International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2018; 3(1): 110-114

www.mathsjournal.com
Received: 12-11-2017
Accepted: 18-12-2017

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Queuing analysis of Geo/Geo/1 queue with catastrophes using Matrix Geometric technique

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Abstract
Catastrophic models and analysis have ample applications in computer science, telecommunications, financial management, insurance sector and many more areas. We consider Geo/Geo/1 with catastrophes. This model is solved by Matrix Geometric technique. Numerical study has been done for various values of parameters.

Keywords: Catastrophes, Matrix Geometric technique

Introduction
Queuing models with catastrophes have gained importance during last few decades due to their applications in many areas viz. computers and telecommunications, financial management, insurance sector, disaster management and many more areas. Whenever a catastrophe occurs at the system, all the customers are forced to leave the system immediately, the server becomes inoperative instantaneously and it becomes operative as soon as a new customer arrives.
Continuous Queuing models with catastrophes have been investigated by many researchers in the past. Indra and Rajan (2017) [4] analysed Markovian queues having two heterogeneous servers with catastrophes using Matrix Geometric technique. Atencia and Moreno (2004) [2] studied Geo/Geo/1 queue with negative customers and disasters. Jeyakumar and Gunasekaran (2017) [3] analysed Geo/G/1 queue with disaster and single vacation.
Matrix Geometric technique was conceptualized by Marcel F. Neuts in 1974. Many researchers used this technique in their respective papers.
In the current paper, we are presenting a simple analysis of Geo/Geo/1 queue with catastrophe using Matrix Geometric technique. The rest of the paper is organized as follows: In section 2, description of the model is presented. In section 3, Expression for probability generating function (PGF) of system-size for Geo/Geo/1 for LAS-DA is presented followed by computation of the system-size steady-state probabilities. In section 4, Expression for PGF of system-size for Geo/Geo/1 for EAS is presented followed by computation of the system-size steady-state probabilities. Finally, in section 5, numerical study has been done for the analysis of the model for various values of the parameters.

Section 2
Model description
Let the time axis be marked by 0, 1, 2,…, m,… In the queue Geo/Geo/1, it is assumed that inter arrival times I are independent and geometrically distributed as $a_n = P(I = n\, slots) = (1-\lambda)^{n-1}\lambda$ with mean inter arrival time = 1/λ. The inter service times S are independent and geometrically distributed as $b_n = P(S = n\, slots) = (1-\mu)^{n-1}\mu$ with mean inter service time = 1/μ. The traffic intensity (ρ) is $\rho = \frac{\lambda}{\mu}$. It is assumed that the customers are served individually according to the first-come, first-served (FCFS) discipline. In late arrival system with delayed access (LAS-DA), the potential arrivals occur in (m-, m). After getting service for n slots (n ≥ 1) a customer leaves in (m, m+). While in an early arrival system (EAS), a potential arrival occurs in (m,m+) and a potential departure occurs in (m-,m).

The catastrophes occur at the system with probability ξ ($0 \leq \xi \leq 1$). If an arriving customer finds that the server is idle, the service of the arriving customer starts immediately. Otherwise, the arriving customer joins the queue.

Section 3

Derivation of expression of PGF for Geo/Geo/1 with catastrophe with LAS-DA

Let n(t) be the number of customers in the system at time t. Let n be the stationary random variable for the number of customers in the system.

We define $\pi_i = \{n = i\} = \lim_{t\to\infty} P\{n(t) = i\}$ where $i\,\epsilon$ set of whole numbers and π_i represents the stationary probability of i customers in the system.

The stationary probability vector Π is given by

$\Pi = (\pi_0, \pi_1, \pi_2, \pi_3, \dots \dots \dots \dots)$

We define $\Pi(s) = \sum_{j=0}^{j=\infty} \pi_j s^j$ to be the probability generating function (PGF) of system-size probabilities.

In Matrix Geometric Method the steady state probabilities π_i are related geometrically to each other as $\pi_i = \pi_1 R^{i-1}$ for $i \geq 2$. Here R is called rate element.

The infinitesimal generator matrix Q of the Queuing system under consideration is given by

$$Q = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \dots & \dots & \dots \\ \bar{\lambda}\mu + \xi & -\bar{\lambda}\mu - \lambda\bar{\mu} - \xi & \lambda\bar{\mu} & 0 & 0 & \dots & \dots & \dots \\ \xi & \bar{\lambda}\mu & -\bar{\lambda}\mu - \lambda\bar{\mu} - \xi & \lambda\bar{\mu} & 0 & \dots & \dots & \dots \\ \xi & 0 & \bar{\lambda}\mu & -\bar{\lambda}\mu - \lambda\bar{\mu} - \xi & \lambda\bar{\mu} & \dots & \dots & \dots \\ \xi & 0 & 0 & \bar{\lambda}\mu & -\bar{\lambda}\mu - \lambda\bar{\mu} - \xi & \lambda\bar{\mu} & \dots & \dots \\ \dots & \dots & \dots & \dots & \bar{\lambda}\mu & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

With the help of stationary probability matrix Π and infinitesimal generator matrix Q we get

$$-\lambda\,\pi_0 + \left(\bar{\lambda}\mu + \xi(1-R)^{-1}\right)\pi_1 = 0 \qquad \dots.(1)$$

$$\lambda\pi_0 + \left(-\bar{\lambda}\mu - \lambda\bar{\mu} - \xi\right)\pi_1 + \bar{\lambda}\mu\,\pi_2 = 0 \qquad \dots.(2)$$

$$\lambda\bar{\mu}\,\pi_{j-1} + \left(-\bar{\lambda}\mu - \lambda\bar{\mu} - \xi\right)\pi_j + \bar{\lambda}\mu\,\pi_{j+1} = 0 \quad (j \geq 2) \qquad \dots.(3)$$

Multiplying above equations with appropriate powers of s and then adding we get

$$-\lambda\pi_0 + \left(\bar{\lambda}\mu + \xi(1-R)^{-1}\right)\pi_1 + \lambda\pi_0 s + \lambda\bar{\mu}s\sum_{j=2}^{j=\infty}\pi_{j-1}s^{j-1} + \left(-\bar{\lambda}\mu - \lambda\bar{\mu} - \xi\right)\sum_{j=1}^{j=\infty}\pi_j s^j + \frac{\bar{\lambda}\mu}{s}\sum_{j=1}^{j=\infty}\pi_{j+1}s^{j+1} = 0 \qquad \dots(4)$$

On further simplification we get

$$\Pi(s) = \frac{\left(-\lambda\mu s^2 + \left(\mu(\lambda-\bar{\lambda}) - \xi\right)s + \bar{\lambda}\mu\right)\pi_0 - \xi(1-R)^{-1}s\,\pi_1}{\lambda\bar{\mu}s^2 - \left(\bar{\lambda}\mu + \lambda\bar{\mu} + \xi\right)s + \bar{\lambda}\mu} \qquad \dots.(5)$$

Equation (5) represents the expression for PGF of Geo/Geo/1 with catastrophe with LAS-DA. If we put $\xi = 0$ in equation (5) we get

$$\Pi(s) = \frac{\left(-\lambda\mu s^2 + \left(\mu(\lambda-\bar{\lambda})\right)s + \bar{\lambda}\mu\right)\pi_0}{\lambda\bar{\mu}s^2 - \left(\bar{\lambda}\mu + \lambda\bar{\mu} + \xi\right)s + \bar{\lambda}\mu} \qquad \dots.(6)$$

Equation (6) represents the expression for PGF of Geo/Geo/1 with LAS-DA, which is same as obtained by Hunter.

Matrix Geometric Solution of Geo/Geo/1 with catastrophe with LAS-DA

The stationary probability matrix Π is obtained by solving ΠQ = 0

$$-\lambda\pi_0 + \pi_1\left(\bar{\lambda}\mu + \xi(1-R)^{-1}\right) = 0 \qquad \dots.(7)$$

$$\lambda\pi_0 + \left(-\bar{\lambda}\mu - \lambda\bar{\mu} - \xi + \bar{\lambda}\mu R\right) = 0 \qquad \dots.(8)$$

$$\bar{\lambda}\mu R^2 + \left(-\bar{\lambda}\mu - \lambda\bar{\mu} - \xi\right)R + \lambda\bar{\mu} = 0 \qquad \dots.(9)$$

$$\pi_i = \pi_1 R^{i-1} \quad \text{for } i \geq 2 \qquad \dots.(10)$$

Here R < 1 is non-negative solution of equation (9) and is given by

$$R = \frac{\left(\bar{\lambda}\mu + \lambda\bar{\mu} + \xi\right) - \sqrt{\left(\bar{\lambda}\mu + \lambda\bar{\mu} + \xi\right)^2 - 4\bar{\lambda}\mu\lambda\bar{\mu}}}{2\bar{\lambda}\mu} \qquad \dots.(11)$$

The normalizing equation is given by

$$\pi_0 + \pi_1(1-R)^{-1} = 1 \qquad \dots.(12)$$

R is calculated using equation (11). π_0 and π_1 are calculated using equations (7), (8) and (12) and are given by

$$\pi_0 = \frac{-\bar{\lambda}\mu - \lambda\bar{\mu} - \xi + \bar{\lambda}\mu R}{-\bar{\lambda}\mu - \lambda\bar{\mu} - \xi + \bar{\lambda}\mu R - \lambda(1-R)^{-1}} \qquad \dots.(13)$$

$$\pi_1 = \frac{-\lambda}{-\bar{\lambda}\mu - \lambda\bar{\mu} - \xi + \bar{\lambda}\mu R - \lambda(1-R)^{-1}} \qquad \dots.(14)$$

Using equation (10) π_i for $i \geq 2$ are calculated.

Substituting $\xi = 0$ in equations (11), (13) and (14) we get

$$R = \frac{\lambda\bar{\mu}}{\bar{\lambda}\mu} \quad \text{…. (15)}$$

$$\pi_0 = 1 - \rho \quad \text{…. (16)}$$

$$\pi_1 = \rho(1 - R) \quad \text{…. (17)}$$

Equations (10), (15), (16) and (17) together represents Matrix Geometric solution of Geo/Geo/1 with LAS-DA. This solution has been obtained from Geo/Geo/1 with catastrophe with LAS-DA by putting $\xi = 0$ and match with the solution of Geo/Geo/1 with LAS-DA given by Hunter.

Section 4

Expression of PGF for Geo/Geo/1 with catastrophe with EAS

The infinitesimal generator matrix Q for the above system is given by

$$Q = \begin{pmatrix} -\lambda\bar{\mu} & \lambda\bar{\mu} & 0 & 0 & 0 & \dots & \dots & \dots \\ \bar{\lambda}\mu + \xi & -\bar{\lambda}\mu - \lambda\bar{\mu} - \xi & \lambda\bar{\mu} & 0 & 0 & \dots & \dots & \dots \\ \xi & \bar{\lambda}\mu & -\bar{\lambda}\mu - \lambda\bar{\mu} - \xi & \lambda\bar{\mu} & 0 & \dots & \dots & \dots \\ \xi & 0 & \bar{\lambda}\mu & -\bar{\lambda}\mu - \lambda\bar{\mu} - \xi & \lambda\bar{\mu} & \dots & \dots & \dots \\ \xi & 0 & 0 & \bar{\lambda}\mu & -\bar{\lambda}\mu - \lambda\bar{\mu} - \xi & \lambda\bar{\mu} & \dots & \dots \\ \dots & \dots & \dots & 0 & \bar{\lambda}\mu & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & 0 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Using stationary probability matrix Π and infinitesimal generator matrix Q we get

$$-\lambda\bar{\mu}\pi_0 + \left(\bar{\lambda}\mu + \xi(1-R)^{-1}\right)\pi_1 = 0 \quad \text{…. (18)}$$

$$\lambda\bar{\mu}\pi_0 + \left(-\bar{\lambda}\mu - \lambda\bar{\mu} - \xi\right)\pi_1 + \bar{\lambda}\mu\pi_2 = 0 \quad \text{…. (19)}$$

$$\lambda\bar{\mu}\pi_{j-1} + \left(-\bar{\lambda}\mu - \lambda\bar{\mu} - \xi\right)\pi_j + \bar{\lambda}\mu\pi_{j+1} = 0\ ; \quad \text{for } j \ge 2 \quad \text{…. (20)}$$

Multiplying above equations with appropriate powers of s and then adding we get

$$-\lambda\bar{\mu}\pi_0 + \left(\bar{\lambda}\mu + \xi(1-R)^{-1}\right)\pi_1 + \lambda\bar{\mu}\pi_0 s + \left(-\bar{\lambda}\mu - \lambda\bar{\mu} - \xi\right)\sum_{j=1}^{j=\infty}\pi_j s^j + \lambda\bar{\mu}s\sum_{j=1}^{j=\infty}\pi_{j-1}s^{j-1} + \frac{\bar{\lambda}\mu}{s}\sum_{j=1}^{j=\infty}\pi_{j+1}s^{j+1} = 0 \quad \text{…. (21)}$$

On further simplification we get

$$\Pi(s) = \frac{\left((-\bar{\lambda}\mu - \xi)s + \bar{\lambda}\mu\right)\pi_0 - s\xi(1-R)^{-1}\pi_1}{\lambda\bar{\mu}s^2 + \left(-\bar{\lambda}\mu - \lambda\bar{\mu} - \xi\right)s + \bar{\lambda}\mu} \quad \text{…. (22)}$$

Equation (22) represents the expression for PGF of Geo/Geo/1 with catastrophe with EAS. If we put $\xi = 0$ in equation (21) we get

$$\Pi(s) = \frac{(-\bar{\lambda}\mu s + \bar{\lambda}\mu)\pi_0}{\lambda\bar{\mu}s^2 + \left(-\bar{\lambda}\mu - \lambda\bar{\mu} - \xi\right)s + \bar{\lambda}\mu} \quad \text{…. (23)}$$

Equation (23) represents the expression for PGF of Geo/Geo/1 with EAS, which is same as obtained by Hunter.

Matrix Geometric Solution of Geo/Geo/1 with catastrophe with EAS

The stationary probability matrix Π is obtained by solving ΠQ = 0

$$-\lambda\bar{\mu}\pi_0 + \left(\bar{\lambda}\mu + \xi(1-R)^{-1}\right)\pi_1 = 0 \quad \text{…. (24)}$$

$$\lambda\bar{\mu}\pi_0 + \left(-\bar{\lambda}\mu - \lambda\bar{\mu} - \xi + \bar{\lambda}\mu R\right)\pi_1 = 0 \quad \text{…. (25)}$$

$$\bar{\lambda}\mu R^2 + \left(-\bar{\lambda}\mu - \lambda\bar{\mu} - \xi\right)R + \lambda\bar{\mu} = 0 \quad \text{…. (26)}$$

$$\pi_i = \pi_1 R^{i-1} \quad \text{for } i \ge 2 \quad \text{…. (27)}$$

Here R < 1 is non-negative solution of equation (26) and is given by

$$R = \frac{(\bar{\lambda}\mu + \lambda\bar{\mu} + \xi) - \sqrt{(\bar{\lambda}\mu + \lambda\bar{\mu} + \xi)^2 - 4\bar{\lambda}\mu\lambda\bar{\mu}}}{2\bar{\lambda}\mu} \quad \text{…. (28)}$$

The normalizing equation is given by

$$\pi_0 + \pi_1(1-R)^{-1} = 1 \quad \text{…. (29)}$$

R is calculated using equation (28). π_0 and π_1 are calculated using equations (24), (25) and (29) and are given by

$$\pi_0 = \frac{-\bar{\lambda}\mu - \lambda\bar{\mu} - \xi + \bar{\lambda}\mu R}{-\bar{\lambda}\mu - \lambda\bar{\mu} - \xi + \bar{\lambda}\mu R - \lambda\bar{\mu}(1-R)^{-1}} \quad \text{…. (30)}$$

$$\pi_1 = \frac{-\lambda\bar{\mu}}{-\bar{\lambda}\mu - \lambda\bar{\mu} - \xi + \bar{\lambda}\mu R - \lambda\bar{\mu}(1-R)^{-1}} \quad \text{…. (31)}$$

Using equation (27) π_i for $i \ge 2$ are calculated.

Substituting $\xi = 0$ in equations (28), (30) and (31) we get

$$R = \frac{\lambda\bar{\mu}}{\bar{\lambda}\mu} \quad \text{…. (32)}$$

$$\pi_0 = 1 - R \quad \text{…. (33)}$$

$$\pi_1 = R(1 - R) \quad \text{…. (34)}$$

Equations (27), (32), (33) and (34) together represents Matrix Geometric solution of Geo/Geo/1 with EAS. This solution has been obtained from Geo/Geo/1 with catastrophe with EAS by putting ξ = 0 and match with the solution of Geo/Geo/1 with EAS given by Hunter.

Numerical Study

The stationary system-size probabilities have been calculated using codes developed in Maple software. All computations have been rounded off to six decimal places. The results have been verified by matching the two sets of means and variances obtained from probability values and from theoretical formulas. The means and variances, as calculated from probability distributions, are given by the following formulas:

Mean = $\sum_{n=1}^{n=N} n\, \pi_n$

Variance = $\sum_{n=1}^{n=N} n^2 \pi_n - \left(\sum_{n=1}^{n=N} n\, \pi_n \right)^2$

Although we are solving infinite capacity model, but for computational purpose we have restricted the system capacity to N.
The means and variances formulas for system-size, as derived from probability generating functions, are given by the following:

Mean and Variance for System Size in LAS-DA

Mean = $\frac{\pi_0(-\mu-\xi)-\pi_1\xi(1-R)^{-1}-(\lambda\bar{\mu}-\bar{\lambda}\mu-\xi)}{-\xi}$

Variance = $\frac{-2\lambda\bar{\mu}-2(\lambda\bar{\mu}-\bar{\lambda}\mu-\xi)Mean-2\lambda\bar{\mu}}{-\xi} + Mean - (Mean)^2$

Mean and Variance for System Size in EAS

Mean = $\frac{\pi_0(-\bar{\lambda}\mu-\xi)-\pi_1\xi(1-R)^{-1}-(\lambda\bar{\mu}-\bar{\lambda}\mu-\xi)}{-\xi}$

Variance = $\frac{2\lambda\bar{\mu}+2\,Mean(\lambda\bar{\mu}-\bar{\lambda}\mu-\xi)}{\xi} + Mean - (Mean)^2$

The results for various values of ξ are shown in the following tables:

Table 1: Results of Geo/Geo/1 for LAS-DA of steady-state system-size probabilities for various values of ξ ($\lambda = 0.2$; $\mu = 0.3$; $\rho = 0.666667$; $N = 100$)

n	π_n		
	$\xi = 0$	$\xi = 0.5$	$\xi = 0.8$
0	0.333333	0.777777	0.834820
1	0.277777	0.185185	0.145084
2	0.162037	0.030864	0.017650
3	0.094521	0.005144	0.002147
4	0.055137	0.000857	0.000261
5	0.032163	0.000142	0.000031
6	0.018762	0.000023	0.000003
7	0.010944	0.000003	0.0000004
8	0.006384	0.0000007	0.00000005
9	0.003724	0.0000001	0.000000006
10	0.002172	0.00000002	0.0000000008
Value of R	0.583333	0.166666	0.121654
Sum of Probabilities	0.999999	1.000000	0.999999
Mean from Proba.	1.600000	0.266666	0.188057
Mean from PGF	1.600000	0.266666	0.188057
Var. from Prob.	3.519999	0.302222	0.204785
Var. from PGF	3.520000	0.302222	0.204785

Table 2: Results of Geo/Geo/1 for EAS of steady-state system-size probabilities for various values of ξ ($\lambda = 0.2$; $\mu = 0.3$; $\rho = 0.666667$; $N = 100$)

n	π_n		
	$\xi = 0$	$\xi = 0.5$	$\xi = 0.8$
0	0.416666	0.833333	0.878345
1	0.243055	0.138888	0.106854
2	0.141782	0.023148	0.012999
3	0.082706	0.003858	0.001581
4	0.048245	0.000643	0.000192
5	0.028143	0.000107	0.000023
6	0.016416	0.000017	0.000002
7	0.009576	0.000002	0.0000003
8	0.005586	0.0000005	0.00000004
9	0.003258	0.00000008	0.000000005
10	0.001900	0.00000001	0.0000000006

Value of R	0.583333	0.166666	0.121654
Sum of Probabilities	1.000000	1.000000	1.000000
Mean from Proba.	1.400000	0.199999	0.138503
Mean from PGF	1.400000	0.199999	0.138503
Var. from Prob.	3.359999	0.240000	0.157687
Var. from PGF	3.360000	0.240000	0.157687

Remarks

It is important to note that steady-state probabilities are same at pre-arrival epoch, at random epoch and at post-departure epoch as the infinitesimal generator matrix is same at these epochs.

Several experiments are done to calculate steady-state probabilities for number in system for low, moderate and high values of ρ and ξ both for model under consideration**.** But due to lack of space, they are given only for $\rho = 0.666667$ for Geo/Geo/1.

All the results are obtained using a software package known as MAPLE.

Extensions

The method discussed here can be applied to more complex models such as bulk-arrivals.

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