Hybrid fixed point theorem for abstract measure
Integro-differential equation

SS Bellale and GB Dapke

Abstract
In this paper, an existence theorem for an abstract measure integro-differential equation is proved via a nonlinear alternative of Leray-Schauder type. An existence result is also proved for extremal solutions for Caratheodory as well as discontinuous cases of the nonlinearities involved in the equations, an example is also proved to illustrate the hypothesis and abstract theory developed in this paper.

Keywords: Abstract measure integro-differential equation, Existence theorem

1. Introduction
The study of abstract measure differential equations is initiated by Sharma [8, 9] and subsequently developed by Joshi [7], Shendge and Joshi [10]. Similarly, the study of abstract measure Integro-differential equation is studied by Bellale and Dhage [4] for various aspect of solutions. In such models of differential equations, the ordinary derivative is replaced by the derivative of the set functions which there by gives the generalization of the ordinary derivative and measure differential equations. The various aspects of the solution of the abstract measure differential equations have been studied in the literature using the fixed point techniques such as Schauder’s fixed point principle and Banach contraction mapping principle etc. In such situation one needs to show that the operator under consideration maps a certain set into itself. This is a serve restriction, which motivated us to persue the study of abstract measure Integro-differential equations using the nonlinear alternative of Leray-Schauder. In the present chapter we shall prove the existence and uniqueness result for an abstract measure-Integro differential equation in question is also proved under certain monotonicity conditions of the nonlinearity involved in the equation. In the following section we give some preliminaries in the sequel.

2. Definitions and Notations
Let $X$ be real Banach space with a norm $\| \cdot \|$. Let $x, y \in X$. Then the line segment $\overline{xy}$ in $X$ is defined by

$$\overline{xy} = \{ z \in X \mid z = x + r(y - x), 0 \leq r \leq 1 \}.$$  \hspace{1cm} (2.1)

Let $x_0 \in X$ be a fixed point and $z \in X$. Then for any $x \in \overline{x_0 z}$, we define the sets $S_x$ and $\overline{S_x}$ in $X$ by

$$S_x = \{ r x : -\infty < r < 1 \}$$ \hspace{1cm} (2.2)

and

$$\overline{S_x} = \{ r x : -\infty < r \leq 1 \}.$$ \hspace{1cm} (2.3)

Thus we have

$$\overline{xy} = \overline{S_x} - S_y$$

for all $x, y \in \overline{x_0 z}$ be arbitrary. We say $x_1 < x_2$ if $S_{x_1} \subset S_{x_2}$ or equivalently $x_1 x_2 \subset x_0 x_2$. In this case we also write $x_2 > x_1$. 

x_1
Let $M$ denote the $\sigma$-algebra of all subsets of $X$ so that $ca(X,M)$ becomes a measurable space. Let $ca(X,M)$ be the space of all vector measures (signed measures) and define a norm $\| \cdot \|$ on $ca(X,M)$ by

$$\|P\| = \|P\|(X)$$

(2.4)

Where $\|P\|$ is a total variation measure of $P$ and is given by

$$\|P\|(X) = \sup_{\sigma} \sum_{i=1}^{n} |P(E_i)|, \forall E_i \subseteq X$$

(2.5)

Where supremum is taken over all partition $\{E_i : i \in N\}$ of $X$. It is known that $ca(X,M)$ is a Banach space with respect to the norm $\| \cdot \|$ defined by (2.4). Let $\mu$ be a $\sigma$-finite measure on $X$ and let $P \in ca(X,M)$. We say $P$ is absolutely continuous with respect to the measure $\mu$ if $\mu(E) = 0$ implies $P(E) = 0$ for some $E \in M$. In this case we write $P \ll \mu$.

For a fixed $x_0 \in X$, Let $M_0$ denote the $\sigma$-algebra on $S_{x_0}$. Let $z \in X$ be such that $z > x_0$ and Let $M_z$ denote the $\sigma$-algebra all sets containing $M_0$ and the sets of the form $S_x$ for $x \in x_0 z$. Finally, Let $L_\mu(S_z, IR)$ denote the space of all $\mu$-integrable real valued function $h$ on $S_z$ with the norm $\|h\|_\mu$ defined by $\|h\|_\mu = \int_{S_z} |h(x)| d\mu$.

3. Statement of the problem:

Let $\mu$ be a real or $\sigma$-finite positive measure on $X$. Given a $p \in ca(X,M)$ with $P \ll \mu$, consider the abstract measure differential equation (in short AMIDE)

$$\frac{dp}{d\mu} = f \left( x, p(S_x), \frac{1}{x}, k(t, p(S_x)) \right) d\mu + g \left( x, p(S_x), \frac{1}{x}, k(t, p(S_x)) \right) d\mu$$

$$P(E) = q(E), \quad E \in M_0$$

(3.1)

Where $q$ is a given known vector measure, $\frac{dp}{d\mu}$ is a Radon-Nikodym derivative of $P$ with respect to $\mu$ and $f, g : S_z \times IR \times IR \rightarrow IR$ is such that

$$x \rightarrow f \left( x, p(S_x), \frac{1}{x}, k(t, p(S_x)) \right) + g \left( x, p(S_x), \frac{1}{x}, k(t, p(S_x)) \right)$$

is $\mu$-integrable for each $p \in ca(S_z, M_z)$.

**Definition 3.1:** Given an initial real measure $q$ on $M_0$, a vector $p \in ca(S_z, M_z)$ ($z > x_0$) is said to be a solution of AMIDE(3.1) if

(i) $P(E) = q(E), E \in M_0$

(ii) $p \ll \mu$ on $x_0 z$

(iii) $P$ satisfies (3.1) a.e. $\mu$ on $x_0 z$

**Remark 3.1:** The AMIDE (3.1) is equivalent to the abstract measure integral equation

$$P(E) = \int_E f \left( x, p(S_x), \frac{1}{x}, k(t, p(S_x)) \right) d\mu + \int_E g \left( x, p(S_x), \frac{1}{x}, k(t, p(S_x)) \right) d\mu$$

$$q(E), \quad E \in M_0$$

(3.2)

A solution $p$ of AMIDE (3.1) on $x_0 z$ will be denoted by $p(S_{x_0}, q)$.

In the following section we shall prove the main existence theorem for AMIDE (3.1) under suitable conditions on $f$. We shall use the following form of the Leray-Schauder’s nonlinear alternative. See Dugundji and Granas [6].

**Theorem 3.1:** Let $B(0, r)$ and $B(0, r)$ denote respectively the open and closed balls in a Banach space $X$ centered at the origin $0$ of radius $r$, for some $r > 0$. Let $T : B(0, r) \rightarrow X$ be a completely continuous operator. Then either

(i) the operator equation $Tx = x$ has a solution in $B(0, r)$,

(ii) there exists $u \in X$ with $\|u\| = r$ such that $\|x\| = \lambda Tu$ for some $0 < \lambda < 1$. 

"102"
4. Existence theorems:
We need the following definition in the sequel.

**Definition 4.1:** A function \( \psi : S_z \times IR \rightarrow IR \) is called a \( D \)-function if it satisfies

(i) \( \psi \) is continuous,

(ii) \( \psi \) is non-decreasing, and

(iii) \( \psi \) is scalar submultiplicative, that is \( \psi(\lambda r) \leq \lambda \psi(r) \) for all \( \lambda > 0 \) and \( r \in IR^{+} \).

The class of all \( D \)-function on \( IR \) is denoted by \( \Psi \). There does not exist a \( D \)-function on \( IR \).

Indeed the function \( \psi : S_z \times IR \rightarrow IR \) defined by \( \psi(x) = r, \) for all \( x \in IR \), satisfies the conditions (i)–(iii).

Mentioned above and hence a \( D \)-function on \( IR \). Note that \( \psi \) is an element of \( \Psi \).

**Definition 4.2:** A function \( f : S_z \times IR \rightarrow IR \) is said to satisfy condition of Caratheodory or simply Caratheodory if

(i) \( f(x, y_1, y_2) \) is \( \mu \)-measurable for each \( y_1, y_2 \in IR \).

(ii) \( f(x, y_1, y_2) \) is a continuous for almost everywhere \( \mu \) on \( x \in S_z \).

(iii) For each given real number \( \rho > 0 \) there exists a function \( h_p \in L_{\mu}^{1}(S_z, IR) \) such that \( |f(x, y_1, y_2)| \leq h_p(x) \) a.e. \( \mu \), for all \( y_1, y_2 \in IR \) with \( |y_1| \leq \rho \) and \( |y_2| \leq \rho \).

We consider the following set of assumptions.

\( (A_1) \): For any \( z > x_0 \) the \( \sigma \)-algebra \( M_z \) is compact with respect to the topology generated by the Pseudo-metric \( d \) defined by \( d(E_1, E_2) = |\mu|(E_1 \Delta E_2) \), \( E_1, E_2 \in M_z \).

\( (A_2) \): \( \mu(x_0) = 0 \).

\( (A_3) \): \( q \) is continuous on \( M_z \) with respect to the Pseudo-metric \( d \) defined in \( (A_4) \).

\( (A_4) \): The function \( f(x, y_1, y_2) \) is \( L_{\mu}^{1} \)-Caratheodory.

The function \( k : J \times IR \rightarrow IR \) is continuous function and there exists a function \( a \in L_{\mu}^{1}(J, IR) \), such that \( k(x, y_1, y_2) \leq a(x) \) a.e. \( \mu \), \( y_1, y_2 \in J \) and \( \forall y \in IR \).

\( (A_3) \): There exists a function \( \phi \in L_{\mu}^{1}(S_z, IR) \) such that \( \phi(x) > 0 \) a.e. \( \mu \), \( x \in S_z \) and a continuous and non-decreasing function \( \psi : [0, \infty) \to [0, \infty) \), such that \( |f(x, y_1, y_2)| \leq \phi(x) \psi(|y_1| + |y_2|) \) a.e. \( \mu \) on \( x_0 \).

We shall show that the operator \( T \) satisfies all the conditions of Theorem (3.1) on \( B[0, r] \).

**Theorem 4.1:** Suppose that assumptions \( (A_1)-(A_5) \) hold. Further if there exists a real number \( r > 0 \), such that

\[ r > \|q\| + \phi \left( \|1 + a\| \right) \psi(r) \]  

(4.1)

then AMIDE (3.1) has a solution on \( M_z \).

**Proof:** Let \( X = ca(S_z, M_z) \) and consider an open ball \( B(0, r) \) in \( ca(S_z, M_z) \) centered at the origin \( O \) and radius \( r \), where the real number \( r > 0 \) satisfies (4.1). Define an operator \( T \) from \( B[0, r] \) into \( ca(S_z, M_z) \) by

\[ Tp(E) = \int_{E} \left( f \left( x, p \left( \frac{S_z}{S_z} \right), k(t, p \left( \frac{S_z}{S_z} \right)) \right) d\mu + \int_{E} \left( g \left( x, p \left( \frac{S_z}{S_z} \right), k(t, p \left( \frac{S_z}{S_z} \right)) \right) d\mu \right), \quad E \in M_z, \quad E \subset x_0 \]

We shall show that the operator \( T \) satisfies all the conditions of Theorem (3.1) on \( B[0, r] \).

**Step 1:** First we show that \( T \) is a continuous on \( B[0, r] \).
Let \( \{ p_n \} \) be a sequence of vector Measures in \( B[0, r] \) converging to a vector measure \( p \). Then by dominated convergence theorem,

\[
\lim_{n \to \infty} TP_n(E) = \lim_{n \to \infty} \left( \int_E f(x, p_n(\overline{S}_x), \int k(t, p_n(\overline{S}_x))d\mu) d\mu + \int_E g(x, p_n(\overline{S}_x), \int k(t, p_n(\overline{S}_x))d\mu) d\mu \right)
\]

\[
= \int_E f(x, p(\overline{S}_x), \int k(t, p(\overline{S}_x))d\mu) d\mu + \int_E g(x, f(\overline{S}_x), \int k(t, p(\overline{S}_x))d\mu) d\mu.
\]

For all \( E \in M_z, E \subset x_0z \), similarly if \( E \in M_0 \), then

\[
\lim_{n \to \infty} TP_n(E) = q(E) - TP(E)
\]

and, so \( T \) is a continuous operator on \( B[0, r] \).

**Step II:** Now we show that \( T( B[0, r] ) \) is a uniformly bounded set in \( ca(S_z, M_z) \).

Let \( p \in B[0, r] \) be arbitrary. Then we have \( \| p \| \leq r \). Now by definition of the map \( T \) one has

\[
TP_n(E) = \int_E f(x, p(\overline{S}_x), \int k(t, p(\overline{S}_x))d\mu) d\mu + \int_E g(x, p(\overline{S}_x), \int k(t, p(\overline{S}_x))d\mu) d\mu, E \in M_z, E \subset x_0z, E \in M_0
\]

Therefore for any \( E = F \cup G, F \in M_0 \) and \( G \in M_z, G \subset x_0z \),

\[
\left| TP_n(E) \right| \leq \left| q(E) \right| + \int_E f(x, p(\overline{S}_x), \int k(t, p(\overline{S}_x))d\mu) d\mu + \int_E g(x, p(\overline{S}_x), \int k(t, p(\overline{S}_x))d\mu) d\mu
\]

We have,

\[
\left| TP_n(E) \right| \leq \left| q(E) \right| + \int_E f(x, p(\overline{S}_x), \int k(t, p(\overline{S}_x))d\mu) d\mu + \int_E g(x, p(\overline{S}_x), \int k(t, p(\overline{S}_x))d\mu) d\mu
\]

\[
\leq \| q \| + \| \phi \| L_1 \left[ \| 1 + \| \alpha \| L_1 \right] \psi(r)
\]

For all \( E \in M_z \). By definition of the norm \( \| . \| \) we have

\[
\| TP_n \| = \| TP_n(\overline{S}_x) \| \leq \| q \| + \| \phi \| L_1 \left[ \| 1 + \| \alpha \| L_1 \right] \psi(r)
\]

This shows that the set \( T( B[0, r] ) \) is uniformly bounded in \( ca(S_z, M_z) \).

**Step III:** we show that \( T( B[0, r] ) \) is an equi-continuous set in \( ca(S_z, M_z) \).

Now we show that \( T( B[0, r] ) \) is an equi-continuous set in \( ca(S_z, M_z) \). Let \( E_1, E_2 \in M_z \). Then there are sets \( F_1, F_2 \in M_1 \) and \( G_1, G_2 \in M_z \) with \( G_1, G_2 \subset x_0z \), and \( F_i \cap G_i = \phi \), for \( i = 1, 2 \).

We know the set-identities

\[
G_1 = (G_1 - G_2) \cup (G_2 \cap G_1)
\]

\[
G_2 = (G_2 - G_1) \cup (G_2 \cap G_1)
\]

Therefore, we have

\[
TP_n(E_1) - TP_n(E_2) = q(F_1) - q(F_2) + \int_{G_1 - G_2} f(x, p(\overline{S}_x), \int k(t, p(\overline{S}_x))d\mu) d\mu + \int_{G_1 - G_2} g(x, p(\overline{S}_x), \int k(t, p(\overline{S}_x))d\mu) d\mu
\]

\[
- \int_{G_2 - G_1} f(x, p(\overline{S}_x), \int k(t, p(\overline{S}_x))d\mu) + \int_{G_2 - G_1} g(x, p(\overline{S}_x), \int k(t, p(\overline{S}_x))d\mu) d\mu
\]
Since \( f(x, y) \) is \( L^1 \) Caratheodory, we have that
\[
\left| Tp_n\left(E_1\right) - Tp_n\left(E_2\right) \right| \leq q\left(F_1\right) - q\left(F_2\right) + \int_{G_1 \Delta G_2} \left[ f \left( x, p \left( \overline{S}_t \right) , \int k \left( t, p \left( \overline{S}_t \right) \right) d\mu \right] + g \left( x, p \left( \overline{S}_t \right) , \int k \left( t, p \left( \overline{S}_t \right) \right) d\mu \right] d\mu
\]
\[
\leq \left| q\left(F_1\right) - q\left(F_2\right) \right| + \int_{G_1 \Delta G_2} h_r (x) d\mu
\]
Assume that \( d(E_1, E_2) = \mu \left| (E_1 \setminus E_2) \right| \to 0 \).

Then we have \( E_1 \to E_2 \) and consequently \( F_1 \to F_2 \) and \( \mu \left| (G_1 \Delta G_2) \right| \to 0 \). From the continuity of \( q \)
on \( M_0 \) it follows that
\[
\left| Tp_n\left(E_1\right) - Tp_n\left(E_2\right) \right| \leq \left| q\left(F_1\right) - q\left(F_2\right) \right| + \int_{G_1 \Delta G_2} h_r (x) d\mu \to 0 \quad \text{as} \quad E_1 \to E_2
\]

This shows that \( T(B\left[0, r\right]) \) is an equi-continuous set in \( ca(S_z, M_z) \). Now \( T\left(B\left[0, r\right]\right) \) is uniformly bounded and equi-continuous set in \( ca(S_z, M_z) \), so it is compact in the norm in the topology on \( ca(S_z, M_z) \). Now an application of Arzela-Ascoli theorem yields that \( T(B\left[0, r\right]) \) is a compact subset of \( ca(S_z, M_z) \). As a result \( T \) is a continuous and totally bounded operator on \( B[0, r] \). Hence an application of theorem (3.1) yields that either \( T(x) = x \) has a solution or the operator equation \( \lambda T x = x \) has a solution \( u \) with \( \|u\| = r \) for some \( 0 < \lambda < 1 \). We shall show that this later assertion is not possible. We assume the contrary. Then there is an \( u \in X \) with \( \|u\| = r \) satisfying \( u = \lambda T u \) for some \( \lambda, \ 0 < \lambda < 1 \). Now for any \( E \in M_z \), we have \( E = F \cup G \), where \( F \in M_0 \) and \( G \subset x_0 \), \( z \), satisfying \( F \cap G = \phi \).

Now, \( u(E) = \lambda T u (E) \)
\[
\begin{align*}
\lambda \left| q\left(F\right) \right| & \quad \text{if} \quad F \in M_0 \\
\lambda \int \left[ f \left( x, u \left( \overline{S}_t \right) , \int k \left( t, u \left( \overline{S}_t \right) \right) d\mu \right] + g \left( x, u \left( \overline{S}_t \right) , \int k \left( t, u \left( \overline{S}_t \right) \right) d\mu \right] d\mu, \ \text{if} \quad F \in M_z \quad \text{and} \quad G \subset x_0 \zeta
\end{align*}
\]

Therefore,
\[
\left| u(E) \right| = \left| \lambda \left| q\left(F\right) \right| \right| + \lambda \int \left[ f \left( x, u \left( \overline{S}_t \right) , \int k \left( t, u \left( \overline{S}_t \right) \right) d\mu \right] + g \left( x, u \left( \overline{S}_t \right) , \int k \left( t, u \left( \overline{S}_t \right) \right) d\mu \right] d\mu
\]
\[
\leq \left| q \right| + \lambda \int \left( \phi \left( r \right) \right) \left( \left| u \left( \overline{S}_t \right) \right| \right) d\mu
\]
\[
\leq \left| q \right| + \lambda \int \left( \phi \left( r \right) \right) \left( \left| u \right| \right) d\mu
\]
\[
= \left| q \right| + \lambda \left| u \right| _{L^\psi} \left( \left| u \right| \right)
\]

This further implies that
\[
\|u\| = \left| u \right| \left( S_z \right) \leq u(E)
\]
\[
\leq \left| q \right| + \lambda \left| u \right| _{L^\psi} \left( \left| u \right| \right)
\]

Substituting \( \|u\| = r \) in the above inequality yields that
\[
r \leq \left| q \right| + \lambda \left| u \right| _{L^\psi} \left( \left| u \right| \right)
\]

(4.3)
Which is a contradiction to the inequality (4.1). Hence the operator equation $Tp = p$ has a solution in $v$ with $\|v\| \leq r$. Consequently the AMIDE (3.1) has a solution $p = p(S_{x_0}, q)$ in $B[\sigma, r]$. This completes the proof.

**Example 4.1:** Given $p \in ca(S_z, M_z)$ with $p < \mu$, consider the AMIGDE (3.1)

$$\frac{d}{d\mu} \left( \frac{p \left( \frac{\mu}{p} \right)}{1 + p \left( \frac{\mu}{p} \right)^2} \right) = \phi \left( \frac{\mu}{p} \right) \text{ a.e. } \mu \text{ on } x_0 z.$$ (4.4)

$p \left( \frac{\mu}{p} \right) = q \in \mathbb{R}$ (4.5) where $q$ is a given known vector measure, $\lambda \left( \frac{\mu}{p} \right) = \frac{p \left( \frac{\mu}{p} \right)}{f \left( x, p \left( \frac{\mu}{p} \right) \right)}$ is a signed measure such that $\lambda < \mu$.

Define the functions $f : S_z \times \mathbb{R} \rightarrow \mathbb{R}$ and $g : S_z \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x, y) = 1 + |y|$$ and $$g(x, y) = \frac{\phi(x)y}{1 + y^2}$$ respectively.

Below we shall show that the functions $f$ and $g$ satisfy all the conditions of theorem (4.1).

Obviously $f$ is continuous on the domain of its definition. Let $y_1, y_2 \in \mathbb{R}$. Then we have

$$|f(x, y_1) - f(x, y_2)| = |1 + y_1 - 1 - y_2| = |y_1 - y_2| \leq |y_1 - y_2|$$

Which show that $f(x, y)$ satisfies the Lipschitz condition in $y$ with the Lipschitz constant $\alpha = 1$.

Obviously the function $g(x, y)$ is Caratheodory on $x_0 z$. To see this, note that the function $x \rightarrow \frac{\phi(x)y}{1 + y^2}$ is obviously $\mu$-measurable for all $y \in \mathbb{R}$ and the function $y \rightarrow \frac{\phi(x)y}{1 + y^2}$ is continuous for all $x \in x_0 z$. Again

$$g(x, y_1, y_2) = g(x, y_1) = \left| \frac{\phi(x)y_1}{1 + y_1^2} \right| \leq \left| \phi(x) \right| = \left| \phi(x) \psi \left( \|y_1\| \right) \right|,$$

Where $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is defined by $\psi(r) = 1.$

Thus, if $\|q\|_1 \|\phi\|_1 < 1$, then all the assumptions $(A_1)-(A_5)$ of theorem (4.1) are satisfied. Hence, the AMIGDE (4.4) has solution $p \left( \frac{\mu}{p} \right)$ defined on $x_0 z$.

## References