The proof of trigonometric formula for the sum and difference of the two angles by Ptolemy's theorem

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Abstract

The trigonometric formula for the sum and difference of two angles is an important set of formulas in senior high school mathematics. It included four formulas totally, namely the sine formula of the sum of two angles, the sine formula of the difference of two angles, the cosine formula of the sum of two angles and the cosine formula of the difference of two angles. At present, there have been many proofs of them, including the proof with Ptolemy's theorem.[1-3] However, the proofs with Ptolemy's theorem we found by now only is the proof of the sine formula of the sum of two angles, namely \( \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \). The proofs with Ptolemy's theorem about other three formulas have not been found yet. In fact, they could also be proved with the Ptolemy's theorem.

1. Introduction

The trigonometric formula for the sum and difference of two angles is an important set of formulas in senior high school mathematics. It included four formulas totally, namely the sine formula of the sum of two angles, the sine formula of the difference of two angles, the cosine formula of the sum of two angles and the cosine formula of the difference of two angles. At present, there have been many proofs of them, including the proof with Ptolemy's theorem. [1-3] However, the proofs with Ptolemy's theorem we found by now only is the proof of the sine formula of the sum of two angles, namely \( \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \). The proofs with Ptolemy's theorem about other three formulas have not been found yet. In fact, they could also be proved with the Ptolemy's theorem.

2. The proof of the sine formula for the difference of two angles

Constructing a geometric figure firstly as shown in figure 1. In which the segment \( AD \) is the diameter of the circle \( O \). Meanwhile, it is also a side of the quadrilateral \( ABCD \). Connecting the point \( C \) and \( O \), and extending line segment \( CO \). The point \( P \) is the intersection of line segment \( CO \) and circle \( O \). Connecting \( B \) and \( P \).
Letting \( \angle CDA = \alpha \), \( \angle CDB = \beta \), \( AD = 1 \).
\[ \therefore \ \angle ACD = 90^\circ, \angle ABD = 90^\circ, \]
\[ \therefore A = \sin \beta, BD = \cos \beta, CD = \cos \alpha, AC = \sin \alpha. \]
\[ \therefore CP \text{ is the diameter of the circle } O \text{ too,} \]
\[ \therefore \angle CBP = 90^\circ. \]
\[ \therefore BC = \sin \angle BPC = \sin \angle BDC = \sin(\alpha - \beta). \]

Then we can know form the Ptolemy’s theorem:
\[ BD \cdot AC = AB \cdot CD + BC \cdot AD, \text{ namely } \cos \beta \cdot \sin \alpha = \sin \beta \cdot \cos \alpha + \sin(\alpha - \beta) \]
That is \( \sin(\alpha - \beta) = \sin \alpha \cdot \cos \beta - \cos \alpha \cdot \sin \beta \)

3. The proof of the cosine formula for the sum of two angles
Constructing a geometric figure firstly as shown in figure 2. In which the line segment \( AD \) is the diameter of the circle \( O \).
Meanwhile, it is also a side of the quadrilateral \( ABCD \). Connecting the point \( C \) and \( O \), and extending line segment \( CO \). The point \( P \) is the intersection of segment \( CO \) and circle \( O \). The point \( P \) is the intersection of line segment \( CO \) and circle \( O \). Connecting \( B \) and \( P \).

4. The proof of the cosine formula for the difference of two angles
Constructing a a geometric figure firstly as shown in figure 3. In which the line segment \( AC \) is the diameter of the circle \( O \).
Meanwhile, it is also a diagonal of the quadrilateral \( ABCD \). Connecting the point \( B \) and \( O \), and extending line segment \( BO \). The point \( P \) is the intersection of line segment \( BO \) and circle \( O \). Connecting \( D \) and \( P \).
Letting $\angle ACD = \alpha$, $\angle BAC = \beta$, $AC = 1$.

$\therefore AC$ is the diameter of the circle $O$, $\therefore \angle ADC = 90^\circ$, $\angle ABC = 90^\circ$, $\therefore AB = \cos \beta$, $BC = \sin \beta$, $AD = \sin \alpha$, $CD = \cos \alpha$. 

$\therefore BP$ is the diameter of the circle $O$ too, $\therefore \angle BDP = 90^\circ$.

$\therefore AO = BO$, $\therefore \angle ABP = \angle BAC = \beta$.

$\therefore \angle ABD = \angle ACD = \alpha$, $\therefore \angle PBD = \angle ABD - \angle ABP = \alpha - \beta$, $\therefore BD = \cos(\alpha - \beta)$

Then we can know form the Ptolemy’s theorem:

$AC \cdot BD = AB \cdot CD + BC \cdot AD$, namely $\cos(\alpha - \beta) = \cos \beta \cdot \cos \alpha + \sin \beta \cdot \sin \alpha$

That is $\cos(\alpha - \beta) = \cos \alpha \cdot \cos \beta + \sin \alpha \cdot \sin \beta$

5. References


2. Wang XW. Comparison of the proof methods of the cosine formula for the sum of the two angles, Vocational and Technical Education. 1998; (12):30-31.