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## The length-biased weighted frechet distribution: Properties and estimation

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### Abstract

In this paper, we introduce a new class of length-biased weighted Fréchet (LBWF) distribution. The statistical properties of this distribution including the mean, variance, coefficient of variation, harmonic mean, raw and incomplete moments, skewness, kurtosis, mode, moment and probability generating functions, reliability, hazard and reverse hazard functions, Lorenz, Bonferroni and Zenga curves, Renyi of entropy, order statistics and record statistics are studied. The moment and maximum likelihood estimates are obtained for the model parameters.

**Keywords:** Fréchet distribution, weighted fréchet distribution, maximum likelihood estimation, moment estimation, order statistics, record statistics

### 1. Introduction

The Fréchet distribution has many applications ranging from life testing through rainfall, queues, wind speeds, earthquakes, sea currents, horse racing, floods and track race records. Historically it was first presented by Maurice Fréchet (1927) [10]. Many authors have studied the Fréchet distribution; for examples, Mubarak (2012) [18] obtained the maximum likelihood and least squares estimates for Fréchet distribution based on progressive type-II censoring. Abbas & Yincai (2012) [1] compared the maximum likelihood, probability weighted moments and Bayes estimates for the scale parameter of Fréchet distribution in complete samples. Nasir & Aslam (2015) [20] estimated the shape parameter of Fréchet distribution via Bayesian scheme based on different prior distribution and various loss functions. Reyad *et al.* (2016) [25] obtained the QE-Bayes and E-Bayes estimates of scale parameter corresponding to the Fréchet distribution.

Recently, some extensions of the Fréchet distribution are introduced. Nadarajah & Kotz (2003) [19] presented the exponentiated Fréchet distribution. Mahmoud & Mandouh (2013) [15] proposed the transmuted Fréchet distribution. Mead *et al.* (2016) [16] suggested the beta exponential Fréchet distribution. Afify *et al.* (2016) [2] introduced the Weibull Fréchet distribution. The probability density function (pdf) of the Fréchet distribution is

$$f(x; \alpha, \beta) = \alpha \beta^\alpha x^{-(\alpha+1)} e^{-(\beta/x)^\alpha}, \quad x > 0, \alpha, \beta > 0 \quad (1)$$

Where  $\beta$  and  $\alpha$  are the scale and shape parameters respectively.

The theory of weighted distributions gives a unified approach to dealing with model specification and data interpretation problems. Weighted distributions always occur in researches related to survival analysis, analysis of intervention data, ecology and biomedicine (refer to Stene (1981) [27], Gupta & Nadarajah (1985) [11] and Patil & Taillie (1989) [22]). The idea of weighted distributions was first emerged by Fisher (1934) [9]. Numerous studies on weighted distributions were published, for example; Rao (1965) [23] explained that weighted distributions can be used when the recorded observations cannot be considered as a random sample from original distributions, which means that sometimes it is not possible to work with truly random sample from the underlying population. Gupta & Tripathi (1996) [12] introduced the weighted bivariate logarithmic series distribution.

Shaban & Boudrissa (2007) [26] presented the length biased Weibull distribution and estimated its parameters. Das & Roy (2011) [8] pointed out the length biased weighted generalized Rayleigh distribution and studied its properties. Rashwan (2013) [24] introduced the generalized gamma length biased distribution and investigated its properties. Ahmed *et al.* (2016) [4] suggested the length biased weighted Lomax distribution with its properties and estimated its parameters.

Suppose that  $X$  is a non-negative random variable and its probability density function (pdf) is  $f(x)$ , then the pdf of the weighted random variable  $X_w$  is called the weighted distribution which is given by

$$f_w(x) = \frac{w(x)f(x)}{E(w(x))}, \quad x > 0 \tag{2}$$

where  $w(x)$  be a non-negative weight function and  $E(w(x)) = \int_0^\infty w(x)f(x) dx$ ,  $E(w(x)) > 0$ .

We can note that different choices of the weight function  $w(x)$  gave different weighted distributions. If  $w(x) = x$ , then the resulting distribution is called length-biased distribution with pdf given by

$$f_l(x) = \frac{x f(x)}{E(x)}, \quad x > 0 \tag{3}$$

where  $E(x) = \int_0^\infty x f(x) dx$ ,  $E(x) > 0$ .

The main objective of this paper is to propose a new distribution that is called length-biased weighted Frechet (LBWF) distribution and to study some of its various statistical properties. In addition, the parameters of the new distribution are estimated by using the moments and maximum likelihood estimation methods.

This paper is organized as follows: In Section 2, the length-biased weighted Frechet distribution is defined. The limit of the LBWF distribution is studied in Section 3. The survival analysis is discussed in Section 4. In Section 5, some statistical properties of the the new model are investigated. In Section 6, the moment and maximum likelihood estimates for the model parameters are obtained. Some concluding remarks have been given in the last Section.

### 2. Length-Biased Weighted Frechet Distribution

In this section, we derive the pdf and the cumulative distribution function (cdf) of the LBWF. The mean of the Frechet distribution given in Eq. (1) is

$$E(x) = \beta \Gamma(1 - 1/\alpha) \tag{4}$$

Then, the pdf of LBWF can be obtained by using Eq. (1) and Eq. (4) in Eq. (3) to be

$$f(x; \alpha, \beta) = \frac{\alpha \beta^{\alpha-1}}{\Gamma(1 - 1/\alpha)} x^{-\alpha} e^{-(\beta/x)^\alpha}, \quad x > 0, \alpha > 1, \beta > 0, \tag{5}$$

The cdf of the LBWF is given by

$$F(x; \alpha, \beta) = \frac{\alpha \beta^{\alpha-1}}{\Gamma(1 - 1/\alpha)} \int_0^x y^{-\alpha} e^{-(\beta/y)^\alpha} dy$$

Substituting  $z = (\beta/y)^\alpha$ , then we have  $F(x; \alpha, \beta) = \frac{1}{\Gamma(1 - 1/\alpha)} \int_{(\beta/x)^\alpha}^\infty z^{-1/\alpha} e^{-z} dz$  and

$$F(x; \alpha, \beta) = \frac{\Gamma(1 - 1/\alpha, (\beta/x)^\alpha)}{\Gamma(1 - 1/\alpha)}, \quad x > 0, \alpha > 1, \beta > 0 \tag{6}$$

where the upper incomplete gamma function is  $\Gamma(a, b) = \int_b^\infty x^{a-1} e^{-x} dx$ ,  $a > 0, b \geq 0$ .

### 3. The Limit of the Length-Biased Weighted Frechet Distribution

The limit of the length-biased weighted Frechet distribution when  $x \rightarrow 0$  is 0 and when  $x \rightarrow \infty$  is 0. We can show this by taking the limit of Eq. (5) as follows:

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \left[ \frac{\alpha \beta^{\alpha-1} x^{-\alpha} e^{-(\beta/x)^\alpha}}{\Gamma(1 - 1/\alpha)} \right] = \left[ \frac{\alpha \beta^{\alpha-1}}{\Gamma(1 - 1/\alpha)} \right] \lim_{x \rightarrow 0} \left[ x^{-\alpha} e^{-(\beta/x)^\alpha} \right]$$

$$= \left( \frac{\alpha \beta^{\alpha-1}}{\Gamma(1-1/\alpha)} \right) \left( \lim_{x \rightarrow 0} x^{-\alpha} \right) \left( \lim_{x \rightarrow 0} e^{-(\beta/x)^\alpha} \right) = 0$$

Because  $\lim_{x \rightarrow \infty} x^{-\alpha} = 0$  and  $\lim_{x \rightarrow \infty} e^{-(\beta/x)^\alpha} = 0$ .

Similarly, as  $x \rightarrow \infty$ , we can observe that by replacing the limit  $x \rightarrow 0$  with  $x \rightarrow \infty$ , we get

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \left( \frac{\alpha \beta^{\alpha-1} x^{-\alpha} e^{-(\beta/x)^\alpha}}{\Gamma(1-1/\alpha)} \right) = \left( \frac{\alpha \beta^{\alpha-1}}{\Gamma(1-1/\alpha)} \right) \lim_{x \rightarrow \infty} \left[ x^{-\alpha} e^{-(\beta/x)^\alpha} \right]$$

$$= \left( \frac{\alpha \beta^{\alpha-1}}{\Gamma(1-1/\alpha)} \right) \left( \lim_{x \rightarrow \infty} x^{-\alpha} \right) \left( \lim_{x \rightarrow \infty} e^{-(\beta/x)^\alpha} \right) = 0$$

Because  $\lim_{x \rightarrow \infty} x^{-\alpha} = 0$ .

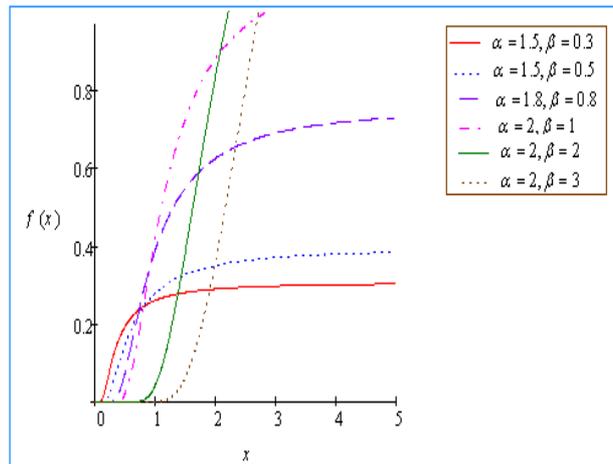


Fig 1: The pdf of the LBWFD for different values of the parameters.

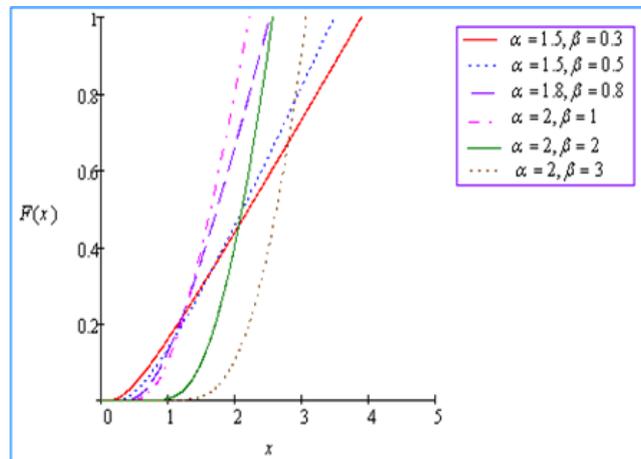


Fig 2: The cdf of the LBWFD for different values of the parameters.

#### 4. Survival Analysis

In this section, we obtain the reliability function, the hazard function and the inverse hazard function for the LBWF distribution. In general, the reliability function,  $R(x)$  of a random variable  $X$  can be given from

$$R(x) = 1 - F(x)$$

Using Eq. (6) in the above equation, we get

$$R(x) = 1 - \left[ \frac{\Gamma(1-1/\alpha, (\beta/x)^\alpha)}{\Gamma(1-1/\alpha)} \right] \tag{7}$$

Also, the hazard function  $h(x)$ , of random variable  $X$  is defined as follows:

$$h(x) = \frac{f(x)}{R(x)}$$

Substituting from Eq. (5) and Eq. (7) into the above equation yields

$$h(x) = \frac{\alpha \beta^{\alpha-1} x^{-\alpha} e^{-(\beta/x)^\alpha}}{\Gamma(1-1/\alpha) - \Gamma(1-1/\alpha, (\beta/x)^\alpha)} \tag{8}$$

In addition, the reverse hazard function  $h_r(x)$ , can be obtained as

$$h_r(x) = \frac{f(x)}{F(x)}$$

Inserting Eq. (5) and Eq. (6) into the last equation yields

$$h_r(x) = \frac{\alpha \beta^{\alpha-1} x^{-\alpha} e^{-(\beta/x)^\alpha}}{\Gamma(1-1/\alpha, (\beta/x)^\alpha)} \tag{9}$$

**5. Statistical Properties**

In this section, we present a mathematical treatment of the proposed distribution such as; mean, variance, coefficient of variation, harmonic mean, raw and incomplete moments, skewness, kurtosis, mode, moment and probability generating functions, reliability, hazard and reverse hazard functions, Lorenz, Bonferroni and Zenga curves, Renyi of entropy, order statistics and record statistics.

**5.1 Raw Moments**

Suppose  $X$  is a random variable distributed according to LBWF distribution, then the raw moments is given by

$$\begin{aligned} \mu'_r &= E(X^r) = \frac{\alpha \beta^{\alpha-1}}{\Gamma(1-1/\alpha)} \int_0^\infty x^{r-\alpha} e^{-(\beta/x)^\alpha} dx \\ &= \frac{\beta^r \Gamma(1-(r+1)/\alpha)}{\Gamma(1-1/\alpha)}, \quad \alpha > r+1 \end{aligned} \tag{10}$$

Substituting  $r = 1, 2, 3, 4$  in Eq. (10), then we get

$$\text{Mean} = \mu'_1 = \frac{\beta \Gamma(1-2/\alpha)}{\Gamma(1-1/\alpha)}, \quad \alpha > 2$$

$$\mu'_2 = \frac{\beta^2 \Gamma(1-3/\alpha)}{\Gamma(1-1/\alpha)}, \quad \alpha > 3$$

$$\mu'_3 = \frac{\beta^3 \Gamma(1-4/\alpha)}{\Gamma(1-1/\alpha)}, \quad \alpha > 4$$

$$\mu'_4 = \frac{\beta^4 \Gamma(1-5/\alpha)}{\Gamma(1-1/\alpha)}, \quad \alpha > 5$$

From above, we can observe that

$$\mu'_2 - \mu'_1 = \frac{\beta [\beta \Gamma(1-3/\alpha) - \Gamma(1-2/\alpha)]}{\Gamma(1-1/\alpha)}, \quad \alpha > 3,$$

$$\mu'_3 - \mu'_2 = \frac{\beta^2 [\beta \Gamma(1-4/\alpha) - \Gamma(1-3/\alpha)]}{\Gamma(1-1/\alpha)}, \quad \alpha > 4$$

and

$$\mu'_4 - \mu'_3 = \frac{\beta^3 [\beta \Gamma(1-5/\alpha) - \Gamma(1-4/\alpha)]}{\Gamma(1-1/\alpha)}, \quad \alpha > 5$$

Consequently, we can establish a recurrence relation of raw moments for the LBWF distribution to be

$$\mu'_{r+1} = \left[ \frac{\beta^r [\beta \Gamma(1 - (r + 2)/\alpha) - \Gamma(1 - (r + 1)/\alpha)]}{\Gamma(1 - 1/\alpha)} \right] + \mu'_r, \quad \alpha > r + 2 \tag{11}$$

Therefore, the variance and the standard deviation are given by

$$v(x) = \frac{\beta^2 [\Gamma(1 - 1/\alpha) \Gamma(1 - 3/\alpha) - \Gamma^2(1 - 2/\alpha)]}{\Gamma^2(1 - 1/\alpha)}, \quad \alpha > 3 \tag{12}$$

and

$$\sigma = \frac{\beta [\Gamma(1 - 1/\alpha) \Gamma(1 - 3/\alpha) - \Gamma^2(1 - 2/\alpha)]^{1/2}}{\Gamma(1 - 1/\alpha)}, \quad \alpha > 3 \tag{13}$$

**5.2 Incomplete Moments**

Suppose  $X$  is a random variable obeying the LBWF distribution, then the  $r^{\text{th}}$  incomplete moments denoted as  $m_r(z)$  can be obtained as follows:

$$m_r(z) = \int_0^z x^r f(x) dx = \frac{\alpha \beta^{\alpha-1}}{\Gamma(1 - 1/\alpha)} \int_0^z x^{r-\alpha} e^{-(\beta/x)^\alpha} dx$$

Put  $u = (\beta/x)^\alpha$  in the previous integration, then we get

$$m_r(z) = \frac{\beta^r}{\Gamma(1 - 1/\alpha)} \int_{(\beta/z)^\alpha}^\infty u^{-(r+1)/\alpha} e^{-u} du$$

or

$$m_r(z) = \frac{\beta^r \Gamma(1 - (r + 1)/\alpha, (\beta/z)^\alpha)}{\Gamma(1 - 1/\alpha)}, \quad \alpha > r + 1, r = 1, 2, 3, \dots \tag{14}$$

**5.3 Harmonic Mean**

The harmonic mean of the LBWF distribution is derived as follows:

$$H = [E(1/x)]^{-1} = \left[ \frac{\alpha \beta^{\alpha-1}}{\Gamma(1 - 1/\alpha)} \int_0^\infty x^{-\alpha-1} e^{-(\beta/x)^\alpha} dx \right]^{-1} = \beta \Gamma(1 - 1/\alpha), \quad \alpha > 1 \tag{15}$$

**5.4 Coefficient of Variation**

The coefficient of variation of the LBWF distribution is given by

$$C_V = \frac{\sigma}{\mu} = \frac{[\Gamma(1 - 1/\alpha) \Gamma(1 - 3/\alpha) - \Gamma^2(1 - 2/\alpha)]^{1/2}}{\Gamma(1 - 2/\alpha)}, \quad \alpha > 3 \tag{16}$$

**5.5 Skewness and Kurtosis**

The coefficients of skewness and kurtosis of the LBWF distribution are given respectively as follows:

$$\omega_1 = \frac{\mu'_3}{(\mu'_2)^{3/2}} = \frac{\Gamma^{1/2}(1 - 1/\alpha) \Gamma(1 - 4/\alpha)}{\Gamma^{3/2}(1 - 3/\alpha)}, \quad \alpha > 4 \tag{17}$$

and

$$\omega_2 = \frac{\mu'_4}{(\mu'_2)^2} = \frac{\Gamma(1 - 1/\alpha) \Gamma(1 - 5/\alpha)}{\Gamma^2(1 - 3/\alpha)}, \quad \alpha > 5 \tag{18}$$

**5.6. Moment and Probability Generating Functions**

The moment generating function of LBWF distribution, say  $M_x(t)$  can be obtained as follows:

$$M_x(t) = E(e^{tx}) = \frac{\alpha \beta^{\alpha-1}}{\Gamma(1 - 1/\alpha)} \int_0^\infty e^{tx} x^{-\alpha} e^{-(\beta/x)^\alpha} dx$$

Using  $e^{tx} = \sum_{h=0}^\infty \frac{t^h x^h}{h!}$  we get

$$M_x(t) = \frac{\alpha \beta^{\alpha-1}}{\Gamma(1-1/\alpha)} \sum_{h=0}^{\infty} \frac{t^h}{h!} \int_0^{\infty} x^{h-\alpha} e^{-(\beta/x)^\alpha} dx$$

or

$$M_x(t) = \frac{1}{\Gamma(1-1/\alpha)} \sum_{h=0}^{\infty} \frac{t^h \beta^h \Gamma(1-(h+1)/\alpha)}{\Gamma(h+1)}, \alpha > h+1 \tag{19}$$

Likewise, the probability generating function denoted as  $M_{[x]}(t)$  of the LBWF distribution can be derived as below

$$M_{[x]}(t) = E(t^x) = \frac{\alpha \beta^{\alpha-1}}{\Gamma(1-1/\alpha)} \int_0^{\infty} t^x x^{-\alpha} e^{-(\beta/x)^\alpha} dx$$

$$= \frac{\alpha \beta^{\alpha-1}}{\Gamma(1-1/\alpha)} \int_0^{\infty} e^{x \ln t} x^{-\alpha} e^{-(\beta/x)^\alpha} dx$$

Using  $e^{x \ln t} = \sum_{w=0}^{\infty} \frac{(\ln t)^w x^w}{w!}$  we get

$$M_{[x]}(t) = \frac{\alpha \beta^{\alpha-1}}{\Gamma(1-1/\alpha)} \sum_{w=0}^{\infty} \frac{(\ln t)^w}{w!} \int_0^{\infty} x^{w-\alpha} e^{-(\beta/x)^\alpha} dx$$

or

$$M_x(t) = \frac{1}{\Gamma(1-1/\alpha)} \sum_{w=0}^{\infty} \frac{(\ln t)^w \beta^w \Gamma(1-(w+1)/\alpha)}{\Gamma(w+1)}, \alpha > w+1 \tag{20}$$

### 5.7 Mode

The mode of the LBWF distribution can be obtained by taking the logarithm of Eq. (5) as

$$\log f(x; \alpha, \beta) = \log \left[ \frac{\alpha \beta^{\alpha-1}}{\Gamma(1-1/\alpha)} \right] - \alpha \log x - (\beta/x)^\alpha$$

Differentiating the above equation with respect to  $x$  and equating this derivate to zero, we get

$$\frac{\partial \log f(x; \alpha, \beta)}{\partial x} = -\frac{\alpha}{x} + \frac{\alpha \beta^\alpha}{x^{\alpha+1}} = 0$$

Solving the previous nonlinear equation for  $x$ , then the mode is given by

$$x = \beta \tag{21}$$

### 5.8 Lorenz, Bonferroni and Zenga Curves

The Lorenz and Bonferroni curves have been used in different fields such as demography, insurance, reliability, medicine and economics [for more details see Kleiber & Kotz (2003)]<sup>[13]</sup>. The Lorenz curve,  $L_F(x)$  is defined by Oluyede & Rajasooriya (2013)<sup>[21]</sup> as follows:

$$L_F(x) = \frac{1}{E(x)} \int_0^x t f(t) dt$$

$$= \frac{\alpha \beta^{\alpha-2}}{\Gamma(1-2/\alpha)} \int_0^x t^{1-\alpha} e^{-(\beta/t)^\alpha} dt$$

Put  $y = (\beta/x)^\alpha$  in the above equation, then we get

$$L_F(x) = \int_{(\beta/x)^\alpha}^{\infty} y^{-2/\alpha} e^{-y} dy = \Gamma(1-2/\alpha, (\beta/x)^\alpha), \quad \alpha > 2 \tag{22}$$

The Bonferroni curve,  $B(F(x))$  is given by Oluyede & Rajasooriya (2013)<sup>[21]</sup> as below:

$$B(F(x)) = \frac{1}{F(x)E(x)} \int_0^x t f(t) dt = \frac{L_F(x)}{F(x)}$$

Using Eq. (6) and Eq. (22) in the above, then we have

$$B(F(x)) = \frac{\Gamma(1-1/\alpha) \Gamma(1-2/\alpha, (\beta/x)^\alpha)}{\Gamma(1-1/\alpha, (\beta/x)^\alpha)}, \quad \alpha > 2 \tag{23}$$

The Zenga curve  $A(x)$  is presented by Oluyede & Rajasooriya (2013)<sup>[21]</sup> as follows:

$$A(x) = 1 - \left[ \frac{M^-(x)}{M^+(x)} \right]$$

Where  $M^-(x) = \frac{1}{F(x)} \int_0^x t f(t) dt$  and  $M^+(x) = \frac{1}{1-F(x)} \int_x^\infty t f(t) dt$ . Then, for the new model we have

$$M^-(x) = \frac{\alpha \beta^{\alpha-1}}{\Gamma(1-1/\alpha, (\beta/x)^\alpha)} \int_0^x t^{1-\alpha} e^{-(\beta/t)^\alpha} dt$$

or

$$M^-(x) = \frac{\beta \Gamma(1-2/\alpha, (\beta/x)^\alpha)}{\Gamma(1-1/\alpha, (\beta/x)^\alpha)}, \alpha > 2$$

Also, we have

$$\begin{aligned} M^+(x) &= \frac{\alpha \beta^{\alpha-1}}{\left[ \Gamma(1-1/\alpha) - \Gamma(1-1/\alpha, (\beta/x)^\alpha) \right]} \int_x^\infty t^{1-\alpha} e^{-(\beta/t)^\alpha} dt \\ &= \frac{\beta}{\left[ \Gamma(1-1/\alpha) - \Gamma(1-1/\alpha, (\beta/x)^\alpha) \right]} \int_0^{(\beta/x)^\alpha} y^{-2/\alpha} e^{-y} dy \\ &= \frac{\beta \gamma(1-1/\alpha, (\beta/x)^\alpha)}{\left[ \Gamma(1-1/\alpha) - \Gamma(1-1/\alpha, (\beta/x)^\alpha) \right]} \end{aligned}$$

Where the lower incomplete gamma function is  $\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt$ ,  $a > 0$ ,  $b \geq 0$ .

Consequently, the Zenga curve is given below

$$A(x) = 1 - \left[ \frac{\Gamma(1-2/\alpha, (\beta/x)^\alpha) \left[ \Gamma(1-1/\alpha) - \Gamma(1-1/\alpha, (\beta/x)^\alpha) \right]}{\gamma(1-2/\alpha, (\beta/x)^\alpha) \Gamma(1-1/\alpha, (\beta/x)^\alpha)} \right] \tag{24}$$

### 5.9. Rényi Entropy

The entropy of a random variable  $X$  is a measure of randomness contained in a probability distribution. The Rényi entropy is

defined as  $I_R(\delta) = \frac{1}{1-\delta} [\log I(\delta)]$ , where  $I(\delta) = \int f^\delta(x) dx$ ,  $\delta > 0$  and  $\delta \neq 1$ . Using Eq. (5) yields

$$I(\delta) = \frac{\alpha^\delta \beta^{\delta(\alpha-1)}}{(\Gamma(1-1/\alpha))^\delta} \int_0^\infty x^{-\alpha\delta} e^{-\delta(\beta/x)^\alpha} dx$$

$$I(\delta) = \frac{\alpha^{\delta-2} \beta^{1-\delta} \delta^{(1/\alpha)-\delta} \Gamma(\delta-1/\alpha)}{(\Gamma(1-1/\alpha))^\delta}$$

Therefore, the Rényi entropy is given from

$$\begin{aligned} I_R(\delta) &= \frac{1}{1-\delta} \log \left[ \frac{\alpha^{\delta-2} \beta^{1-\delta} \delta^{(1/\alpha)-\delta} \Gamma(\delta-1/\alpha)}{(\Gamma(1-1/\alpha))^\delta} \right] \\ &= \frac{1}{\delta-1} \left[ (\delta-2) \log \alpha + (1-\delta) \log \beta + ((1/\alpha)-\delta) \log \delta \right. \\ &\quad \left. + \log \Gamma(\delta-1/\alpha) - \delta \log \Gamma(1-1/\alpha) \right] \end{aligned} \tag{25}$$

**5.10 Order Statistics**

Order statistics play an important role in probability and statistics. Let  $x_{1:n} \leq x_{2:n}, \dots \leq x_{n:n}$  be the ordered sample from a continuous population with pdf  $f(x)$  and cdf  $F(x)$ . The pdf of  $X_{k:n}$ , the  $k^{th}$  order statistics is given by

$$f_{X_{k:n}}(x) = \frac{n!}{(k-1)!(n-k)!} f(x) [F(x)]^{k-1} [1-F(x)]^{n-k} \tag{26}$$

Then, the pdf of the  $k^{th}$  order LBWF random variable  $X_{k:n}$  can be obtained by using Eqs.(5) and (6) in Eq. (26) to be

$$f_{X_{k:n}}(x) = \frac{n! \alpha \beta^{\alpha-1}}{(k-1)!(n-k)!} \Delta_1(x | \alpha, \beta) x^{-\alpha} e^{-(\beta/x)^\alpha} \tag{27}$$

Where

$$\Delta_1(x | \alpha, \beta) = \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j [\Gamma(1-1/\alpha)]^{-(k+j)} \left[ \Gamma\left(1-1/\alpha, (\beta/x)^\alpha\right) \right]^{k+j-1}$$

Therefore, the pdf of the  $1^{th}$  order LBWF random variable  $X_{1:n}$  is given by

$$f_{X_{1:n}}(x) = n \alpha \beta^{\alpha-1} \Delta_2(x | \alpha, \beta) x^{-\alpha} e^{-(\beta/x)^\alpha} \tag{28}$$

Where

$$\Delta_2(x | \alpha, \beta) = \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j [\Gamma(1-1/\alpha)]^{-(j+1)} \left[ \Gamma\left(1-1/\alpha, (\beta/x)^\alpha\right) \right]^j$$

Also, the pdf of the  $n^{th}$  order LBWF random variable  $X_{n:n}$  is given by

$$f_{X_{n:n}}(x) = n \alpha \beta^{\alpha-1} \Delta_3(x | \alpha, \beta) x^{-\alpha} e^{-(\beta/x)^\alpha} \tag{29}$$

Where

$$\Delta_3(x | \alpha, \beta) = [\Gamma(1-1/\alpha)]^{-n} \left[ \Gamma\left(1-1/\alpha, (\beta/x)^\alpha\right) \right]^{n-1}$$

Moreover, the joint distribution of two order statistics  $X_{k:n} \leq X_{s:n}$  is given as

$$f_{X_{k:n}, X_{s:n}}(x_1, x_2) = \frac{n!}{(k-1)!(s-k-1)!(n-s)!} f(x_1) f(x_2) [F(x_1)]^{k-1} \times [F(x_2) - F(x_1)]^{s-k-1} [1-F(x_2)]^{n-s} \tag{30}$$

Then, for the LBWF distribution we have  $f_{X_{k:n}, X_{s:n}}(x_1, x_2) = \frac{n! \alpha^2 \beta^{2(\alpha-1)}}{(k-1)!(s-k-1)!(n-s)!} \Delta_4(x_1, x_2 | \alpha, \beta) \times x_1^{-\alpha} x_2^{-\alpha} \exp\left[(-\beta^\alpha)(x_1^{-\alpha} + x_2^{-\alpha})\right]$  (31)

Where

$$\Delta_4(x_1, x_2 | \alpha, \beta) = \sum_{i=0}^{n-s} \sum_{j=0}^{s-k-1} \binom{n-s}{i} \binom{s-k-1}{j} (-1)^{i+j} \times [\Gamma(1-1/\alpha)]^{-(s+i)} \left[ \Gamma\left(1-1/\alpha, (\beta/x_1)^\alpha\right) \right]^{k+j-1} \times \left[ \Gamma\left(1-1/\alpha, (\beta/x_2)^\alpha\right) \right]^{s+i-(k+j)-1}$$

**5.11 Record Statistics**

Record values and the corresponding statistics are of interest and important in many real life applications including meteorological analysis, industrial stress testing, hydrology and athletic events. Let  $X_1, X_2, \dots, X_n$  be a sequence of independent and identically distributed (iid) random variables having cdf  $F(x)$  and pdf  $f(x)$ . Set  $Y_n = \max(\min)\{X_1, X_2, \dots, X_n\}$  for  $n \geq 1$ . We say  $X_j$  is an upper (lower) record value of this sequence if  $Y_j > (<) Y_{j-1}, j > 1$ . Thus  $X_j$  will be called an upper (lower) record value if its value exceeds (is lower than) that of all previous observations.

The pdf of  $X_{U(r)}$ , the  $r^{th}$  upper record is given as [see Ahsanullah (1995) and Arnold *et al.* (1998)]<sup>[5, 6]</sup>

$$f_{X_{U(r)}}(x) = \frac{1}{(r-1)!} [R(x)]^{r-1} f(x) \tag{32}$$

Where

$$R(x) = -\ln[1 - F(x)]$$

Then, the pdf of the  $r^{th}$  upper record LBWF random variable  $X_{U(r)}$  can be obtained by using Eqs. (5) and (6) in Eq. (32) to be

$$f_{X_{U(r)}}(x) = \frac{\alpha \beta^\alpha}{(r-1)! \Gamma(1-1/\alpha)} \Delta_5^{r-1}(x | \alpha, \beta) x^{-\alpha} e^{-(\beta/x)^\alpha} \tag{33}$$

Where

$$\Delta_5(x | \alpha, \beta) = \ln[\Gamma(1-1/\alpha)] - \ln[\Gamma(1-1/\alpha) - \Gamma(1-1/\alpha, (\beta/x)^\alpha)]$$

Furthermore, the joint distribution of the first  $n$  upper record values  $x \equiv (x_{U(1)}, x_{U(2)}, \dots, x_{U(n)})$  is given by [see Ahsanullah (1995)]<sup>[5]</sup>

$$f_{1,2,\dots,n}(x_{U(1)}, x_{U(2)}, \dots, x_{U(n)}) = f(x_{U(n)}) \prod_{i=1}^{n-1} \frac{f(x_{U(i)})}{1 - F(x_{U(i)})} \tag{34}$$

Then, for the LBWF distribution we get 
$$f_{1,2,\dots,n}(x_{U(1)}, x_{U(2)}, \dots, x_{U(n)}) = \frac{\alpha^n \beta^{n\alpha}}{\Gamma(1-1/\alpha)} \exp \left[ (-) \left( \begin{array}{l} \Delta_6(x | \alpha, \beta) - \Delta_7(x | \alpha) \\ + \Delta_8(x | \alpha, \beta) \end{array} \right) \right] \tag{35}$$

Where

$$\Delta_6(x | \alpha, \beta) = \beta^\alpha \sum_{i=1}^n \frac{1}{x_i^\alpha}, \quad \Delta_7(x | \alpha) = \alpha \sum_{i=1}^n \ln x_i$$

and

$$\Delta_8(x | \alpha, \beta) = \sum_{i=1}^{n-1} \ln \left[ \Gamma(1-1/\alpha) - \Gamma(1-1/\alpha, (\beta/x_i)^\alpha) \right]$$

Furthermore, the pdf of  $X_{L(r)}$ , the  $r^{th}$  lower record is given as [see Ahsanullah (1995) and Arnold *et al.* (1998)]<sup>[5, 6]</sup>

$$f_{X_{L(r)}}(x) = \frac{1}{(r-1)!} [H(x)]^{r-1} f(x) \tag{36}$$

Where

$$H(x) = -\ln[F(x)]$$

Consequently, for the LBWF distribution we obtain

$$f_{X_{L(r)}}(x) = \frac{\alpha \beta^\alpha}{(r-1)! \Gamma(1-1/\alpha)} \Delta_9^{r-1}(x | \alpha, \beta) x^{-\alpha} e^{-(\beta/x)^\alpha} \tag{37}$$

Where

$$\Delta_9(x|\alpha, \beta) = \ln[\Gamma(1-1/\alpha)] - \ln\left[\Gamma\left(1-1/\alpha, (\beta/x)^\alpha\right)\right]$$

Moreover, the joint distribution of the first  $n$  lower record values  $x \equiv (x_{L(1)}, x_{L(2)}, \dots, x_{L(n)})$  is given by [see Ahsanullah (1995)] [5]

$$f_{1,2,\dots,n}(x_{L(1)}, x_{L(2)}, \dots, x_{L(n)}) = f(x_{L(n)}) \prod_{i=1}^{n-1} \frac{f(x_{L(i)})}{F(x_{L(i)})} \tag{38}$$

Then, for the proposed model we get

$$f_{1,2,\dots,n}(x_{L(1)}, x_{L(2)}, \dots, x_{L(n)}) = \frac{\alpha^n \beta^{n\alpha}}{\Gamma(1-1/\alpha)} \exp\left[(-) \left[ \begin{array}{l} \Delta_6(x|\alpha, \beta) - \Delta_7(x|\alpha) \\ + \Delta_{10}(x|\alpha, \beta) \end{array} \right] \right] \tag{39}$$

Where

$$\Delta_{10}(x|\alpha, \beta) = \sum_{i=1}^{n-1} \ln\left[\Gamma\left(1-1/\alpha, (\beta/x_i)^\alpha\right)\right]$$

**6. Methods of Estimation**

In this section we describe two estimation methods to obtain the estimates for the model parameters associated to the LBWF distribution.

**6.1. Moments Estimation**

The moment estimators of  $\alpha$  and  $\beta$  denoted as  $\hat{\alpha}_{MME}$  and  $\hat{\beta}_{MME}$  respectively can be obtained by equating population moments with the sample moments as follows:

$$\mu'_r = \frac{1}{n} \sum_{i=1}^n x_i^r$$

Then, we have

$$\frac{\beta \Gamma(1-2/\alpha)}{\Gamma(1-1/\alpha)} = \bar{x}, \quad \bar{x} = \frac{1}{n} \sum_{i=1}^n x_i \tag{40}$$

and

$$\frac{\beta^2 \Gamma(1-3/\alpha)}{\Gamma(1-1/\alpha)} = \frac{1}{n} \sum_{i=1}^n x_i^2 \tag{41}$$

Eq. (40) yields

$$\beta = \frac{\Gamma(1-1/\alpha)}{\Gamma(1-2/\alpha)} \bar{x} \tag{42}$$

Using Eq. (42) in Eq. (41), we obtain

$$\frac{\Gamma(1-1/\alpha) \Gamma(1-3/\alpha)}{\Gamma^2(1-2/\alpha)} = \frac{s^2}{\bar{x}^2} + 1, \quad s^2 = \frac{1}{n} \sum_{i=1}^n x_i^2 - \bar{x}^2 \tag{43}$$

Solving Eq. (43) for  $\alpha$  by numerical methods, then we get  $\hat{\alpha}_{MME}$  and then substituting in Eq. (42) to obtain  $\hat{\beta}_{MME}$ .

**6.2. Maximum Likelihood Estimation**

Let  $x_1, x_2, \dots, x_n$  be a random sample from the LBWF, then the corresponding likelihood function is given by

$$L(x; \alpha, \beta) = \prod_{i=1}^n \frac{\alpha \beta^{\alpha-1}}{\Gamma(1-1/\alpha)} x^{-\alpha} e^{-(\beta/x_i)^\alpha}$$

$$= \alpha^n \beta^{n(\alpha-1)} (\Gamma(1-1/\alpha))^{-n} \exp(-) \left[ \alpha \sum_{i=1}^n \log x_i + \beta^\alpha \sum_{i=1}^n x_i^{-\alpha} \right] \tag{44}$$

Taking the logarithm of Eq. (44), we have

$$\log L(x; \alpha, \beta) = n \log \alpha + n(\alpha - 1) \log \beta - n \log \Gamma(1 - 1/\alpha)$$

$$- \alpha \sum_{i=1}^n \log x_i - \beta^\alpha \sum_{i=1}^n x_i^{-\alpha} \tag{45}$$

Now; differentiate Eq. (45) with respect to  $\alpha$  and  $\beta$ , then equate this derivative to zero, we obtain the following normal equations:

$$\frac{\partial \log L(x; \beta)}{\partial \alpha} = \frac{n}{\alpha} + n \log \beta - n \Psi(1 - 1/\alpha) - \sum_{i=1}^n \log x_i$$

$$- \beta^\alpha \left[ \sum_{i=1}^n x_i^{-\alpha} \log x_i + \log \beta \sum_{i=1}^n x_i^{-\alpha} \right] = 0 \tag{46}$$

And

$$\frac{\partial \log L(x; \beta)}{\partial \beta} = \frac{n(\alpha - 1)}{\beta} - \alpha \beta^{\alpha-1} \sum_{i=1}^n x_i^{-\alpha} = 0 \tag{47}$$

Where  $\Psi(x)$  is the digamma function.

The maximum likelihood estimates (MLEs);  $\hat{\alpha}_{MLE}$  and  $\hat{\beta}_{MLE}$  of  $\alpha$  and  $\beta$  respectively can be obtained by solving Eqs (46) and (47) simultaneously with numeric iteration techniques. Moreover, the Fisher information matrix is given by  $I(\alpha, \beta) = -E \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix}$

With

$$I_{11} = \frac{\partial^2}{\partial \alpha^2} \log L(x; \alpha, \beta) = -n \left\{ \begin{array}{l} \left[ \frac{1}{\alpha^2} + \frac{1}{\Gamma(1-1/\alpha)} \left[ \Psi^{(1)}(1-1/\alpha) - \frac{(\Psi(1-1/\alpha))^2}{\Gamma(1-1/\alpha)} \right] \right] \\ - \beta^\alpha \sum_{i=1}^n x_i^{-\alpha} [\ln(\beta) - \ln(x_i)]^2 \end{array} \right\}$$

$$I_{12} = I_{21} = \frac{\partial^2}{\partial \alpha \partial \beta} \log L(x; \alpha, \beta) = -\frac{n}{\beta} - \beta^{\alpha-1} \left[ \sum_{i=1}^n x_i^{-\alpha} + \alpha \sum_{i=1}^n x_i^{-\alpha} [\ln(\beta) - \ln(x_i)] \right]$$

and

$$I_{22} = \frac{\partial^2}{\partial \beta^2} \log L(x; \alpha, \beta) = \frac{-(\alpha - 1)}{\beta^2} \left[ n + \alpha \beta^\alpha \sum_{i=1}^n x_i^{-\alpha} \right]$$

Where  $\Psi^{(n)}(x)$  is the polygamma function.

### 7. Conclusion

This paper introduced a new distribution named the length-biased weighted Frechet distribution. Different properties of this distribution were studied including the mean, variance, coefficient of variation, harmonic mean, raw and incomplete moments, skewness, kurtosis, mode, moment and probability generating functions, reliability, hazard and reverse hazard functions, Lorenz, Bonferroni and Zenga curves, Renyi of entropy, order statistics and record statistics. The parameters of the new distribution are estimated by using moment and maximum likelihood approaches and the observed Fisher information matrix is obtained.

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