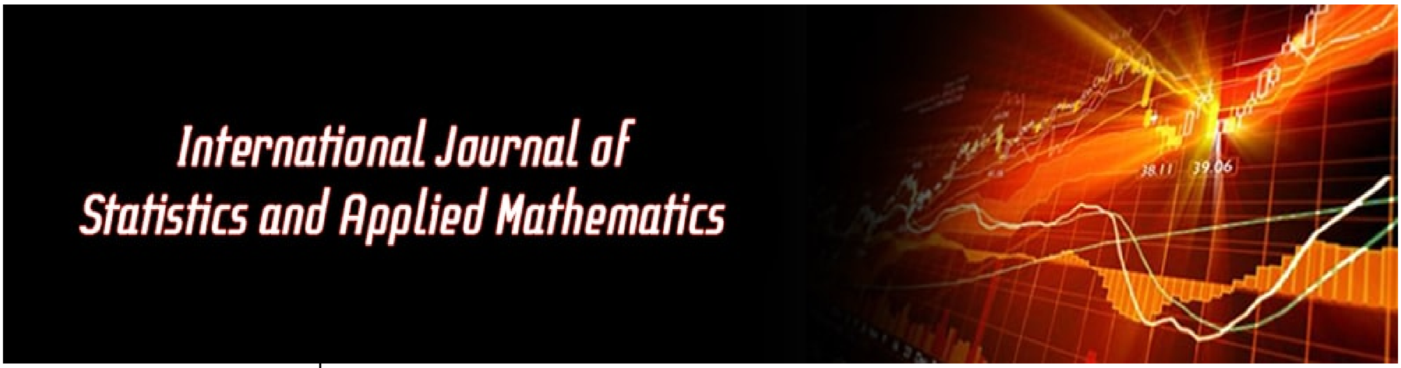
International Journal of Statistics and Applied Mathematics 2018; 3(1): 101-106



ISSN: 2456-1452
Maths 2018; 3(1): 101-106

www.mathsjournal.com
Received: 15-11-2017
Accepted: 22-12-2017

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Hybrid fixed point theorem for abstract measure Integro-differential equation

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Abstract
In this paper, an existence theorem for an abstract measure integro-differential equation is proved via a nonlinear alternative of Leray-Schauder type. An existence result is also proved for extremal solutions for Caratheodory as well as discontinuous cases of the nonlinearities involved in the equations, an example is also proved to illustrate the hypothesis and abstract theory developed in this paper.

Keywords: Abstract measure integro-differential equation, Existence theorem

1. Introduction

The study of abstract measure differential equations is iniated by Sharma [8, 9] and subsequently developed by Joshi [7], Shendge and Joshi [10], Similarly, the study of abstract measure Integro-differential equation is studied by Bellale and Dhage [4] for various aspect of solutions. In such models of differential equations, the ordinary derivative is replaced by the derivative of the set functions which there by gives the generalization of the ordinary derivative and measure differential equations. The various aspects of the solution of the abstract measure differential equations have been studied in the literature using the fixed point techniques such as Schauder's fixed point principle and Banach contraction mapping principle etc. In such situation one needs to show that the operator under consideration maps a certain set into itself. This is a serve restriction, which motivated us to persue the study of abstract measure Integro-differential equations using the nonlinear alternative of Leray-Schauder. In the present chapter we shall prove the existence and uniqueness result for an abstract measure-Integro differential equation in question is also proved under certain monotonicity conditions of the nonlinearity involved in the equation. In the following section we give some preliminaries in the sequel.

2. Definitions and Notations

Let X be real Banach space with a norm $\| . \|$. Let $x, y \in X$. Then the line segment $\overline{x\,y}$ in X is defined by

$$\overline{x\,y} = \{ z \in X \mid z = x + r(y - x) ,\ 0 \le r \le 1 \}. \tag{2.1}$$

Let $x_0 \in X$ be a fixed point and $z \in X$. Then for any $x \in \overline{x_0\, z}$, we define the sets S_x and $\overline{S}_x$ in X by

$$S_x = \{ r\,x : -\infty < r < 1 \} \tag{2.2}$$

And $\overline{S}_x = \{ r\,x : -\infty < r \le 1 \}$ (2.3)

Thus we have

$\overline{x\,y} = \overline{S}_x - S_x$ for all $x, y \in \overline{x_0\, z}$ be arbitrary. We say $x_1 < x_2$ if $Sx_1 \subset Sx_2$ or equivalently $\overline{x_0\, x_1} \subset \overline{x_0\, x_2}$. In this case we also write $x_2 > x_1$.

Let M denote the $\sigma-$ algebra of all subsets of X so that $ca(X,M)$ becomes a measurable space. Let $ca(X,M)$ be the space of all vector measures (signed measures) and define a norm $|\,.\,|$ on $ca(X,M)$ by

$$\|P\| = |p|(X) \tag{2.4}$$

Where $|p|$ is a total variation measure of p and is given by

$$|p|(X) = \sup_{\sigma} \sum_{i=1}^{\infty} |p(E_i)|, \ \forall E_i \subset X \tag{2.5}$$

Where supremum is taken over all partition $\{E_i : i \in N\}$ of X. It is known that $ca(X,M)$ is a Banach space with respect to the norm $\|\,.\,\|$ defined by (2.4). Let μ be a σ - finite measure on X and let $p \in ca(X,M)$. We say p is absolutely continuous with respect to the measure μ if $\mu(E) = 0$ implies $p(E) = 0$ for some $E \in M$. In this case we write $p << \mu$.

For a fixed $x_0 \in X$, Let M_0 denote the $\sigma-$algebra on S_{x0}. Let $z \in X$ be such that $z > x_0$ and Let M_z denote the $\sigma-$ algebra all sets containing M_0 and the sets of the form S_x for $x \in \overline{x_0 z}$. Finaly, Let $L^1_\mu(S_z, IR)$ denote the space of all μ integrable real valued function h on S_z with the norm $\|.\|_{L^1_\mu}$ defined by $\|\,.\,\|_{L^1_\mu} = \int_{S_z} |h(x)|\,d\mu$.

3. Statement of the problem:

Let μ be a real or σ - finite positive measure on X. Given a $p \in ca(X,M)$ with $p << \mu$, consider the abstract measure differential equation (in short AMIDE)

$$\left.\begin{aligned} \frac{dp}{d\mu} &= f\left(x,\, p(\overline{S}_x),\, \int_{S_x} k\left(t, p(\overline{S}_t)\right) d\mu\right) + g\left(x,\, p(\overline{S}_x),\, \int_{S_x} k\left(t, p(\overline{S}_t)\right) d\mu\right) \quad a.e.[\mu] \text{ on } \overline{x_0 z} \\ p(E) &= q(E), \qquad E \in M_0 \end{aligned}\right\} \tag{3.1}$$

Where q is a given known vector measure, $\frac{dp}{d\mu}$ is a Radon-Nikodym derivative of p with respect to μ and $f, g : S_z \times IR \times IR \to IR$ is such that

$$x \to f\left(x,\, p(\overline{S}_x),\, \int_{S} k\left(t, p(\overline{S}_t)\right) d\mu\right) + g\left(x,\, p(\overline{S}_x),\, \int_{S_x} k\left(t, p(\overline{S}_t)\right) d\mu\right)$$

is μ - integrable for each $p \in ca(S_z, M_z)$.

Definition 3.1: Given an initial real measure q on M_0, a vector $p \in ca(S_z, M_z)(z > x_0)$ is said to be a solution of AMIDE(3.1) if

(i) $p(E) = q(E), E \in M_0$

(ii) $p << \mu$ on $\overline{x_0 z}$

(iii) P satisfies (3.1) a.e. $[\mu]$ on $\overline{x_0 z}$

Remark 3.1: The AMIDE (3.1) is equivalent to the abstract measure integral equation

$$p(E) = \begin{cases} \int_E f\left(x, p(\overline{S}_x), \int_{S_x} k\left(t, p(\overline{S}_t)\right) d\mu\right) d\mu + \int_E g\left(x, p(\overline{S}_x), \int_E k\left(x, p(\overline{S}_x)\right) d\mu\right) d\mu, & E \in M_z, E \subset \overline{x_0 z} \\ q(E), & E \in M_0 \end{cases} \tag{3.2}$$

A solution p of AMIDE (3.1) on $\overline{x_0 z}$ will be denoted by $p(\overline{S}_{x_0}, q)$.

In the following section we shall prove the main existence theorem for AMIDE (3.1) under suitable conditions on f. We shall use the following form of the Leray-Schauder's nonlinear alternative. See Dugundji and Granas [6].

Theorem 3.1: Let $B(0,r)$ and $B[0,r]$ denote respectively the open and closed balls in a Banach space X centered at the origin 0 of radius r, for some $r > 0$. Let $T : B[0,r] \to X$ be a completely continuous operator. Then either

(i) the operator equation $Tx = x$ has a solution in $B[o,r]$, or

(ii) there exists $u \in X$ with $\|u\| = r$ such that $\|u\| = \lambda Tu$ for some $0 < \lambda < 1$.

4. Existence theorems:
We need the following definition in the sequel.

Definition 4.1: A function $\psi : S_z \times IR^+ \to IR^+$ is called a D **- function** if it satisfies
(i) ψ is continuous,
(ii) ψ is non decreasing, and
(iii) ψ is scalar submultiplicative, that is $\psi(\lambda r) \leq \lambda(\psi r)$ for all $\lambda > 0$ and $r \in IR^+$.
The class of all D -function on IR^+ is denoted by ψ. There do not exists D - function on IR.
Indeed the function $\psi : S_z \times IR^+ \to IR^+$ defined by $\psi(r) = l\, r$, $l > 0$ satisfies the conditions $(i)-(iii)$
Mentioned above and hence a D - function on IR^+. Note that $\psi \in \psi$ *then* $\psi(0) = 0.$

Definition 4.2: A function $f : S_z \times IR \times IR \to IR$ is said to satisfy condition of Caratheodory or simply Caratheodory if
(i) $x \to f(x, y_1, y_2)$ is μ - measurable for each $y_1, y_2 \in IR$.
(ii) $(y_1, y_2) \to f(x, y_1, y_2)$ is a continuous for almost everywhere $[\mu]$ on $x \in \overline{x_0\, z}$
Again, a Caratheodory function f on $S_z \times IR \times IR$ is called L^1_μ-Caratheodory, if
(iii) For each given real number $\rho > 0$ there exists a function $h_p \in L^1_\mu(S_z, IR)$ such that $|f(x, y_1, y_2)| \leq h_p(x)$ *a. e.* $[\mu]$, $x \in \overline{x_0\, z}$, for all $y_1, y_2 \in IR$ *with* $|y_1| \leq \rho$ *and* $|y_2| \leq \rho$ We consider the following set of assumptions.

(A_1) For any $z > x_0$ the σ -algebra M_z is compact with respect to the topology generated by the Pseudo-metric d defined by $d(E_1, E_2) = |\mu|(E_1 \Delta E_2)$, $E_1, E_2 \in M_z$.
(A_2) $\mu(\{x_0\}) = 0$
(A_3) q is continuous on M_z with respect to the Pseudo-metric d defined in (A_1)
(A_4) The function $f(x, y_1, y_2)$ is L^1_μ - Caratheodory.
The function $k : J \times J \times IR \to IR$ is continuous function and there exists a function $a \in L^1_\mu(J, IR^+)$, such that $|k(x, y_1, y_2)| \leq \alpha(x)|y|$ *a. e.* $y_1, y_2 \in J$ *and* $\forall y \in IR$
(A_5) There exists a function $\phi \in L^1_\mu(S_z, IR^+)$ such that $\phi(x) > 0$ *a. e.* $[\mu]$, $x \in S_z$ and a continuous and nondecreasing function $\psi : [0, \infty) \to (0, \infty)$, such that $|f(x, y_1, y_2)| \leq \phi(x)\, \psi(|y_1| + |y_2|)$ *a.e.* $[\mu]$ *on* $\overline{x_0 z}$. For all $y_1, y_2 \in IR$

Theorem 4.1: suppose that assumptions $(A_1)-(A_5)$ hold. Further if there exists a real number $r > 0$ such that

$$r > \|q\| + \|\phi\|_{L^1_\mu}([1 + \|\alpha\|])\, \psi(r) \tag{4.1}$$

then AMIDE (3.1) has a solution on M_z.

Proof: Let $X = ca(S_z, M_z)$ and consider an open ball $B(0, r)$ in $ca(S_z, M_z)$ centered at the origin '0' and radius r, where the real number $r > 0$ satisfies (4.1). Define an operator T from $B[0, r]$ into $ca(S_z, M_z)$ by

$$Tp(E) = \begin{cases} \int_E f\left(x,\ p(\overline{S}_x),\ \int_{S_x} k(t, p(\overline{S}_t))\right) d\mu + \int_E g\left(x, p(\overline{S}_x), \int_{S_x} k(t,\ p(\overline{S}_t))\right) d\mu, & E \in M_z,\ E \subset \overline{x_0\, z} \\ q(E), & \text{if } E \in M_0 \end{cases}$$

We shall show that the operator T satisfies all the conditions of Theorem (3.1) on $B[0, r]$.

Step I: *First we show that* T *is a continuous on* $B[0, r]$.

Let $\{p_n\}$ be a sequence of vector Measures in $B[0,r]$ converging to a vector measure p. Then by dominated convergence theorem,

$$\lim_{n\to\infty} Tp_n(E) = \lim_{n\to\infty}\left(\int_E f\left(x, p_n(\overline{S}_x), \int_{\overline{S}_x} k(t, p_n(\overline{S}_t))d\mu\right)d\mu + \int_E g\left(x, p_n(\overline{S}_x), \int_{\overline{S}_x} k(t, p_n(\overline{S}_t))d\mu\right)d\mu,\right.$$

$$= \int_E f\left(x, p(\overline{S}_x), \int_{\overline{S}_x} k(t, p(\overline{S}_t)d\mu)d\mu\right) + \int_E g\left(x, f(\overline{S}_x), \int_{\overline{S}_x} k(t, p(\overline{S}_x))d\mu\right)d\mu, \quad = Tp(E)$$

For all $E \in M_z$, $E \subset \overline{x_0 z}$. Similarly if $E \in M_0$, then

$$\lim_{n\to\infty} Tp_n(E) = q(E) = Tp(E)$$

And, so T is a continuous operator on $B[0,r]$.

Step II: *Now we show that $T(B[0,r])$ is a uniformly bounded set in $ca(S_z, M_z)$.*

Let $p \in B[0,r]$ be arbitrary. Then we have $\|p\| \le r$. Now by definition of the map T one has

$$Tp_n(E) = \begin{cases} \int_E f\left(x, p(\overline{S}_x), \int_{\overline{S}_x} k(t, p(\overline{S}_t))d\mu\right)d\mu + \int_E g\left(x, p(\overline{S}_x), \int_{\overline{S}_x} k(t, p(\overline{S}_t))d\mu\right)d\mu, & E \in M_z,\ E \subset \overline{x_0 z} \\ q(E), & E \in M_0 \end{cases}$$

Therefore for any $E = F \cup G$, $F \in M_0$ *and* $G \in M_z$, $G \subset \overline{x_0 z}$,

We have,

$$|Tp_n(E)| \le |q(E)| + \left|\int_E f\left(x, p(\overline{S}_x), \int_{Sx} k(t, p(\overline{S}_t))d\mu\right)d\mu + \int_E g\left(x, p(\overline{S}_t), \int_{Sx} k(x, p(\overline{S}_t))d\mu\right)d\mu\right|$$

$$\le \|q\| + \int_E \left| f\left(x, p(\overline{S}_t), \int_{Sx} k(t, p(\overline{S}_t))d\mu\right) + g\left(x, p(\overline{S}_t), \int_{\overline{S}_x} k(t, p(\overline{S}_t))d\mu\right)\right| d\mu$$

$$\le \|q\| + \|\phi\|_{L^1_\mu}\left(\left[1 + \|\alpha\|_{L^1_\mu}\right]\right)\psi(r)$$

For all $E \in M_z$, By definition of the norm $\|\cdot\|$ we have

$$\|Tp_n\| = |Tp_n|(S_z) \le \|q\| + \|\phi\|_{L^1_\mu}\left(\left[1 + \|\alpha\|_{L^1_\mu}\right]\right)\psi(r)$$

This shows that the set $T(B[0,r])$ is uniformly bounded in $ca(S_z, M_z)$.

Step III: *we show that $T(B[0,r])$ is an equi-continuous set in $ca(S_z, M_z)$.*

Now we show that $T(B[0,r])$ is an equi-continuous set in $ca(S_z, M_z)$. Let $E_1, E_2 \in M_z$. Then there are sets $F_1, F_2 \in M_0$ *and* $G_1, G_2 \in M_z$ with $G_1, G_2 \subset \overline{x_0 z}$, and $F_i \cap G_i = \phi$, for $i = 1, 2$.

We know the set-identities

$$\left.\begin{aligned} G_1 &= (G_1 - G_2) \cup (G_2 \cap G_1) \\ &\quad and \\ G_2 &= (G_2 - G_1) \cup (G_2 \cap G_1) \end{aligned}\right\} \tag{4.2}$$

Therefore, we have

$$Tp_n(E_1) - Tp_n(E_2) = q(F_1) - q(F_2) + \int_{G_1 - G_2}\left[f\left(x, p(\overline{S}_x), \int_{\overline{S}_x} k(t, p(\overline{S}_t))d\mu\right) + g\left(x, p(\overline{S}_x), \int_{\overline{S}_x} k(t, p(\overline{S}_t))d\mu\right)\right]d\mu$$

$$- \int_{G_2 - G_1}\left[f(x, p(\overline{S}_x)), \int_{\overline{S}_x} k(t, p(\overline{S}_t))d\mu + g\left(x, p(\overline{S}_x), \int_{\overline{S}_x} k(t, p(\overline{S}_t))d\mu\right)\right]d\mu$$

Since $f(x,y)$ is L^1_μ - Caratheodory, we have that

$$| Tp_n(E_1) - Tp_n(E_2)| \le | q(F_1) - q(F_2)| + \int_{G_1 \Delta G_2} \left[\left| f\left(x, p(\overline{S}_x), \int_{\overline{S}_x} k\left(t, p(\overline{S}_t)\right) d\mu\right) + g\left(x, p(\overline{S}_t), \int_{\overline{S}_x} k\left(t, p(\overline{S}_t)\right) d\mu\right) \right| \right] d\mu$$

$$\le | q(F_1) - q(F_2)| + \int_{G \Delta G_2} h_r(x) d\mu$$

Assume that $d(E_1, E_2) = |\mu|(E_1 \Delta E_2) \to 0$.

Then we have $E_1 \to E_2$ and consequently $F_1 \to F_2$ and $|\mu|(G_1 \Delta G_2) \to 0$. From the continuity of q on M_0 it follows that

$$| Tp_n(E_1) - Tp_n(E_2)| \le | q(F_1) - q(F_2)| + \int_{G_1 \Delta G_2} h_r(x) d\mu \to 0 \ \ as \ E_1 \to E_2 .$$

This shows that $T(B[0,r])$ is an equi-continuous set in $ca(S_z, M_z)$. Now $T(B[0,r])$ is uniformly bounded and equi-continuous set in $ca(S_z, M_z)$, so it is compact in the norm in the topology on $ca(S_z, M_z)$. Now an application of Arzela-Ascolli theorem yields that $T(B[0,r])$ is a compact subset of $ca(S_z, M_z)$. As a results T is a continuous and totally bounded operator on $B[0,r]$. Hence an application of theorem (3.1) yields that either $Tx = x$ has a solution or the operator equation $\lambda Tx = x$ has a solution u with $\|u\| = r$ for some $0 < \lambda < 1$. We shall show that this later assertion is not possible. We assume the contrary. Then there is an $u \in X$ with $\|u\| = r$ satisfying $u = \lambda Tu$ for some λ, $0 < \lambda < 1$. Now for any $E \in M_z$, we have $E = F \cup G$, where $F \in M_0$ and $G \subset \overline{x_0 z}$, satisfying $F \cap G = \phi$.

Now, $u(E) = \lambda Tu(E)$

$$u(E) = \begin{cases} \lambda q(F), & if \quad F \in M_0 \\ \lambda \int_G \left[f\left(x, u(\overline{S}_x), \int_{\overline{S}_x} k\left(t, u(\overline{S}_t)\right) d\mu\right) + g\left(x, u(\overline{S}_t), \int_{\overline{S}_x} k\left(t, u(\overline{S}_t)\right) d\mu\right) \right] d\mu, & G \in M_z \ and \ G \subset \overline{x_0 z} \end{cases}$$

Therefore,

$$|u(E)| = |\lambda q(F)| + \left| \lambda \int_G \left[f\left(x, u(\overline{S}_x), \int_{\overline{S}_x} k\left(t, u(\overline{S}_t)\right) d\mu\right) + g\left(x, u(\overline{S}_x), \int_{\overline{S}_x} k\left(t, u(\overline{S}_t)\right) d\mu\right) \right] \right| d\mu$$

$$\le \|q\| + \left| \int_G \left[f\left(x, u(\overline{S}_x), \int_{\overline{S}_x} k\left(t, u(\overline{S}_t)\right) d\mu\right) + g\left(x, u(\overline{S}_x), \int_{\overline{S}_x} k\left(t, u(\overline{S}_t)\right) d\mu\right) \right] d\mu \right|$$

$$\le \|q\| + \int_G \phi(t) \psi\left(\left| u(\overline{S}_t) \right| \right) d\mu$$

$$\le \|q\| + \int_G \phi(t) \psi(\|u\|) d\mu$$

$$= \|q\| + \|\phi\|_{L^1_\mu} \psi(\|u\|).$$

This further implies that

$$\|\mu\| = |u|(S_z) \le |u(E)|$$

$$\le \|q\| + \|\phi\|_{L^1_\mu} \psi(\|u\|).$$

Substituting $\|u\| = r$ in the above inequality yields that

$$r \le \|q\| + \|\phi\|_{L^1_\mu} \psi(r) \tag{4.3}$$

Which is a contradiction to the inequality (4.1).

Hence the operator equation $Tp = p$ has a solution in v with $\| v \| \le r$. Consequently the AMIDE(3.1) has a solution $p = p(S_{x_0}, q)$ in $B[o, r]$. This completes the proof.

Example 4.1: Given $p \in ca(S_z, M_z)$ with $p << \mu$, consider the AMIGDE(3.1)

$$\frac{d}{d\mu}\left(\frac{p(\bar{S}_x)}{1+|p(\bar{S}_x)|}\right) = \frac{\phi(x)\, p(\bar{S}_x)}{1+p^2(\bar{S}_x)} \quad a.e.[\mu] \quad on \ \overline{x_0 z} \tag{4.4}$$

$p(\bar{S}_{x0}) = q \in IR$ (4.5) where q is a given known vector measure, $\lambda(\bar{S}_x) = \frac{p(\bar{S}_x)}{f(x, p(\bar{S}_x))}$ is a signed measure such that $\lambda << \mu$.

Define the functions $f : S_z \times IR \to IR$ and $g : S_z \times IR \to IR$ by

$f(x, y) = 1 + |y|$ and $g(x, y) = \frac{\phi(x) y}{1+y^2}$ respectively.

Below we shall show that the functions f and g satisfy all the conditions of theorem (4.1).

Obviously f is continuous on the domain of its definition. Let $y_1, y_2 \in IR$. Then we have

$$|f(x.y_1) - f(x, y_2)| = \left|1 + |y_1| - 1 - |y_2|\right|$$
$$= \left| |y_1| - |y_2| \right|$$
$$\le |y_1 - y_2|$$

Which show that $f(x, y)$ satisfies the Lipschitz condition in y with the Lipschitz constant $\alpha = 1$.

Obviously the function $g(x, y)$ is Caratheodory on $\overline{x_0 z}$. To see this, note that the function $x \to \frac{\phi(x) y}{1+y^2}$ is obviously $\mu-$ measurable for all $y \in IR$ and the function $y \to \frac{\phi(x) y}{1+y^2}$ is continuous for all $x \in \overline{x_0 z}$. Again

$$g(x, y_1, y_2) = g(x, y_1) = \left|\frac{\phi(x) y_1}{1+y_1^2}\right| \le |\phi(x)| = \phi(x)\psi(\| y_1 \|),$$

Where $\psi : IR^+ \to IR^+$ is defined by $\psi(r) = 1$.

Thus, if $\| q \| + \| \phi \|_{L^1} < 1$, then all the assumptions $(A_1)-(A_5)$ of theorem (4.1) are satisfied. Hence, the AMIGDE (4.4) has solution $p(\bar{S}_{x0}, z)$ defined on $\overline{x_0 z}$.

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