.
Therefore $rf(G - v) \leq w(g) \leq w(f) = rf(G)$.
i.e. $rf(G - v) \leq rf(G)$; which is a contradiction.
Therefore v is not isolated vertex in G.
ii) Suppose for some *rf*-function f on $G, f(v) = 0$ then v cannot be an isolated vertex on G.
Now define g on $G - v$ as follows:
$$g(w) = f(w); \forall\, w \in V(G - v)$$
Then obviously g is a Maximal Roman Free Function on $G - v$.
Also $rf(G - v) \leq w(g) = w(f) = rf(G)$; which is a contradiction.
Now suppose for some *rf*- function f on $G, f(v) = 1$.
Now define g on $G - v$ as follows:
a) If w is a neighbour of v such that for every neighbour x of w in $G - v, f(x) = 0$ or w is an isolated vertex in $G - v$ then define $g(w) = 2$.
b) If w is a neighbour of v such that for every neighbour x of w in $G - v, f(x) = 1$ then define $g(w) = 1$.
c) For all other vertices t, define $g(t) = f(t)$.
Then g is a Maximal Roman Free Function on $G - v$.
Also $rf(G - v) \leq w(g) \leq w(f) = rf(G)$; which is again a contradiction. Thus $f(v) = 1$ is also not possible for any *rf*-function f on G.
Hence $f(v) = 2$ for every *rf*- function f on G.
iii) Suppose there is a Maximal Roman Free Function g on $G - v$ with $w(g) \leq rf(G)$ and $V_2(g)$ is a subset of $V(G) - N[v]$.
Then obviously $rf(G - v) \leq w(g) \leq rf(G)$; which is a contradiction.
Conversely suppose conditions (i), (ii) and (iii) are satisfied.
Suppose $rf(G - v) = rf(G)$. Let f be an *rf*-function on $G - v$.
Now define $g: V(G) \to \{0,1,2\}$ as follows:
$$g(v) = 0 \text{ and}$$
$$g(w) = f(w); \forall\, w \neq v$$
Then g is a Roman Free Function on G.

Case-I: Suppose there is a neighbour x of v such that $g(x) = 2$. Then g is a Maximal Roman Free Function on G. Since $w(g) = w(f) = rf(G)$ and also we have $rf(G) = rf(G - v), g$ is an *rf*-function on G with $g(v) = 0$.

Case-II: For every neighbour w of $v, f(w) \neq 2$. Then $g(w) \neq 2$. Then $V_2(g)$ is a subset of $V(G) - N[v]$ and g is a Maximal Roman Free Function on $G - v$ such that $w(f) \leq rf(G)$; which contradicts condition (iii).
Thus $rf(G - v) = rf(G)$ is not possible.
Suppose $rf(G - v) < rf(G)$. Let g be an *rf*-function on $G - v$.
Now define $f: V(G) \to \{0,1,2\}$ as follows:
$$f(v) = 0 \text{ and}$$
$$f(w) = g(w); \forall\, w \neq v$$
Then f is a Roman Free Function on G.
If there is a neighbour x of v such that $f(x) = g(x) = 2$ then f is a Maximal Roman Free Function on G. This implies $rf(G) \leq w(f) = w(g) = rf(G - v)$.
i.e. $rf(G) \leq rf(G - v)$; which is a contradiction.
Therefore for every neighbour x of v such that $f(x) = g(x) \neq 2$.
Thus there is a Maximal Roman Free Function g on $G - v$ such that $V_2(g)$ is a subset of $V(G) - N[v]$ and $w(g) \leq rf(G)$; this again contradicts the condition (iii).
Thus $rf(G - v) < rf(G)$ is also not possible.
Hence $rf(G - v) > rf(G)$. □

Example 4.5: Consider the following example in which the vertex set is $\{v_1, v_2, v_3, v_4\}$.

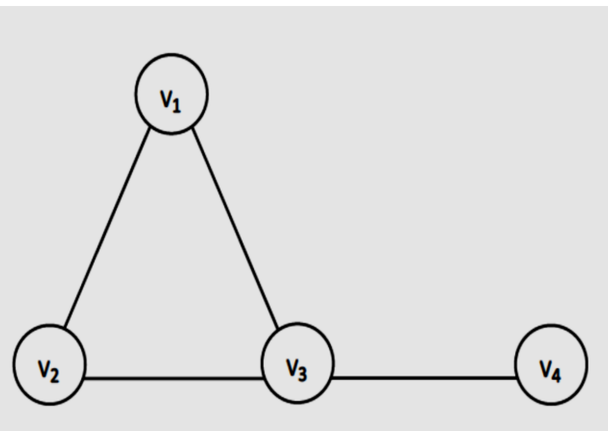

Fig 3: (GRAPH G)

Define $f: V(G) \rightarrow \{0,1,2\}$ be any function such that $f(v_1) = 0, f(v_2) = 0, f(v_3) = 2$ *and* $f(v_4) = 0$
Then f is an *rf*-function and $rf(G) = 2$.
Now consider the graph $G - v_3$ and define g on $G - v_3$ as follows:
$g(v_1) = 2, g(v_2) = 0$ *and* $g(v_4) = 2$
Then g is an *rf*-function and $rf(G - v_3) = 4$
Thus $rf(G - v_3) > rf(G)$. ◻

Corollary 4.6: Let G be a graph and $u, v \in V(G)$ such that $rf(G - u) > rf(G)$ and $rf(G - v) > rf(G)$ then u and v are non-adjacent vertices.

Proof: Let f be any *rf*-function on $V(G)$ then $f(v) = 2$ and $f(u) = 2$ by the theorem 4.4. If u and v are adjacent then $f(u) + f(v) = 2 + 2 > 2$; which contradicts the fact that f is a Roman Free Function. So u and v can not be adjacent vertices. ◻

Remark 4.7: From the above corollary it follows that the set of all vertices for which $rf(G - v) > rf(G)$ is an Independent set. ◻
Now we state and prove the necessary and sufficient conditions under which the *rf*-number decreases when a vertex is removed from the graph.

Theorem 4.8: Let G be a graph and $v \in V(G)$ then $rf(G - v) < rf(G)$ if and only if for every *rf*-function g on $G - v$ there is an *rf*-function h on G such that the restriction of h on $G - v$ is equal to g and $h(v) = 1$ or 2.

Proof: First suppose that the condition is satisfied. Let g be an *rf*-function on $G - v$ then there is an *rf*-function h on G such that $h(v) = 1$ or 2 and the restriction of h on $G - v$ is equal to g.
Then $rf(G - v) = w(g) < w(h) = rf(G)$.
i.e. $rf(G - v) < rf(G)$.
Conversely suppose $rf(G - v) < rf(G)$. Let g be an *rf*-function on $G - v$.
Define $h': V(G) \rightarrow \{0,1,2\}$ as follows:

$$h'^{(v)} = 0 \text{ and } h'(w) = g(w); \forall\, w \neq v$$

Then obviously h' is a Roman Free Function on G. But h' cannot be a Maximal Roman Free Function on G because it would imply that $rf(G) \leq w(h') = w(g) = rf(G - v)$; which is not true.
Therefore there is a Maximal Roman Free Function h'' on G such that $h' < h''$. Therefore there is a vertex x in G such that $h'(x) < h''(x)$.
If $x \neq v$ then this would imply that $g(x) < g'(x)$ where g' is the restriction of h'' on $G - v$. This means g is not a Maximal Roman Free Function on $G - v$ which is a contradiction.
Therefore $x = v$.
Thus $h'(v) < h''(v)$. Therefore $h''(v) = 1$ or 2.
Suppose $h''(v) = 1$.
Since $rf(G - v) < rf(G), h''$ is an *rf*-function on G with $h''(v) = 1$ and the restriction of h'' on $G - v$ is equal to g.
Suppose $h''(v) = 2$.
Then again by the similar argument h'' is an *rf*-function on G with $h''(v) = 2$ and the restriction of h'' on $G - v$ is equal to g. Thus the theorem. ◻

Example 4.9: Consider the graph $G = C_4$ with vertices $\{v_1, v_2, v_3, v_4\}$.
Now consider the graph $G - v_4$ which is a path graph P_3 with vertices $\{v_1, v_2, v_3\}$.

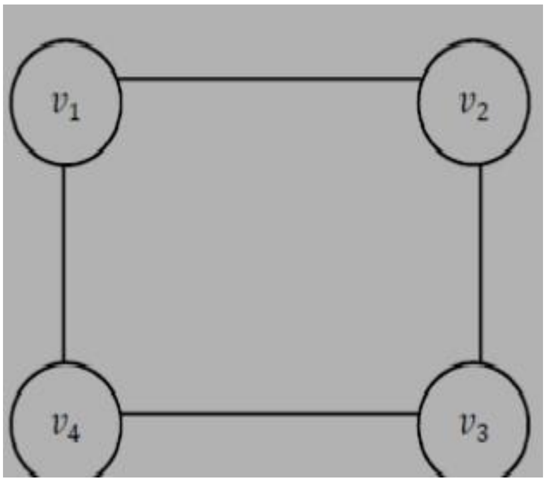

Fig 4: (GRAPH G)

Define g on $G - v_4$ as follows:
$g(v_1) = 0, g(v_2) = 2$ *and* $g(v_3) = 0$
Then g is an *rf*-function on $G - v_4$ and $rf(G - v_4) = 2$
Now define h on G as follows:

$$h(v_4) = 2 \text{ and}$$
$$h(w) = g(w); \forall w \neq v_4$$

Then h is an *rf*-function on G and $rf(G) = 4$.
Further the restriction of h on $G - v_4$ is equal to g with $h(v_4) = 2$.
Thus $rf(G - v_4) < rf(G)$.

Corollary 4.10: Let G be a graph and $u, v \in V(G)$ such that $rf(G - u) < rf(G)$ and $rf(G - v) > rf(G)$ then u and v are non adjacent vertices.

Proof: Let g be an *rf*-function on $G - u$.
Since $rf(G - u) < rf(G)$ there is an *rf*-function f on G such that $f(u) = 1$ or 2 and restriction of f on $G - u$ is equal to g.
Since $rf(G - v) > rf(G), f(v) = 2$.
If u and v are adjacent then either $f^*(uv) = f(u) + f(v) = 3$ or 4; which contradicts the fact that f is a Roman Free Function.
Therefore u and v cannot be adjacent vertices.

5. Concluding Remarks

The restriction of a Roman Free Function on its subgraph is a Roman Free Function. However the restriction of a Maximal Roman Free Function on its subgraph need not be a Maximal Roman Free Function. Thus we may consider the following problem:

Problem 1: Under what conditions the restriction of a Maximal Roman Free Functions on its subgraph is a Maximal Roman Free Functions?
In particular we may consider the following problem:

Problem 2: Let v be a vertex of the graph G. Under what conditions the restriction of a Maximal Roman Free Function on $G - v$ is a Maximal Roman Free Function.
Further the similar problem can be asked for *rf*-functions also. There is a scope for investigation in this direction.

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