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The proof of trigonometric formula for the sum and difference of the two angles by Ptolemy's theorem

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Abstract
The trigonometric formula for the sum and difference of two angles is an important set of formulas in senior high school mathematics. Based on the proof of the sine formula of two angles provided by former researchers, we gave the proofs of they other three formulas with the Ptolemy's theorem. These proof contributed to the teaching and learning of senior high school mathematics teachers and students.

Keywords: Mathematics formulas, sine, cosine, Ptolemy's theorem

1. Introduction
The trigonometric formula for the sum and difference of two angles is an important set of formulas in senior high school mathematics. It included four formulas totally, namely the sine formula of the sum of two angles, the sine formula of the difference of two angles, the cosine formula of the sum of two angles and the cosine formula of the difference of two angles. At present, there have been many proofs of them, including the proof with Ptolemy's theorem. [1-3] However, the proofs with the Ptolemy's theorem we found by now only is the proof of the sine formula of the sum of two angles, namely $\sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$. The proofs with Ptolemy's theorem about other three formulas have not been found yet. In fact, they could also be proved with the Ptolemy's theorem.

2. The proof of the sine formula for the difference of two angles
Constructing a geometric figure firstly as shown in figure 1. In which the segment AD is the diameter of the circle O. Meanwhile, it is also a side of the quadrilateral $ABCD$. Connecting the point C and O, and extending line segment CO. The point P is the intersection of line segment CO and circle O. Connecting B and P.

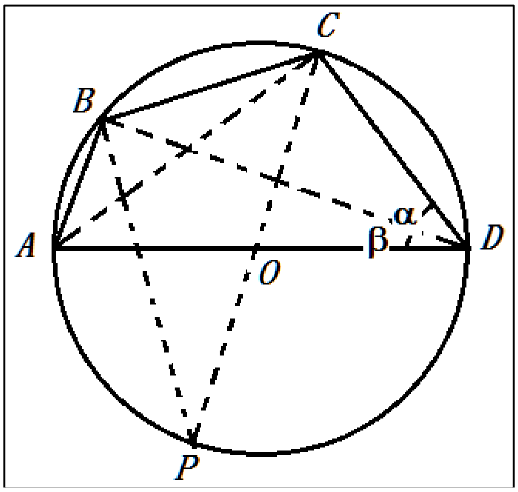

Fig 1

Letting $\angle CDA = \alpha$, $\angle CDB = \beta$, $AD = 1$.

$\because$ AD is the diameter of the circle O,

$\therefore$ $\angle ACD = 90°$, $\angle ABD = 90°$,

$\therefore$ $AB = \sin\beta$, $BD = \cos\beta$, $CD = \cos\alpha$, $AC = \sin\alpha$.

$\because$ CP is the diameter of the circle O too,

$\therefore$ $\angle CBP = 90°$,

$\therefore$ $BC = \sin\angle BPC = \sin\angle BDC = \sin(\alpha - \beta)$

Then we can know form the Ptolemy's theorem:

$BD \cdot AC = AB \cdot CD + BC \cdot AD$, namely $\cos\beta \cdot \sin\alpha = \sin\beta \cdot \cos\alpha + \sin(\alpha - \beta)$

That is $\sin(\alpha - \beta) = \sin\alpha \cdot \cos\beta - \cos\alpha \cdot \sin\beta$

3. The proof of the cosine formula for the sum of two angles

Constructing a geometric figure firstly as shown in figure 2. In which the line segment AD is the diameter of the circle O. Meanwhile, it is also a side of the quadrilateral $ABCD$. Connecting the point C and O, and extending line segment CO. The point P is the intersection of segment CO and circle O. The point P is the intersection of line segment CO and circle O. Connecting B and P.

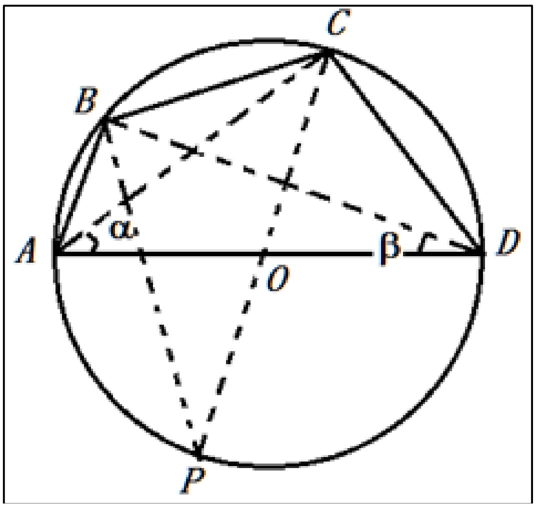

Fig 2

Letting $\angle CAD = \alpha$, $\angle ADB = \beta$, $AD = 1$.

$\because$ AD is the diameter of the circle O,

$\therefore$ $\angle ACD = 90°$, $\angle ABD = 90°$,

$\therefore$ $AB = \sin\beta$, $BD = \cos\beta$, $CD = \sin\alpha$, $AC = \cos\alpha$.

$\because$ CP is the diameter of the circle O too,

$\therefore$ $\angle CBP = 90°$.

$\because$ $AO = CO$, $\therefore$ $\angle ACP = \angle CAD = \alpha$.

$\because$ $\angle BCA = \angle BDA = \beta$,

$\therefore$ $\angle BCP = \angle BCA + \angle ACP = \alpha + \beta$, $\therefore$ $BC = \cos(\alpha + \beta)$

Then we can know form the Ptolemy's theorem

$AC \cdot BD = AB \cdot CD + BC \cdot AD$, namely $\cos\alpha \cdot \cos\beta = \sin\beta \cdot \sin\alpha + \cos(\alpha + \beta)$

That is $\cos(\alpha + \beta) = \cos\alpha \cdot \cos\beta - \sin\alpha \cdot \sin\beta$

4. The proof of the cosine formula for the difference of two angles

Constructing a a geometric figure firstly as shown in figure 3. In which the line segment AC is the diameter of the circle O. Meanwhile, it is also a diagonal of the quadrilateral $ABCD$. Connecting the point B and O, and extending line segment BO. The point P is the intersection of line segment BO and circle O. Connecting D and P.

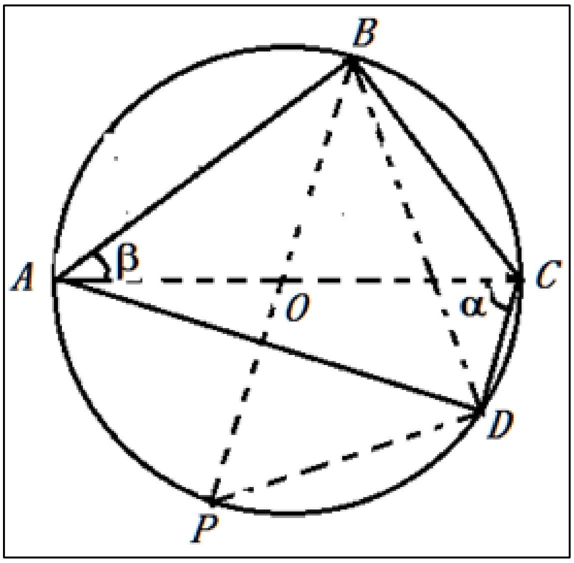

Fig 3

Letting $\angle ACD = \alpha$, $\angle BAC = \beta$, $AC = 1$.

$\because$ AC is the diameter of the circle O, $\therefore \angle ADC = 90°$, $\angle ABC = 90°$, $\therefore AB = \cos\beta$, $BC = \sin\beta$, $AD = \sin\alpha$, $CD = \cos\alpha$.

$\because$ BP is the diameter of the circle O too,

$\therefore \angle BDP = 90°$.

$\because AO = BO$, $\therefore \angle ABP = \angle BAC = \beta$.

$\because \angle ABD = \angle ACD = \alpha$,

$\therefore \angle PBD = \angle ABD - \angle ABP = \alpha - \beta$,

$\therefore BD = \cos(\alpha - \beta)$

Then we can know form the Ptolemy's theorem:

$AC \cdot BD = AB \cdot CD + BC \cdot AD$, namely $\cos(\alpha - \beta) = \cos\beta \cdot \cos\alpha + \sin\beta \cdot \sin\alpha$

That is $\cos(\alpha - \beta) = \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta$

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