International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2018; 3(1): 126-131

www.mathsjournal.com
Received: 18-11-2017
Accepted: 19-12-2017

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Note on laguerre polynomial of two variable $\mathcal{L}_n$ (x, y)

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Abstract
In this paper we obtain properties, expansion of polynomials involving the generalized associated Laguerre Polynomials which are closely related to generalized Lagguerre polynomials of Dattoli *et al*. These results provide useful extensions of the well-known results of Lagguerre Polynomials $L_n(x)$.

Keywords: Laguerre Polynomials, Dattoli *et al*., recurrence relation

1. Introduction

Two variable one index Laguerre polynomials have been given by Dattoli *et al* [1-4] and by Pathan, Khan and Yasmin [5].
Two variable one index Laguerre polynomials defined as

$$\mathcal{L}_n(x, y) = n! \sum_{r=0}^{n} \frac{(-1)^r x^r y^{n-r}}{(n-r)!\,(r!)^2} \quad \ldots (1.1)$$

$$\mathcal{L}_n(x, 0) = (-1)^n \frac{x^n}{n!}, \quad \ldots (1.2)$$

and are specified by the generating function

$$\sum_{n=0}^{\infty} t^n \mathcal{L}_n(x, y) = \frac{1}{1-yt} \exp\left(\frac{-xt}{1-yt}\right); \; |yt| < 1 \quad \ldots (1.3)$$

$\mathcal{L}_n(x, y)$ are linked to the ordinary Laguerre polynomials $L_n(x)$ by

$$\mathcal{L}_n(x, 1) = L_n(x) \quad \ldots (1.4)$$

$$\mathcal{L}_n(x, y) = y^n L_n\left(\frac{x}{y}\right) \quad \ldots (1.5)$$

A generalization of (1.1) provided by the following definition of the generalized associated Laguerre polyniomials

$$\mathcal{L}_n^{(\alpha)}(x, y) = \sum_{r=0}^{n} \frac{(-1)^r (1+\alpha)_n\, y^{n-r} x^r}{(n-r)!\, r!\, (1+\alpha)_r} \quad \ldots (1.6)$$

$$= \sum_{r=0}^{n} \frac{(-1)^r (\alpha+n)!\, y^{n-r} x^r}{r!\,(n-r)!\,(\alpha+r)!},$$

and the generating function, we get

$$\sum_{n=0}^{\infty} \mathcal{L}_n^{(\alpha)}(x, y)\, t^n = (1-yt)^{-\alpha-1} \exp\left(\frac{-xt}{1-yt}\right), \qquad \ldots (1.7)$$

and $$\sum (c)_n\, \mathcal{L}_n^{(\alpha)}(x, y)\, t^n / (1+\alpha)_n\, (1-yt)^{-c}\; {}_1F^1 \left[\begin{matrix} c; & \dfrac{-xt}{} \\ 1+\alpha; & 1-yt \end{matrix}\right] \qquad \ldots (1.8)$$

where c be an arbitirary number
so that for α = 0, (1.6) and (1.7) reduces to (1.1) and (1.3)
Now using expansion on R.H.s. in (1.7) and after some calculation, we get

$$x^n = \sum_{r=0}^{n} \frac{(-1)^r\, n!\,(1+\alpha)_n\, y^{n-r}}{(n-r)!\,(1+\alpha)_r}\, \mathcal{L}_r^{(\alpha)}(x, y) \qquad \ldots (1.9)$$

and by [8; P.21 (1.11a)]
Kummer's first transformation formula

$${}_1F_1(\alpha; \gamma; z) = e^z\, {}_1F_1(\gamma - \alpha; \gamma; -z) \qquad \ldots (1.10)$$

where λ is neither zero nor a negative integer
and $(\lambda)_{m+n} = (\lambda)_m (\lambda+m)_n$ … (1.11)

$$(n-k)! = \frac{(-1)^k\, n!}{(-n)_k};\; 0 \le k \le n \qquad \ldots (1.12)$$

In this paper we shall give some basic relation and properties then obtain integral and differentiation involving the generalized associated Laguerre polynomials $\mathcal{L}_n^{(\alpha)}(x, y)$

Some Properties

Theorem – 1: If α and β are arbitrary and positive integer then

(i) $$\mathcal{L}_n^{(\alpha)}(x, y) = \sum_{k=0}^{n} (\alpha-\beta)_k\, y^k\, \mathcal{L}_{n-k}^{(\beta)}(x, y)/k! \qquad \ldots (2.1)$$

(ii) $$\mathcal{L}_n^{(\alpha+\beta+1)}(x+z, y) = \sum_{k=0}^{n} \mathcal{L}_k^{\alpha}(x, y)\, \mathcal{L}_{n-k}^{(\beta)}(z, y) \qquad \ldots (2.2)$$

(iii) $$\mathcal{L}_n^{(\alpha)}(x, y) = \frac{(1+\alpha)_n}{(c)_n} \sum_{k=0}^{n} \frac{(1+\alpha-c)_k}{(1+\alpha)_k}\, \mathcal{L}_k^{(\alpha)}(-x, y)\, \mathcal{L}_{n-k}^{(2c-\alpha-2)}(x, y) \qquad \ldots (2.3)$$

(iv) $$\mathcal{L}_n^{(\alpha)}(xy, z) = \sum_{k=0}^{n} \frac{y^k \left[z(1-y)^{n-k}\right](1+\alpha)_n}{(n-k)!\,(1+\alpha)_k}\, \mathcal{L}_k^{(\alpha)}(x, y) \qquad \ldots (2.4)$$

Proof: (i) consider

$$(1-yt)^{-1-\alpha} \exp\left(\frac{-xt}{1-yt}\right) = (1-yt)^{-(\alpha-\beta)}\, (1-yt)^{1-\beta} \exp\left(\frac{-xt}{1-yt}\right).$$

Now using (1.7) and some series rearrangements

$$\sum_{n=0}^{\infty} \mathcal{L}_n^{(\alpha)}(x, y)\, t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(\alpha-\beta)_k\, (yt)^k}{k!}\, \mathcal{L}_{n-k}^{(\beta)}(x, y)\, t^{n-k}$$

Equating the coefficient of t^n on both side, we get required result (2.1)

(ii) Consider an equation

$$(1-yt)^{-1-\alpha} \exp\left(\frac{-xt}{1-yt}\right) (1-yt)^{-1-\beta} \exp\left(\frac{-zt}{1-yt}\right)$$

$$= (1-yt)^{-1-(\alpha+\beta+1)} \exp\left(\frac{-(x+z)t}{1-yt}\right)$$

Now using (1.7) and using some series rearrangement

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \mathcal{L}_k^{(\alpha)}(x,y)\,t^k\,\mathcal{L}_{n-k}^{(\beta)}(z,y)\,t^{n-k} = \sum_{n=0}^{\infty}\mathcal{L}_n^{(\alpha+\beta+1)}(x+z,y)\,t^n$$

Equation the coefficient of t^n, we get required result (2.2)

(iii) Since linear generating function of Laguerre polynomial defined by

$$\sum_{n=0}^{\infty}(c)_n\,\mathcal{L}_n^{(\alpha)}(x,y)\,t^n/(1+\alpha)_n = (1-yt)^{-c}\,{}_1F_1\left[\begin{matrix} c; & \frac{-xt}{1-yt} \\ 1+\alpha; & \end{matrix}\right]$$

Using Kummer's first formula (1.10) in R.H.S. and after some calculation

$$\sum_{n=0}^{\infty}(c)_n\,\mathcal{L}_n^{(\alpha)}(x,y)\,t^n/(1+\alpha)_n$$

$$= (1-yt)^{-1-(2c-\alpha-2)} \exp\left(\frac{-xt}{1-yt}\right)(1-yt)^{-(1+\alpha-c)}\,{}_1F_1\left[\begin{matrix} 1+\alpha-c; & \frac{-xt}{1-yt} \\ 1+\alpha; & \end{matrix}\right]$$

using (1.7) and (1.8), we get

$$\sum_{n=0}^{\infty}(c)_n\,\mathcal{L}_n^{(\alpha)}(x,y)\,t^n/(1+\alpha)_n$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\mathcal{L}_{n-k}^{(2c-\alpha-2)}(x,y)\frac{(1+\alpha-c)_k}{(1+\alpha)_k}\mathcal{L}_k^{(\alpha)}(-x,y)\,t^n$$

Equating the coefficient of t^n on both side, we get required result (2.3)
Put $c = 1 + \alpha/2 + m/2$ in (2.3), we get

$$\mathcal{L}_n^{(\alpha)}(x,y) = \frac{(1+\alpha)_n}{(1+\alpha/2+m/2)_n}\sum_{k=0}^{n}\frac{\left(\frac{\alpha}{2}-\frac{m}{2}\right)_k}{(1+\alpha)_k}\mathcal{L}_k^{(\alpha)}(-x,y)\,\mathcal{L}_{n-k}^{(m)}(x,y) \quad \ldots (2.5)$$

Again put $c = 1 + \alpha + m$ in (2.3) and using (1.11) we get

$$\mathcal{L}_n^{(\alpha)}(x,y) = (1+\alpha)_n\,(1+\alpha)_m\,/(1+\alpha)_{m+n}$$

$$\sum_{k=0}^{n}(-m)_k\,\mathcal{L}_k^{(\alpha)}(-x,y)\,\mathcal{L}_{n-k}^{(\alpha+2m)}(x,y)/(1+\alpha)_k \quad \ldots (2.6)$$

(iv) Consider the series

$$\sum_{n=0}^{\infty}\mathcal{L}_n^{(\alpha)}(xy,z)\,t^n/(1+\alpha)_n = \sum_{n=0}^{\infty}\mathcal{L}_n^{(\alpha)}\left(\frac{xy}{z}\right)(tz)^n(1+\alpha)_n$$

using [6; P.113(1)], we get

$$\sum_{n=0}^{\infty}\mathcal{L}_n^{(\alpha)}(xy,z)\,t^n/(1+\alpha)_n = e^{tz}\,{}_0F_1\left[\begin{matrix} -; & \frac{-x}{z}(tzy) \\ 1+\alpha; & \end{matrix}\right]$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}[tz(1-y)]^{n-k}/(n-k)!\,\mathcal{L}_k^{(\alpha)}\left(\frac{x}{z}\right)(tzy)^k/(1+\alpha)_k$$

using (1.5) then equating the coefficient of t^n on both side, we get required result (2.4)

Expansions of polynomials

Theorem – 2: If α is a positive and arbitrary integer, then

(i) $$H_n(x,y) = 2^n (1+\alpha)_n \sum_{k=0}^{\infty} {}_2F_2 \left[\begin{matrix} -(n-k)/2, -(n-k-1)/2; \\ -(\alpha+n)/2, -(\alpha+n-1)/2; \end{matrix} -\frac{1}{4y} \right]$$
$$\times (-n)_k \, y^{n-k} \mathcal{L}_k^{(\alpha)}(x,y)/(1+\alpha)_k \qquad \ldots (3.1)$$

(ii) $$P_n(x) = 2^n (\tfrac{1}{2})_n (1+\alpha)_n / n! \sum_{k=0}^{n} (-n)_k \, y^{n-k} \mathcal{L}_k^{(\alpha)}(x,y)/(1+\alpha)_k$$
$$\times {}_2F_3 \left[\begin{matrix} -(n-k)/2, -(n-k-1)/2; \\ (\tfrac{1}{2}-n), -\tfrac{1}{2}(\alpha+n), -\tfrac{1}{2}(\alpha+n-1); \end{matrix} \frac{1}{4y^2} \right] \qquad \ldots (3.2)$$

(iii) $$\mathcal{L}_n^{(\alpha)}(x,y) = \sum_{k=0}^{n} \frac{(1+\alpha)_n (-1)^k y^{n-k}}{(n-k)!\,k!\,2^k (1+\alpha)_k} H_k(x,y)$$
$$\times {}_2F_2 \left[\begin{matrix} -(n-k)/2, -(n-k-1)/2; \\ (1+\alpha+k)/2, 2+\alpha+k/2; \end{matrix} \frac{1}{4y} \right] \qquad \ldots (3.3)$$

(iv) $$\mathcal{L}_n^{(\alpha)}(x,y) = \sum_{k=0}^{n} {}_2F_3 \left[\begin{matrix} \frac{-(n-k)}{2}, \frac{-(n-k-1)}{2}; \\ \frac{(1+\alpha+k)}{2}, \frac{2+\alpha+k}{2}, \frac{3}{2}+k; \end{matrix} \frac{1}{4y^2} \right]$$
$$\times \frac{(-1)^k y^{n-k} (1+\alpha)_n (2k+1) P_n(x)}{2^k (n-k)!\,(\tfrac{3}{2})_k (1+\alpha)_k} \qquad \ldots (3.4)$$

Proof: (i) Consider the following series

$$S = \sum_{n=0}^{\infty} H_n(x,y) \frac{t^n}{n!} \qquad \ldots (3.5)$$

$$S = \sum_{n=0}^{\infty} \sum_{s=0}^{[n/2]} \frac{(-y)^s (2x)^{n-2s} t^n}{s!\,(n-2s)!}$$

Replacing n by n + 2s and using (1.9), and again replacing n by n + k, we get

$$S = \sum_{n=0}^{\infty} \sum_{s=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{k+s} (1+\alpha)_{n+k}}{s!\,n!\,(1+\alpha)_k} 2^{n+k} y^{n+s} \mathcal{L}_k^{(\alpha)}(x,y)\, t^{n+2s+k}$$

Now replacing n by n – 2s and by some calculations we get

$$S = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{s=0}^{[n/2]} \frac{(-1)^s (4y)^{-s} (-n)_{2s}}{s!\,(-\alpha-n-k)_{2s}} \frac{(-1)^k (1+\alpha)_{n+k}}{n!\,(1+\alpha)_k} 2^{n+k} y^n$$
$$\times \mathcal{L}_k^{(\alpha)}(x,y)\, t^{n+k}$$

using Legendre's duplication formula and replacing n by n – k we get

$$S = \sum_{n=0}^{\infty} \sum_{k=0}^{n} {}_2F_2 \left[\begin{matrix} -\tfrac{1}{2}(n-k), -\tfrac{1}{2}(n-k-1); \\ -\left(\frac{\alpha+n}{2}\right), -\tfrac{1}{2}(\alpha+n-1); \end{matrix} \frac{-1}{4y} \right]$$

$$\times (-1)^k 2^n y^{n-k} (1+\alpha)_n \mathcal{L}_n^{(\alpha)}(x,y)\, t^n (n-k)!\,(1+\alpha)_k \qquad \ldots (3.6)$$

Equating the coefficient of t^n of equations (3.5) and (3.6), we get required result (3.1)

(ii) Now consider following the series

$$S = \sum_{n,s=0}^{\infty} P_n(x)\, t^n \qquad \ldots (3.7)$$

$$= \sum_{n,s=0}^{\infty} \frac{(-1)^s (1/2)_{n+s} (2x)^n t^{n+2s}}{s!\,n!}$$

using (1.9) and the same procedure as [6; P.208 (4)], we get
then n by n – k, we get

$$S = \sum_{n=0}^{\infty}\sum_{k=0}^{n} {}_2F_3\left[\begin{matrix} -(n-k)/2,\ -(n-k-1)/2\ ; \\ \tfrac{1}{2}-n,\ -(\alpha+n)/2,\ -(\alpha+n-1)/2\ ; \end{matrix}\ \frac{1}{4y^2}\right]$$

$$\times (-1)^k\, 2^n\, (\tfrac{1}{2})_n\, (1+\alpha)_n\, y^{n-k}\, \mathcal{L}_k^{(\alpha)}(x, y)\, t^n /(n-k)!\,(1+\alpha)_k \qquad \ldots (3.8)$$

Equating the coefficient of t^n of equations (3.7) and (3.8) and using (1,12)then, we get required result (3.2)

(iii) Consider the following series

$$S = \sum_{n=0}^{\infty} \mathcal{L}_n^{(\alpha)}(x, y)\, t^n /(1+\alpha)_n \qquad \ldots (3.9)$$

$$S = \sum_{n,k=0}^{\infty} (-1)^k\, y^n\, x^k\, t^{n+k} / k!\,n!\,(1+\alpha)_k$$

$$= \sum_{n,k=0}^{\infty}\sum_{r=0}^{[k/2]} \frac{(-1)^k\, y^n\, H_{k-2r}(x, y)\, t^{n+k}}{n!\,(1+\alpha)_k\, 2^k\, (k-2r)!\,r!}$$

Now using the same procedure as in part (i) and (ii), we get

$$S = \sum_{n=0}^{\infty}\sum_{k=0}^{n} {}_2F_2\left[\begin{matrix} \dfrac{-(n-k)}{2},\ \dfrac{-(n-k-1)}{2}; \\ \left(\dfrac{1+\alpha+k}{2}\right),\ \left(\dfrac{2+\alpha+k}{2}\right); \end{matrix}\ \frac{1}{4y}\right]$$

$$\text{x}\ \frac{(-1)^k\, y^{n-k}\, H_k(x, y)}{(n-k)!\,k!\,(1+\alpha)_k}\, \frac{t^n}{2^k} \qquad \ldots (3.10)$$

Equating the coefficient of t^n of equation (3.9) and (3.10), we get required result (3.3).
(iv) For proof of (3.4) consider the following series

$$\sum_{n=0}^{\infty} \mathcal{L}_n^{(\alpha)}(x, y)\, t^n (1+\alpha)_n = \sum_{n,k=0}^{\infty} (-1)^k\, y^n\, x^k\, t^{n+k} / k!\,n!\,(1+\alpha)_k$$

Now using [6; P.181 (4)], we get

$$\sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x, y)\, t^n}{(1+\alpha)_n} = \sum_{n,k=0}^{\infty}\sum_{r=0}^{[k/2]} \frac{(-1)^k\, y^n\, (2k-4r+1)}{n!\,(1+\alpha)_k\, 2^k\, r!}\, \frac{P_{k-2r}(x)\, t^{n+k}}{(\tfrac{3}{2})_{k-r}},$$

$$\text{x}\ \frac{(-1)^k\, y^n\, (2k+1)\, P_k(x)\, t^n}{n!\, 2^k \left(\dfrac{3}{2}\right)_k (1+\alpha)_k}$$

Equating the coefficient of t^n on both side, we get required result (3.4)

Special cases

I. For y = 1, then (2.1) and (2.2) reduces to a known result [6; P. 209 (283)]
For y = 1, then (2.3) reduces to a known result [6; P.210 (8)]
For z =1, then (2.4) reduces to a known result [6; P. 209 (15)].
For y = 1, then (2.5) and (2.6) reduces to a known result [6; P. 216 (6 & 7)].

II For y = 1, then (3.1) reduces to a known result [6; P.207 (3)]
For y = 1, then (3.2) reduces to a known result [6; P. 208 (4)].
For y = 1, then (3.3), (3.4) reduces to a known result [6; P. 216 (2&3)].

Special cases I and II are known formulae for some properties and expansions of polynomials for ordinary Laguerre Polynomials $L_n(x)$

References

1. Dattoli G, Torre A. Operational methods and two vaiable Laguerre polynomials, Acc. Sc. Torino-Atti Sc. Fis. 1998; 132:1-7.
2. Dattoli G, Lorenzutta S, Sacchetti D. A note on operational rules for Hermite and Laguerre polynomials, operational rules and Special polynomials, Internet. J Math. Stat. Sci, 2000; 9(2):227-238.
3. Dattoli G, Torre A. Acc. Sc. di Torino-Atti. Sc. Fis. 132, 1. Operational methods for associated laguerre polynomials have also been youched in A. Wunsche, J. Phys. A32. 1998, 3179.
4. Pathan MA, Subuhi Khan. Speical linear group and modified laguerre functions, Kyungpook Math. J. 2000; 40:1-8.
5. Pathan MA, Subuhi Khan, Ghazala Yasmin. Two variable modified laguerre polynomials and their generating relations, South East Asian Jour. Math. and Math. Sciences, 2002; 1:19-28.
6. Rainville ED. Special Functions. Macmillan, New York; Reprinted by Chelsea Pub. Co., Bronx, New York, 1971.