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An algorithm to find a basis of the fixed point subgroup of an endomorphism of a free group

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Abstract
Here we discuss the way to find the basis of the fixed point subgroup of non-injective & non-surjective endomorphisms for the free group of rank two.

Keywords: free group, rank, generator, endomorphism, automorphism

1. Introduction
Free group and its fixed point subgroups is an interesting domain of research since previous century. A wide body of results on fixed subgroups of free groups by their automorphisms and endomorphisms have been found from 1970s onwards.
Let $\mathbb{F}$ be a free group of finite rank. Then End ($\mathbb{F}$) and Aut ($\mathbb{F}$) represent the set of endomorphisms and the set of automorphisms on $\mathbb{F}$ respectively. Fixed subgroup $\mathrm{Fix}\phi = \{u \in \mathbb{F} | \phi(u) = u\}$, where $\phi \in \mathrm{End}(\mathbb{F})$.
In [4], Gersten showed that $\mathrm{Fix}\phi$ is finitely generated for Aut ($\mathbb{F}$). In [1], Bestvina and Handel improved this result by showing that $r(\mathrm{Fix}\phi) \leq r(\mathbb{F})$, where $r(\mathbb{F})$ denotes the rank of the free group $\mathbb{F}$ etc. Before Bestvina and Handel, Imrich and Turner proved that $r(\mathrm{Fix}\phi) \leq r(\mathbb{F})$ for $\phi \in \mathrm{End}(\mathbb{F})$ in [5]. Now the research was moved to find the rank of $\mathrm{Fix}\phi$ for $\phi \in \mathrm{End}(\mathbb{F})$.

Question 1.1: Is there an algorithm to find the $r(\mathrm{Fix}\phi)$ for $\phi \in \mathrm{End}(\mathbb{F})$? [It was asked by J. R. Stallings in 1984]
In the consequence, this question was resolved by Bogopolski and Maslakova in [2] for $\phi \in \mathrm{Aut}(\mathbb{F})$. In addition Bogopolski and Maslakova invented an algorithm to find the basis of $\mathrm{Fix}\phi$ too. For the related concept one may go through [3] also.
But the finding of basis of $\mathrm{Fix}\phi$ for $\phi \in \mathrm{End}(\mathbb{F})$ was still open. This paper provides the partial answer to the question posed by Stallings for the non-injective & non-surjective endomorphisms on the free group of rank two. Generally $\mathbb{F}_2$ denotes the free group of rank two.

2. Main Theorem
The main theorem of this paper is as follows:-

Theorem 2.1: If ϕ is a non-injective & non-surjective endomorphism on $\mathbb{F}_2$ then there exists an algorithm to find the basis of $\mathrm{Fix}\phi$. i.e.
There exists an algorithm with input a non-injective & non-surjective endomorphism ϕ: $\mathbb{F}(u, v) \rightarrow \mathbb{F}(u, v)$ and output a basis of $\mathrm{Fix}\phi$, where u and v are the generators of the free group $\mathbb{F}$.

3. Preliminaries
For the basic theory of free groups reader may go through Combinatorial Group Theorey by [6], Lyndon & Schupp and [7], Magnus, Karrass & Solitar. Obviously endomorphism ϕ on free groups are four of the types as follows:

1. Injective and surjective endomorphism i.e. automorphism
2. Injective and non- surjective endomorphism
3. Neither injective nor surjective endomorphism
4. Non- injective and surjective endomorphism

In group theory it is a well-known fact that free groups are Hopfian. That means every surjective endomorphism on a free group must be an automorphism. Hence the case (iv) is reduced to case (i) i.e. to Automorphism.
It is already mentioned in introduction that the basis of Fixϕ for $\phi \in \mathrm{Aut}(\mathbb{F})$ has been found by Bogopolski and Maslakova in [BM16]. Obviously the cases (i) and (iv) has been done. So we move to find the basis of Fixϕ for $\phi \in \mathrm{End}(\mathbb{F})$ as of case (iii).

4. Proof of the Main Theorem

Here is the proof of theorem 2.1

Since ϕ is a non-injective endomorphism here. Therefore $\mathrm{Im}(\phi)$ must be cyclic and it may be trivial also. The generator of this cyclic group can be found by the use of Stallings folding techniques. Let Fixϕ is non-trivial then $\phi(w^c) = w^c$ for $c \in \mathbb{Z} - \{0\}$.

$$\phi(w^c) = w^c$$

$$\Rightarrow \phi(wwww \ldots . c\ times) = w^c$$

$$\Rightarrow \phi(w)\phi(w)\,\phi(w)\phi(w) \ldots \ldots .. c\ times = w^c$$

$$\Rightarrow \phi(w)^c = w^c$$

$$\Rightarrow \phi(w) = w \text{ [since roots are unique in free group]}$$

Therefore Fixϕ is non-trivial if and only if $\mathrm{Im}(\phi)$ be generated by w is non-trivial and $\phi(w) = w$ and Fix$\phi = \mathrm{Im}(\phi)$.
Hence the algorithm to find the basis of Fixϕ is in following steps:

1. Find the generator w for $\mathrm{Im}(\phi)$ using standard algorithms.
2. Then find $\phi(w)$, if $\phi(w) = w$, where w is not the identity element, then Fix$\phi = < w >$
3. If w is identity then Fixϕ is just trivial group.

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