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A brief description of wavelet and wavelet transforms and their applications

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Abstract
In this paper we highlight the relevant definitions of wavelet, wavelet transform, parseval formula for the wavelet transform, inversion formula, admissibility condition of wavelets, some examples of wavelets, basic properties of continuous wavelet transforms, applications of wavelet transforms. By using Fourier transform we expressed continuous wavelet transform $(W_\emptyset h)(b,a)$ in the form $(W_\emptyset^{+}h)(b,a)$ and $(W_\emptyset^{-} h)(b,a)$.

Keywords: Wavelet, wavelet transform, continuous wavelet transform (CWT), Fourier Transform (FT), Short Term Fourier Transform (STFT)

1. Introduction
With ever huge demand for an exciting new mathematical tools to introduced both the theory and application for science and engineering the development of wavelet and wavelet transform seems more clearly and effectively established. Keeping in mind, the utility and interest of wavelet analysis in different disciplines, the idea of wavelet can be introduced by several separate chain of thought. As most of the authors developed their own way of defining wavelet and wavelet family. In this paper, we high light some few concepts of different authors who worked in the field of wavelet analysis. In 1942, Haar's work in the field of wavelet and later named as Haar's wavelet. In 1946, Dennis Gabor were introduced Gabor atom which are familiar to wavelet family and used in same purposes. In 1975, Zweg's give the idea of continuous wavelet transform. In 1975, Picree Goupilland, Grossman and Morlet work together to developed the idea of continuous wavelet transform. In 2015, Ashish Pathak, Prabhat Yadav and M.M. Dixit derived an Asymptotic Expansion of Wavelet transform [5]. In 2015, Ashish Pathak, Prabhat Yadav and M.M. Dixit proposed On Convolution for General Novel Fractional Wavelet Transform [6]. Ashish Pathak, Prabhat Yadav and M.M.Dixit also derived An Asymptotic Expansion of Continuous Wavelet Transform for Large Dilation Parameter [7]. In 2016, Ashish Pathak, Prabhat Yadav and M.M. Dixit derived An Asymptotic Expansion of Continuous Wavelet Transform for Large Values of b [8] and soon.

2. Wavelet
The word "wavelet" stands for small wave or part of wavelet component. The smallness refers to the condition that this (window) function is of finite length (compactly support). The wave refers to the condition that this function is oscillatory [1]. Wavelet have introduced for representation of function in a more efficient manner than Fourier series. Historically the concept of 'wavelet' [3] originated from the study of time frequency signal analysis, wave propagation and sampling theory. Wavelet are generated from one single function called "Mother wavelet" by translation and dilation. Wavelets have found much application in data compression, Noise removal, pattern recognition, fast computation etc. Wavelet theory provides a unified frame work and coherent theory for a number of different ideas and techniques that have been independently developed in several fields. In 1982, Jean Morlet in collaborations with a group of French engineers first introduced the idea of wavelet as a family of functions constructed by using translations and dilations of a single function called Mother wavelet for the analysis of non-stationary signals.

However this new concept can be viewed as the synthesis of various ideas originating from different disciplines including mathematics, physics and engineering. So, wavelet analysis is about analyzing the signal with short duration finite energy function. Wavelet analysis is an exciting new method for solving difficult problems in mathematics, physics and engineering with modern applications as diverse as wave propagation, data compression, image processing, pattern recognition, computer graphics, the detection of aircraft and submarine and improvement in CAT scans and other medical image technique. Wavelet allow complex information such as music, speech, image and pattern to be decomposed into elementary form called the fundamental building blocks at different positions and scale and subsequently reconstructed with high precision [3]. A wavelet is a mathematical function used to divide a given function or continuous time signal into different scale components. Usually one can assign a frequency range to each scale component. Each scale component can then be studied with a resolution that matches its scale. A wavelet whose admissible constant satisfy $0 < C_{\emptyset} < \infty$ is called admissible wavelet. An admissible wavelet implies that $\widehat{\emptyset}\,(0) = 0$, so that an admissible wavelet must integrate to zero. To recover the original signal, the second inverse continuous wavelet transform can be exploited. A wavelet is a wave like oscillation with amplitude that begins at zero, increases and then decreases back to zero. It can typically be visualized as a “brief oscillation” like one might see recorded by a seismograph or heart monitor. Generally wavelets are purposefully crafted to have specific properties that make them useful for signal processing. Wavelets can be combined using a “reverse, shift, multiply and integrate” technique called convolution with portions of a known signal to extract information from the unknown signal.

Definition (Wavelet): A family of functions constructed from translations and dilations of a single function $\emptyset$, called the mother wavelet, we define wavelets mathematically by [3].

$$\emptyset_{a,b}\,(\mathrm{t}) = a^{\frac{-1}{2}}\,\emptyset\left(\frac{t-b}{a}\right),\ a, b \in R, a \neq 0, \tag{1}$$

where a is known as scaling parameter which measures the degree of compression or scale and b is a translation parameter which determines the time location of the wavelet. The factor $\frac{1}{\sqrt{a}}$ ensure that the daughter wavelets have the equivalent energy as the mother wavelet, irrespective of the scale used and thus merely normalized the energy. If $a > 1$, there is a stretching of $\emptyset(\mathrm{t})$ along time axis. If $a < 1$, there is a contraction or compression of $\emptyset(\mathrm{t})$ along time axis. The term “Mother” implies that the functions with the different region of support that are used in the transformation process are derived from one main function or the Mother wavelet. In other words, the mother wavelet is merely a prototype for generating the other window function or wavelets termed as daughter wavelets. Daughter wavelet is obtained by scaling and translating the mother wavelet. Example: Mexican Hat wavelet, Morlet wavelet etc. We manipulate wavelets in two ways: i) Translation and (ii) Dilation.

Translation

Translation means shifting of the wavelet from one position to another position. Here we change the central position of the wavelet along the time axis. Translation represents the location of the wavelet in relatives to the original signal. Translation measure the distance of the wavelet that has been move along the signal. It is denoted by ‘b’.

Dilation

Dilation is known as scaling parameter which measures the degree of compression or scale. It is denoted by the term “a”. The width of the mother wavelet are changed by a constant ‘a’ known as Scale factor. The scale factor is inverse of frequency. i.e, a$\approx \frac{1}{f}$. If scale increase → Frequency decrease → Mother wavelet is stretched. If scale decrease → Frequency increase → Mother wavelet is compressed. The calculation of the continuous wavelet transform usually start from the scale value a $= 1$, which is then increment in the integers value. The range of ‘a’ depends on the signal being transformed.

3. Wavelet Transform

Wavelet transform is the representation of a function by wavelet. It provides the time -frequency representation. (There are other transforms which give this information too, such as short Term Fourier Transform (STFT), Wigner distribution etc). Wavelet transform is capable of providing the time and frequency information simultaneously, hence giving a time- frequency representation of the signal [1]. The Wavelet transform was developed as an alternative to the Short Term Fourier Transform. The wavelet transform was developed to overcome some resolution related problems of the Short Term Fourier Transform (STFT). The frequency and time information of a signal at some certain point in the time- frequency plane cannot be known. We cannot know what spectral component exists at any given time instant. The best we can do is to investigate what spectral components exist at any given interval of time. This is a problem of resolution, and it is the main reason why researcher has switched to wavelet transform from short Term Fourier Transform (STFT). Short Term Fourier Transform (STFT) gives a fixed resolution at all times, where as wavelet transform gives a variable resolution. We basically need wavelet transform to analyze non-stationary signals i.e., whose frequency response varies in time. Fourier transform is not suitable for non-stationary signal. Fourier transform is not suitable if the signal has time varying frequency, i.e. the signal is non stationary. If only, the signal has the frequency component at all times, then the result obtained by the Fourier transform make sense. The Fourier transform tells whether a certain frequency component exists or not. This information is independent of where in time this component appears. It is therefore very important to known whether a signal is stationary or not, prior to processing it with the Fourier transform. Short Term Fourier Transform (STFT) is a revised version of the Fourier transform. There is only a minor difference between Short Term Fourier Transform and Fourier Transform. In Short Term Fourier Transform, the signal is divided into small enough segments (portions), where these segments of the signal can be assumed to be stationary. The Short Term Fourier Transform of the signal is nothing but the Fourier Transform of the signal multiplied by a window function. The main difference in general is that wavelet is localized in both time and frequency whereas short Fourier transform is only localized in frequency. The short Fourier transform is similar to the wavelet transform in that, it is also time and frequency localized, but these are issued with the frequency /time resolution trade off. Fourier transform does not give any information about the occurrence of the frequency component at a particular time and is not applicable for non-stationary signals. As Morlet transform the signal under investigation into another representation, which presents the signals in a more useful form. This translation of the signals is called Wavelet transform, unlike Fourier transform. We have a variety of wavelets that are used for signal analysis. So choice of particular wavelet for study signal analysis will depends upon the type of the application in hand. Basically wavelet transform show the local matching of the wavelet with the signal. If the wavelet matches the shape of the signals well at a specific scale and locations, then a large transform value is obtained. If the wavelets and signal can correlated or vary well then large transform value is obtained. If the wavelets and signal do not correlate well, a low value of transform is obtained. The transform is calculated or computed at various location of the signals and for various scale of the wavelets, thus for filling up the transform plane. Wavelet transform have advantage over traditional Fourier transform for representing function that have discontinuous and sharp peaks and for accurately deconstructing and reconstructing finite, non-periodic and or non-stationary signals. Wavelet transform are classified into Discrete Wavelet transform (DWT) and Continuous Wavelet Transform (CWT). As both Discrete wavelet transform and continuous wavelet transform are continuous time (analog) transform. They can be used to represent continuous-time (analog) signals. Continuous wavelet transforms operate over every possible scale and translations, whereas discrete wavelet transforms used a specific subset of scale and translation values or representations. There are a large numbers of wavelet transform each suitable for different application. In numerical analysis, continuous wavelets are functions used by the continuous wavelet transform. These function are defined as analytical expression as function either of time or of frequency. Most of the continuous wavelet are used for both wavelet decomposition and composition transform. That is, they are continuous counterpart of orthogonal wavelets. The following continuous wavelets have been invented for various applications: (i) Poission Wavelet (ii) Morlet wavelet (iii) Modified Morlet wavelet (iv) Mexican-Hat wavelet. Continuous wavelet transforms is very efficient in determining the damping ratio of oscillating signals. For example: Identification of damping in dynamical system.

4. The Continuous Wavelet Transform.

The continuous wavelet transform of a function $h \in L^2(R)$ with respect to the wavelet $\emptyset \in L^2(R)$ is defined by [3]

$$(W_\emptyset\, h)\,(b,a) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} h(t)\, \overline{\emptyset_{a,b}(t)}\, dt,\ b \in R, a > 0, \tag{2}$$

where $\emptyset_{a,b}\,(t)$ satisfy (1) and provided the integral exists [3]. Using Fourier transform it can also be expressed as [7]

$$(W_\emptyset\, h)\,(b,a) = \frac{\sqrt{a}}{2\pi} \int_{-\infty}^{\infty} e^{ibw}\, \hat{h}\,(w)\, \overline{\hat{\emptyset}}\,(aw)\, dw, \tag{3}$$

where $\hat{h}(w) = \int_{-\infty}^{\infty} e^{-\,itw}\ h(t)\, dt.$ (4)

Let us rewrite (3) in the form:

$$(W_\emptyset\, h)\,(b,a) = \frac{\sqrt{a}}{2\pi} \left\{\int_0^{\infty} e^{ibw}\ \hat{h}\,(w)\, \overline{\hat{\emptyset}}\,(aw)\, dw + \int_0^{\infty} e^{\,-ibw}\ \hat{h}\,(-w)\, \overline{\hat{\emptyset}}\,(-aw)\, dw\right\}, \tag{5}$$

$$(W_\emptyset\, h)\,(b,a) \quad = (W_\emptyset{}^{+} h)(\mathrm{b,a}) + (W_\emptyset{}^{-} h)\,(\mathrm{b,a}), \tag{6}$$

where

$$(W_\emptyset{}^{+} h)(\mathrm{b,\ a}) = \frac{\sqrt{a}}{2\pi} \left\{\int_0^{\infty} e^{ibw}\ \hat{h}\,(w)\, \overline{\hat{\emptyset}}\,(aw)\, dw \right\} \tag{7}$$

and $(W_\emptyset{}^{-}\ h)\,(\mathrm{b,a}) = \frac{\sqrt{a}}{2\pi} \left\{ \int_0^{\infty} e^{\,-ibw}\ \hat{h}\,(-w)\, \overline{\hat{\emptyset}}\,(-aw)\, dw\right\}$ (8)

5. Theorem (Parseval Formula for the Wavelet Transform)

If $\emptyset \in \mathrm{L}^2(\mathrm{R})$ and $W_\emptyset h(b,a)$ is the wavelet transform of $\mathrm{h} \in \mathrm{L}^2(\mathrm{R})$ and defined by (2), then for any function $h, g \in \mathrm{L}^2(\mathrm{R})$, we obtain [3]

$$\iint_{-\infty}^{\infty} W_\emptyset h(b,a)\, \overline{W_\emptyset g(b,a)}\, \frac{\mathrm{db\ da}}{\mathrm{a}^2} = C_\emptyset\,(h,g), \tag{9}$$

where $0 < C_\emptyset = \int_{-\infty}^{\infty} \frac{|\hat{\emptyset}(w)|^2}{|w|}\, dw < \infty.$

6. Inversion Formula

For analyzing functions (signals), we often need to transform the functions from time domain to frequency domain and reconstruct them from frequency domain to time domain. Therefore, we need a formula for inverse wavelet transform, as we have to recover the original signal h(t), by using inverse continuous wavelet transform.

Theorem: If $\mathrm{h} \in \mathrm{L}^2(\mathrm{R})$, then h can be reconstructed by the formula [3]

$$\mathrm{h(t)} = \frac{1}{\mathrm{C}_\emptyset} \iint_{-\infty}^{\infty} W_\emptyset h(b,a)\, \emptyset_{b,a}\,(t)\ \frac{db\, da}{a^2}, \tag{10}$$

where the equality holds almost everywhere.

7. Admissibility Condition

A wavelet is a function $\emptyset \in L^2(R)$ which satisfies the condition [3]

$$C_\emptyset = \int_{-\infty}^{\infty} \frac{|\widehat{\emptyset}(w)|^2}{|w|}\, dw < \infty, \tag{11}$$

where $\widehat{\emptyset}(w)$ is the Fourier transform of $\emptyset(t)$. Since $\widehat{\emptyset}(w)$ is continuous, $C_\emptyset$ is finite only if $\widehat{\emptyset}(w) = 0$. Equivalently $\int_{-\infty}^{\infty} \emptyset(t)\, dt = 0$. This means that $\emptyset$ must be an oscillatory function with zero mean. Also $\emptyset(t)$ will decay to 0 as $t \to \pm\infty$.

8. Example: (The Haar wavelet)

The Haar wavelet (1910) is one of the classic examples. It is defined by [3]

$$\emptyset(t) = \begin{Bmatrix} 1, 0 \le t < \frac{1}{2} \\ -1, \frac{1}{2} \le t < 1 \\ 0, otherwise \end{Bmatrix}. \tag{12}$$

The Haar wavelet has compact support. It is obvious that $\int_{-\infty}^{\infty} \emptyset\,(t) dt = 0$, and $\int_{-\infty}^{\infty} |\emptyset(t)|^2\, dt = 1$. This wavelet is very well localized in the time domain but poor frequency localization and it is not continuous. So its Fourier transforms $\widehat{\emptyset}(w)$ is computed as [3]

$$\widehat{\emptyset}(w) = i\, e^{\frac{-iw}{2}} \frac{sin^2\left(\frac{w}{4}\right)}{\left(\frac{w}{4}\right)} \tag{13}$$

and $\int_{-\infty}^{\infty} \frac{|\widehat{\emptyset}(w)|^2}{|w|}\, dw = 16 \int_{-\infty}^{\infty} |w|^{-3} |\sin\left(\frac{w}{4}\right)|^4\, dw < \infty.$ (14)

Hence, the Haar wavelet is one of the most fundamental example that illustrates major features of the general wavelet theory.

The Mexican Hat Wavelet

The Mexican hat wavelet is defined by the second derivative of a Gaussian function as [3] $\emptyset(t) = (1 - t^2)\, \frac{e^{-t^2}}{2} = -\frac{d}{d\,t^2}\left(\frac{e^{-t^2}}{2}\right)$. So its Fourier transform is given as $\widehat{\emptyset}(w) = \sqrt{2\pi}\, w^2\, \frac{e^{-w^2}}{2}$.
As Mexican Hat wavelet is a good localization in both time and frequency domain and hence satisfies the admissibility condition.

The Morlet Wavelet

The Morlet wavelet is defined as [3] $\emptyset(t) = e^{iw_o\, t - \frac{t^2}{2}}$. So its Fourier transform is as
$\widehat{\emptyset}(w) = \sqrt{2\pi}\, e^{\frac{-(w-w_o)^2}{2}}$.

The Shannon Wavelet

The Shannon function $\emptyset$ whose Fourier transform satisfies [3] $\widehat{\emptyset}(w) = \hat{\chi}_I(w)$,
where I = $[-2\pi, -\pi] \cup [2\pi, \pi]$, is called the Shannon wavelet.
The Shannon wavelet $\emptyset(t)$ is directly computed from the inverse Fourier transforms of $\widehat{\emptyset}(w)$ as

$$\emptyset(t) = \frac{1}{2} \int_{-\infty}^{\infty} e^{iwt}\, \widehat{\emptyset}(w)\, dw$$
$$= \frac{1}{2}\left[\int_{-2\pi}^{-\pi} e^{iwt}\, \widehat{\emptyset}(w)\, dw + \int_{\pi}^{2\pi} e^{iwt}\, \widehat{\emptyset}(w)\, dw\right]$$
$$= \frac{1}{\pi t}(\sin 2\pi t - \sin \pi t)$$
$$= \frac{\mathrm{Sin}\left(\frac{\pi t}{2}\right)}{\left(\frac{\pi t}{2}\right)} \cos\left(\frac{3\pi t}{2}\right).$$

9. Basics Properties of Continuous Wavelet Transform

The following basics properties of the continuous wavelet transform are well known.
If $\emptyset$ and Ψ are wavelets and h, g are functions which belong to $L^2(R)$, then [3]

(i). Linearity: $W_\emptyset(\alpha h + \beta g) = \alpha(W_\emptyset h,) + \beta(W_\emptyset g\,)$,
where α and β are any two scalars.

(ii). Translation: $\left(W_\emptyset\,(T_c\, h\,)\right)(a, b) = (\,W_\emptyset h)\;(a, b - c)$,
where, T_c is the translation operator defined by $T_c\, h = h(t - c)$.

(iii). Dilation: $\left(W_{\emptyset}\,(D_c\,h\,)\right)(a,b) = \frac{1}{\sqrt{c}}\,(W_{\emptyset}\,h)\left(\frac{a}{c},\frac{b}{c}\right)$, $c > 0$,
where D_c is a dilation operator defined by $D_c\,h(t) = \frac{1}{c}\,h\left(\frac{t}{c}\right)$, $c > 0$,

(iv). Symmetry: $(W_{\emptyset}\,h)(a,b) = \overline{(W_{h\emptyset})\left(\frac{1}{a},-\frac{b}{a}\right)}$, $a \neq 0$.

(v). Parity: $\left(W_{p\emptyset}\,Ph\right)(a,b) = (W_{\emptyset}h)(a,-b)$,
where P is the parity operator defined by $Ph(t) = h(-t)$.

(vi). Anti Linearity: $\left(W_{(\alpha\emptyset\,+\beta\Psi)}\,Ph\right)(a,b) = \bar{\alpha}\,(W_{\emptyset}h)\,(a,b) + \bar{\beta}\,(W_{\Psi}\,h)(a,b)$.

(vii). $\left(W_{T_c\emptyset}\,h\right)(a,b) = (W_{\emptyset}h)(a,b+ca)$.

(viii). $\left(W_{D_c\emptyset}\,h\right)(a,b) = \frac{1}{\sqrt{c}}\,(W_{\emptyset}\,h)(ac,b)$, $c > 0$.

10. Applications of Wavelet Transforms

One of the most popular applications of wavelet transform is image compression or data compression. The advantage of using wavelet-based coding in image compressions is that it provides significant improvements in picture quality at higher compression ratios over conventional techniques. Since wavelet transform has the ability to decompose complex information and patterns into elementary forms, it is commonly used in acoustics processing and pattern recognition. Moreover wavelet transform can be applied to the following scientific research area: edges and corners detection, filter design, partial differential equation solving, transient detection, electro cardiogram (ECG) analysis, texture analysis, business information analysis, structure of galaxies in the universe, digital communication, approximation theory etc. In recent years, there have been many developments and new applications of wavelet analysis for describing complex algebraic functions and analyzing empirical continuous data obtained from many kinds of signals at different scales of resolution [3].

11. Conclusion

One of the main reasons for the discovery of wavelets and wavelet transform is that Fourier transform analysis does not contain the local information of signals [3]. The main difference in general is that wavelets are localized in both time and frequency whereas Standard Fourier Transform is only localized in frequency. So, the Fourier transform analysis cannot be used for analyzing signals in a joint time and frequency domain. The Continuous Wavelet Transform developed or incorporate multiresolution technique, to overcome the problems occurs in Short Term Fourier Transform. The Continuous Wavelet Transform (CWT) uses a technique called Multi-Resolution Analysis (MRA). Multi-Resolution Analysis varies the window size depending on the frequency of that portions of the signals. Thus the high frequency of a signal can be analyzed by using narrow windows. While the low frequency components can be analyzed by using wide windows. By doing so, Continuous wavelet transform find out the analysis of a signal in a optimum resolution. The process of Multi-Resolution Analysis (MRA) and Continuous Wavelet Transform (CWT) is mathematically defined by (6). The Mathematical term $\emptyset(\mathrm{t})$ is termed as the Mother wavelet and is depend on scale or dilation or compression 'a' and translation 'b'. The term $\overline{\emptyset\left(\frac{\mathrm{t-b}}{\mathrm{a}}\right)} = \overline{\emptyset(t)}$ denotes the complex conjugate of the wavelets.

Just as the Fouier transform is reversible resulting in the original signal. Similar cases happen in the Continuous Wavelet Transform (CWT) provided that the admissibility condition is met. The admissibility condition states that the Continuous Wavelet Transform is reversible, if and only if $C_{\emptyset} = \int_{-\infty}^{\infty} \frac{|\,\widehat{\emptyset}(w)\,|^2}{|w|}\,dw < \infty$, where $C_{\emptyset}$ is termed as the admissiblity constant and $\widehat{\emptyset}(w)$ is the Fourier transform of the wavelet function $\emptyset(t)$. If the above condition is satisfied, then reconstruction and analysis of the original signal is possible without loss of information. The original signal can be easily be computed by using the reconstructed formula (10).

As $C_{\emptyset} = \int_{-\infty}^{\infty} \frac{|\,\widehat{\emptyset}(w)\,|^2}{|w|}\,dw < \infty$, if $\emptyset$ is real. If the wavelet $\emptyset$ is complex, it should belong to the Hilbert space, i.e, $\widehat{\emptyset}(w) = 0$, for $w < 0$ and $C_{\emptyset}$ should be replaced by $\frac{1}{2}\,C_{\emptyset}$.

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