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A study of fuzzy set and its operations

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Abstract
Fuzzy sets allow one to work in uncertain and vague situations and solve those problems which have more than one solution. In real life sometimes we are unable to answer many questions because these answers are depending upon two valued logic which are unable to give clear-cut explanation. The present paper aims to study fuzzy set theory and its examples. Here we are also discussing some standard and non-standard operations on fuzzy sets which are generalizations of the corresponding crisp operations.

Keywords: Fuzzy set, standard operations, non-standard operations, crisp operations

1. Introduction
Several mathematicians have introduced the concept of sets in their own ways. These ways of representing problems are more inflexible. The solutions using this concept are not significant in many real life situations. This difficulty was overcome by the Fuzzy concept which was first introduced by an eminent American Cyberneticist Prof L.A.Zadeh in 1965. Definitions, Theorems, Proofs on fuzzy set theory always hold for non-fuzzy sets. The classical set theory is built on the elementary concept of "set" in which an individual is either a member or not a member. In fuzzy set theory all those members are belongs to that set which are completely or partially belongs in that set.

2. Some definitions
Definition 2.1 Classical sets
An element belongs to the set or it does not, then the set is Called "Classical Set". It is also called
"crisp (sets)". There is no uncertainty in this set theory.

Example 2.1.1
X (Universe of discourse)

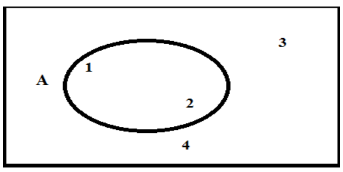

Fig 1: Crisp Set Boundary

Here, $1,2 \in A$, but $3,4 \notin A$.

Definition 2.2 Fuzzy sets
A set in which elements have degree of membership is called "Fuzzy Set". Degree means all tones between black and white. A fuzzy set has a graphical description that expresses how the transition from one to another takes place. This graphical description is called "a membership function".
Example 2.2.1
X (Universe of discourse)

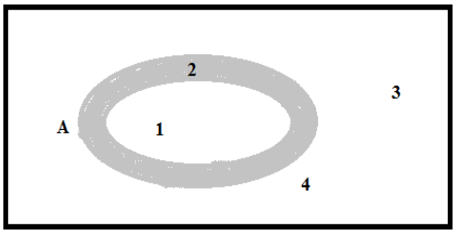

Fig 2: Fuzzy set boundary

Here, 1 $\in$A, 2 partially belongs to A, but 3, 4$\notin$A

3. Mathematical representation of fuzzy set [19]

Let X be the universe of discourse and $[0,1]$ be the closed interval of real numbers. Then a mapping $A : X \to [0,1]$ is called 'a fuzzy set A' or 'fuzzy subset A' of X for which $A(x) = \alpha$ Where, x is a member of X and α is a real number belonging to $[0,1]$

Example 3.1

Let, $X = \{1,2,3,4,5,6,7\}$ be the universal set and $A : X \to [0,1]$ Such that

$A(1) = .2$ i.e. $1 \in .2$ means $1 \in A$ with gradation.2

$A(2) = .5$ i.e. $2 \in .5$ means $2 \in A$ with gradation.5

$A(3) = .6$ i.e. $3 \in .6$ means $3 \in A$ with gradation.6

$A(4) = .7$ i.e. $4 \in .7$ means $4 \in A$ with gradation.7

$A(5) = .8$ i.e. $5 \in .8$ means $5 \in A$ with gradation.8

$A(6) = .9$ i.e. $6 \in .9$ means $6 \in A$ with gradation.9

$A(7) = 1$ i.e. $7 \in 1$ means $7 \in A$ with gradation 1

4. Some standard operations on fuzzy sets

The notion of containment plays a vital role in the case of fuzzy sets. With these notions union, intersection, complement and difference of fuzzy sets are also defined. They are called 'standard operations on fuzzy sets'.

Definition 4.1

Let A and B be two fuzzy sets of the Universal set X. Then

* $A \subset B$ iff $A(x) < B(x)$ for all $x \in X$.
* $A = B$ iff $A(x) = B(x)$ for all $x \in X$.
* A' is the complement of A iff
$A'(x) = 1 - A(x)$, for all $x \in X$.
* $A \cup B : X \to [0,1]$, with $A \cup B(x) = \max\{A(x), B(x)\}$
$= A(x) \vee B(x)$.
* $A \cap B : X \to [0,1]$, with $A \cap B(x) = \min\{A(x), B(x)\} = A(x) \wedge B(x)$.
* $A - B = A \cap B' = A(x) \wedge B(x) = A(x) \wedge B'(x)$

Example 4.1.1

Let $X = \{1,2,3,4,5\}$ be the universal set and A, B, C be three fuzzy sets such that

$A = \{(1,.3),(2,.1),(3,0),(4,.9),(5,0)\}$

$B = \{(1,0),(2,.6),(3,.5),(4,0),(5,.1)\}$ And

$C = \{(1,0),(2,0),(3,.7),(4,0),(5,.4)\}$

Then we have the following:

$A \cup B = \{(1,.3),(2,.6),(3,.5),(4,.9),(5,.1)\}$

$A \cap B = \{(1,0),(2,.1),(3,0),(4,0),(5,0)\}$

$A' = \{(1,.7),(2,.9),(3,1),(4,.1),(5,1)\}$

$B' = \{(1,1),(2,.4),(3,.5),(4,1),(5,.9)\}$

$C' = \{(1,1),(2,1),(3,.3),(4,1),(5,.6)\}$

$$A \cup X = \{(1,1),(2,1),(3,1),(4,1),(5,1)\}$$

$$= \{1,2,3,4,5\} = X$$

$$A \cap X = \{(1,.3),(2,.1),(3,0),(4,.9),(5,0)\} = A$$

$$A - B = A \cap B' = \{(1,.3),(2,.1),(4,.9)\}$$

$$A \cup A' = \{(1,.7),(2,.9),(3,1),(4,.9),(5,1)\} \neq X$$

$$A \cap A' = \{(1,.3),(2,.1),(3,0),(4,.1),(5,0)\} \neq \phi$$

$$A \cap A' = \{(1,.3),(2,.1),(3,0),(4,.1),(5,0)\} \neq \phi$$

We observe that for any three fuzzy sets A, B and C of a universal set X, the following hold [19]

1. Commutativity
$A \cup B = B \cup A$;
$A \cap B = B \cap A$

2. Associativity:
$A \cup (B \cup C) = (A \cup B) \cup C$;

$A \cap (B \cap C) = (A \cap B) \cap C$

3. Distributivity
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

4. Idempotence: $A \cup A = A'$ **;**
$A \cap A = A$.

5. Absorption: $A \cup (A \cap B) = A$;

$A \cap (A \cap B) = A$.

6. Absorption by X & ϕ : $A \cup X = X$,
$A \cap \phi = \phi$.

7. Identity: $A \cup \phi = A$,
$A \cap X = A$.

8. Demorgan's Law: $(A \cup B)' = A' \cap B'$
$(A \cap B)' = A' \cup B'$.

9. Involution: $(A')' = A$

10. Law of contradiction: $A \cap A' = \phi$.
For fuzzy set $A \cap A' \supset \phi$.

11. Law of excluded middle: $A \cup A' = X$
For fuzzy set $A \cup A' \subset X$.

4.2 Product of two fuzzy sets*:* The product of two fuzzy sets A and B of a set X is a new fuzzy set denoted by $A.B$ and is defined by $A.B = \{(x, A(x).B(x)) \mid x \in X\}$

Example 4.2.1 Let $A = \{(x,.2),(y,.8),(z,.4)\}$

$B = \{(x,.4),(y,0),(z,.1)\}$

Then

$A.B(x) = A(x).B(x)$

$= .2 \times .4$
$= .08$

$A.B(y) = A(x).B(x)$

$= .8 \times 0$

$= 0$

$A.B(z) = A(x).B(x)$

$= .4 \times .1$

$= .04$

Therefore,

$A.B = \{(x,.08),(y,0),(z,.04)\}$

4.3 Power of fuzzy sets

The α power of a fuzzy set A is denoted by A^{α} and defined by

$A^{\alpha} = \{(x, A(x))^{\alpha} \mid x \in X\}$

Example 4.3.1 Let $X = \{x, y, z\}$ and $A = \{(x,.4),(y,.2),(z,.7)\}$ and $\alpha = 2$

Then

$A^2 = \{(x,(A(x))^2 \mid x \in X\}$

$= \{(x,(A(x))^2,(y,(A(y))^2,(z,(A(z))^2\}$

$= \{(x,(.4)^2),(y,(.2)^2),(z,(.7)^2)\}$

$= \{(x,.16),(y,.04),(z,.49)\}$

4.4 Concentration of a fuzzy set

The concentration of a fuzzy set A of X is denoted by $con(A)$ and is defined by

$con(A) = \{(x, A(x))^2 \mid x \in X\}$

i.e. $con(A) = A^2$

Example4.4.1 If A denotes the fuzzy set of good students then A^2 stands for fuzzy set of very good students.

4.5 Dilation of a fuzzy set

The dilation of a fuzzy set A of X is denoted by Dil (A) and is defined by

$Dil(A) = \{(x, A(x))^{.5} \mid x \in X\}$

i.e. $Dil(A) = A^{.5}$

Example 4.5.1 If A denotes the fuzzy set of good students then $A^{.5}$ stands for fuzzy set of more or less good students.

5. Non Standard Operations on fuzzy Sets

In previous section we have discussed about standard operations on $PF(X)$ which are nice generalizations of the corresponding crisp operations. There are also other operations called 'T-norms' and 'T-conorms' which the generalizations or extensions are of crisp intersection and crisp union respectively.

5.1 T-norms (fuzzy intersection) [14]

A mapping $T : I \times I \to I$ is called a T-norm if the following holds, for all $a,b,c \in I = [0,1]$

(i) Unit element

$T(a,1) = a$

(ii) Monotonic

$T(a,c) \leq T(b,d)$ Whenever $a \leq b \;\&\; c \leq d$.

(iii) Commutative

$T(a,b) = T(b,a)$

(iv) Associative

$T(a,T(b,c)) = T(T(a,b),c)$

5.2 T-conorms (fuzzy union) [14]

A mapping $S : I \times I \to I$ is called a T-Conorm if the following holds, for all $a,b,c \in I$

(i) Unit element

$S(a,0) = a$

(ii) Monotonic

$S(a,c) \le S(b,c)$ Whenever $a \le b$.

(iii) Commutative

$S(a,b) = S(b,a)$

(iv) Associative:

$S(a, S(b,c)) = S(S(a,b),c)$

T- Conorms associated with T-norm is defined by

$S(x, y) = 1 - T(1-x, 1-y) \ for \ all \ x, y \in I$

The standard Fuzzy union is the smallest Fuzzy union while Fuzzy intersection is the largest Fuzzy intersection [19].

5.3 Fuzzy complement

A mapping $N : I \times I \to I$ is called a complement or negation operator if the following holds, for all $a,b,c \in I$:

(i) Boundary condition

$N(0) = 1$ & $N(1) = 0$

(ii) Monotonic

$N(a) \le N(b)$

Whenever $a \ge b$

(iii) Continuity

N is continuous.

(iv) Involution

$N(N(a)) = a$

The Threshold type compliment is defined by $N(\alpha) = 1$ for $\alpha \le t, \alpha \in [0,1]$ = 0 for $\alpha > t, t \in [0,1[$

$\alpha > t, t \in [0,1[$ Here t is called ‘the threshold of N’.

5.5 Yagar class of compliment

The Yagar class of compliment is defined by $N_w(\alpha) = (1-\alpha^w)^{1/w}$ Where $w \in]0,\infty[$

5.6 Sugeno class of compliment

The Sugeno class of compliment is defined by $C_\lambda(\alpha) = \frac{1-\alpha}{1+\lambda\alpha}$ where $\lambda \in]\ 1, \infty\ [$

5.7 Cartesian product on fuzzy sets

Let $A_1, A_2, A_3, ...A_n$ be the fuzzy sets of $X_1, X_2, X_3, ...X_n$ The Cartesian product of these fuzzy sets is a fuzzy set in the product space $X_1 \times X_2 \times X_3 \times ....\times X_n$ defined by

$A_1 \times A_2 \times A_3 \times ....\times An = Min\{(x, A_i(x_i)) : x = x_{1,} x_{2,} x_3 ...., x_n \in X_i\}$

Where, $X_i = X_1 \times X_2 \times X_3 \times ....\times X_n$

Example 5.7.1 Let $X = \{a,b,c\}$ and $Y = \{l,m\}$ is universe of discourse.

Let $A = \{(a,.2),(b,.4),(c,.5)\}$ be a fuzzy set of X and $B = \{(l,.4),(m,.6)\}$ be a fuzzy set of Y .Then

$Min(A(a), B(l)) = Min(.2,.4) = .2$

$Min(A(a), B(m)) = Min(.2,.6) = .2$

$Min(A(b), B(l)) = Min(.4,.4) = .4$

$Min(A(b), B(m)) = Min(.4,.6) = .4$

$Min(A(c), B(l)) = Min(.5,.4) = .4$

$Min(A(c), B(m)) = Min(.5,.6) = .5$

$\therefore A \times B = \{.2,.2,.4,.4,.4,.5\}$

5.8 Algebraic product

The m^{th} power of a fuzzy set A of X is denoted by A^m and defined by

$A^m = \{X, (A(X))^m\}$

Example5.8.1 Let $X = \{a, b, c\}$ and $A = \{(a, .2), (b, .4), (c, .5)\}$

Then $A^2 = \{(a, (.2)^2), (b, (.4)^2), (c, (.5)^2)\} = \{(a, .4), (b, .16), (c, .25)\}$

5.9 Algebraic sum

The algebraic Sum of two fuzzy sets A and B of X is denoted by $A + B$ and defined by

$A + B = \{(x, A + B(x)) \mid x \in X\}$

Where, $A + B(x) = A(x) + B(x) - A(x).B(x)$

Example5.9.1 Let $X = \{a, b, c\}$ and

$A = \{(a, .2), (b, .3), (c, .6)\}$

$B = \{(a, .4), (b, .5), (c, .3)\}$

Then

$$A + B(a) = A(a) + B(a) - A(a).B(a) \begin{array}{l} = .2 + .4 - (.2).(.4) \\ = .6 - .08 \\ = .52 \end{array}$$

$$A + B(b) = A(b) + B(b) - A(b).B(b) \begin{array}{l} = .3 + .5 - (.3).(.5) \\ = .8 - .15 \\ = .65 \end{array}$$

$$A + B(c) = A(c) + B(c) - A(c).B(c) \begin{array}{l} = .6 + .3 - (.6).(.3) \\ = .9 - .18 \\ = .72 \end{array}$$

$\therefore\ A + B = \{(a, .52), (b, .65), (c, .72)\}$

5.10 Bounded sum

The bounded sum of two fuzzy sets A and B is denoted by $A \oplus B$ and defined by

$A \oplus B = \{(x, A \oplus B(x)) \mid x \in X\}$, where $A \oplus B(x) = \min\{1, A(x) + B(x)\}$

Example5.10.1 Let $X = \{a, b, c\}$ and

$A = \{(a, .2), (b, .4), (c, .9)\}$

$B = \{(a, .4), (b, .7), (c, .2)\}$

Then $A \oplus B = \{(a, .6), (b, 1), (c, 1)\}$

5.11 Bounded difference

The bounded difference of two fuzzy sets A and B of X is denoted by $A \Theta B$ and is defined by

$A \Theta B = \{(x, A \Theta B(x) \mid x \in X\}$

$A \Theta B(x) = \max\{0, A(x) + B(x) - 1\}$

Example5.11.1 Let $X = \{a, b, c\}$ and

$A = \{(a, .2), (b, .4), (c, .9)\}$

$B = \{(a, .4), (b, .7), (c, .3)\}$

Then

$A \Theta B(a) = \max\{0, .2 + .4 - 1\}$

$= \max\{0, -.4\}$

$= 0$

$A \Theta B(b) = \max\{0, .4 + .7 - 1\}$

$= \max\{0, .1\}$

$= .1$

$A \Theta B(c) = \max\{0, .9 + .3 - 1\}$

$= \max\{0, .2\}$

$= .2$

$\therefore\ A \Theta B = \{(a, 0), (b, .1), (c, .2)\}$

5.12 Decomposition of fuzzy sets

Let $X = \{a, b, c\}$ and

$A = \{(a,.2), (b,.4), (c,.6)\}$

Then

$.2_A = \{(a,1), (b,1), (c,1)\}$

$.4_A = \{(a,0), (b,1), (c,1)\}$

$.6_A = \{(a,0), (b,0), (c,1)\}$

For any $\alpha \in I$, the decomposition of A is denoted by α^A and defined by

${}_{\alpha}A = \alpha .{}^{\alpha}A$

Now,

$.2^A = \{(a,.2), (b,.2), (c,.2)\}$

$.4^A = \{(a,0), (b,.4), (c,.4)\}$

$.6^A = \{(a,0), (b,0), (c,.6)\}$

Clearly,

$A = {}_{.2}A \cup {}_{.4}A \cup {}_{.6}A$

6. Some theorems

We now state the following three decomposition theorems for fuzzy sets.

6.1 First decomposition theorem

$A = \bigcup_{\alpha \in I} \alpha^A$ for every fuzzy set.

6.2 Second decomposition theorem

$A = \bigcup_{\alpha \in I} \alpha^{+A}$ for every fuzzy set.

Where $\alpha^{+A} = \alpha .\alpha^{+}A$

6.3. Third Decomposition Theorem

$A = \bigcup_{\alpha \in \wedge(A)} \alpha^A$ for every fuzzy sets A.

Where $\wedge A$ is the level set of fuzzy set A defined by

$\wedge A = \{\alpha \in I \mid A(x) = \alpha \ \ for\ some\ x \in X\}$

7. Extension principle for fuzzy sets [14]

Let $f : X \to Y$ be a crisp function. This function f is said to be 'fuzzified' when it is extended to act on fuzzy sets defined on X and Y then f induces two fuzzified functions f and its inverse f^{-1} of the following forms: $f : F(X) \to F(Y)$ and $f^{-1} : F(Y) \to F(X)$

Defined by $[f(A)](y) = \sup_{x, y = f(x)} A(x) \ \ for\ all\ A \in F(X);$

$\left[f^{-1}(B)\right](x) = B(f(x)) \ \ for\ all\ B \in F(Y)$

Let $A_i \in F(X), B_i \in F(Y) \ \ and \ \ i \in \nabla$ (the indexed set). Then the following properties of Functions obtained by the extension principle hold:

(i) $f(A) = \phi \Leftrightarrow A = \phi$

(ii) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$

(iii) $f(\bigcup_{i \in \nabla} A_i) = \bigcup_{i \in \nabla} f(A_i)$

(iv) $f(\bigcap_{i \in \nabla} A_i) = \bigcap_{i \in \nabla} f(A_i)$

(v) $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$

(vi) $f^{-1}(\bigcup_{i \in \nabla} B_i) = \bigcup_{i \in \nabla} f^{-1}(B_i)$

(vii) $f^{-1}(\bigcap_{i \in \nabla} B_i) = \bigcap_{i \in \nabla} f^{-1}(B_i)$

(viii) $\overline{f^{-1}(B)} = f^{-1}(B)$

(ix) $A \subseteq f^{-1}(f(A))$

(x) $B \supseteq f(f^{-1}(B))$

(xi) $\alpha_{[f(A)]} \supseteq \bigcup f(\alpha_A) \; where \; \alpha \in I$

(xii) $\alpha^{+}_{[f(A)]} = f(\alpha^{+}_{A}) \; where \; \alpha \in I$

(xiii) $f(A) = \bigcup f(\alpha^{+}_{A}) \; where \; \alpha \in I$

When a given function is defined on a Cartesian product as $X = X_1 \times X_2 \times X_3 \times ....\times X_n$ the extension principle is also applicable. In this case, $x \in X$ is replaced by $x = (x_1, x_2, x_3, ...x_n) \; where \; x_i \in X_i \; and \; i \in \nabla$

8. Conclusion

It follows that operations on fuzzy sets are nice generalization of the corresponding crisp operations.

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