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A tri-neuron network model and bifurcation phenomenon with respect to synaptic weight and time delay

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Abstract
We consider a tri-neuron network model and calculate the condition of steady state stability of the equilibrium points analytically. We have also focused on investigating the conditions for single Hopf-Bifurcation with respect to synaptic weight and also the conditions for resonant co-dimension two bifurcations with respect to time delay.

Keywords: Tri-neuron network model, time delay, synaptic weight, Double Hopf-bifurcation

Introduction

It is well known that the human brain is made up of large number of cells called neurons and their interaction. An artificial neural network is an information processing system that has certain characteristic in common with biological neural network. The neural network model occupies a unique and fascinating position in the development of human memory, which was proposed in 1980 by Hopfield, constructing a simplified model with electronic circuit implementation to store certain memories or patterns in a manner rather similar to our brain. Recently E. Bas *et al.* [2] consider the two most commonly used types of artificial neural networks are the multilayer feed-forward and multiplicative neuron model ANNs. In this paper they to evaluate the performance the proposed method to the well known real world time series datasets and also a simulation study is performed. The algorithm has superior performance both when it is applied to real world time series datasets and the simulation study when compared with other ANNs reported in the literature. Another of its advantages is that, for datasets with outliers, the results are very close to the results obtained from the original datasets. They also demonstrate that the algorithm is unaffected by outliers and has a robust structure. F. Mesiti *et al.* [3] shown that the intracellular calcium dynamics of atrocities is a critical process with significant impact on the regulation of the neural activity. In a tripartite synapse the atrocity of the cell actively interacts with the neural response of adjacent pre-synaptic neurons providing a feedback which is believed to regulate several brain processes. They investigated the special link between neuron and glia-cells from the mathematical and electronic point of view, providing an alternative representation of the glia-neuron system. Main gliotransmitters are glutamate ATP, D-serine and adenosine S. Paixao *et al.* [4] which have direct impact on adjacent neurons equipped with compatible receptors located on the cell membrane. Glutamate is one of the most important abundant neurotransmitter in the brain and neurons have several types of glutamate sensitive receptors such as AMPAR and NMDAR and mGluR metabotropic glutamate receptors. In [5] the authors described the communication between neurons in terms of input or output blocks with the purpose of nano machine neuron communications. A more theoretical analysis of multiple inputs single out put synaptic communications has been described in [6], where as a block description of the neuronal process accounting for the stochastic release of neurotransmitters has been proposed in [7] with an extension to the tripartite synapse proposed in [8]. Successful attempts in using intercellular calcium signaling to establish a communication channel for neuronal applications have been reported by Nakano *et al.* in [9]. To our knowledge no other existing investigations in the engineering of neuronal communication systems have consider the active presence of the atrocyte.

Other authors [1, 12] have found that crushed brick aggregate concrete has a relatively lower strength at early age than normal aggregate concrete. They attributed this characteristic to the higher water absorption of crushed brick aggregate compared to gravel, which was used as the control aggregate. The compressive strength is lower for crushed brick concrete; the higher the rate of substitution normal aggregate with brick, the lower the compressive strength. The use of bricks as concrete aggregate caused a 40% reduction in compressive strength [13, 14]. Concrete produced with these aggregates did not perform as well as concrete produced with natural aggregates in terms of strength. Strength properties of specimens with crushed brick aggregate are lower compared to the same properties for the reference mixtures with natural aggregate. Results of experimental investigations in [15] revealed lower compressive strength of concrete with recycled brick aggregate. These results are expected due to the lower grain hardness of a crushed brick/tile aggregate compared to a river aggregate

Moreover, our approach can provide further tools for design and implementation of biomimetic devices mimicking the behavior of neurons and atrocyte at micro and nano scale as well as for the implementation of innovative biomedical tools for the stimulation of neurons at nanoscale, as envisioned in [10, 11]. In modeling ANN it is necessary to incorporate the processing time of each neuron to make the model more realistic. In the past two decades the analysis of stability of steady states for different classes of neural networks with or without time delays such as additive neural networks, cellular neural networks, bi-directional neural networks has been extensively investigated. In some artificial neural network information is store as stable equilibrium points of the system. Retrieval occurs when the system is initialized within the basin of attraction of one of the equilibrium and the network is allowed to stabilize in its steady state. Delay may render such works more versatile. Nevertheless uncontrolled delays may degrade network performance by rending the equilibrium unstable and makes the retrieval of the corresponding information impossible. Thus delay is an important parameter in neural system.

1. Mathematical formulation of the model

Here we consider a three unit network model by the following system of delay differential equations in a parameter space consisting of internal decay rate, self connection strength, connection strength of one neuron to other and delay;

$$\frac{dx_1}{dt} = -kx_1(t) + \beta \tanh[x_1(t-\tau)] + a \tanh[x_2(t-\tau)] + a \tanh[x_3(t-\tau)]$$
$$\frac{dx_2}{dt} = -kx_2(t) + a \tanh[x_1(t-\tau)] + \beta \tanh[x_2(t-\tau)] + a \tanh[x_3(t-\tau)]$$
$$\frac{dx_3}{dt} = -kx_3(t) + a \tanh[x_1(t-\tau)] + a \tanh[x_2(t-\tau)] + \beta \tanh[x_3(t-\tau)]$$

………….……… (2)

Where $x_i(t)$ is the activation state of neuron i(i=1, 2, 3) at time t, k>0 is the decay rate of neurons, $\beta \neq 0$ is self synaptic weight, a is the weight of synaptic connections from one neuron to another, $\tau \geq 0$ corresponds to the transmission delay along the axon.

2. Schematic representation of three neurons model:

The fundamental questions we would like to answer in order to understand the behavior of a model with time delays are the following:

i. What equilibrium solutions occur in the system?
ii. Are these stable or unstable?
iii. Whether periodic solutions occur in the system?
iv. How do the answers to these questions change as parameter s like synaptic weight, delay parameter is varied?
v. Whether the theoretical results can be validated by numerical example?

3. Steady state and stability analysis

Obviously (0, 0, 0) is the trivial steady state of the system (2). Linear zing (2) about (0, 0, 0) it gives

$$\frac{dy_1}{dt} = -ky_1(t) + \beta[y_1(t-\tau)] + a[y_2(t-\tau)] + a[y_3(t-\tau)]$$
$$\frac{dy_2}{dt} = -ky_1(t) + a[y_1(t-\tau)] + \beta[y_2(t-\tau)] + a[y_3(t-\tau)]$$
$$\frac{dy_3}{dt} = -ky_3(t) + a[y_1(t-\tau)] + a[y_2(t-\tau)] + \beta[y_3(t-\tau)]$$

……………….. (3)

The characteristic equation of (3) is

$$\begin{vmatrix} -k-\lambda+\beta e^{-\lambda\tau} & ae^{-\lambda\tau} & ae^{-\lambda\tau} \\ ae^{-\lambda\tau} & -k-\lambda+\beta e^{-\lambda\tau} & ae^{-\lambda\tau} \\ ae^{-\lambda\tau} & ae^{-\lambda\tau} & -k-\lambda+\beta e^{-\lambda\tau} \end{vmatrix} = 0$$

$$[-k-\lambda+\beta e^{-\lambda\tau}][(-k-\lambda+\beta e^{-\lambda\tau})(-k-\lambda+\beta e^{-\lambda\tau}) - a^2e^{-2\lambda\tau} - ae^{-\lambda\tau}\{ae^{-\lambda\tau}(-k-\lambda+\beta e^{-\lambda\tau}) - a^2e^{-2\lambda\tau}\} + ae^{-\lambda\tau}\{a^2e^{-2\lambda\tau} - ae^{-\lambda\tau}(-k-\lambda+\beta e^{-\lambda\tau})\}] = 0$$

$$(-k-\lambda+\beta e^{-\lambda\tau})^3 - 3a^2e^{-2\lambda\tau}(-k-\lambda+\beta e^{-\lambda\tau}) + 2a^3e^{-3\lambda\tau} = 0$$

$$\Rightarrow [\lambda+k-(\beta+2a)e^{-\lambda\tau}][\lambda+k-(\beta-a)e^{-\lambda\tau}]^2 = 0$$

……………….. (4)

Let $\Delta_1(\lambda) = \lambda + k - (\beta+2a)e^{-\lambda\tau}$ and $\Delta_2(\lambda) = [\lambda + k - (\beta - a)e^{-\lambda\tau}]$. We get the following theorem,

Theorem 1

(i) If $k > (|\beta| + 2|a|)$ then trivial equilibrium (0, 0, 0) of (3) is locally asymptotically stable for all $\tau \geq 0$.
(ii) If $k > (\beta + 2a)$ then trivial equilibrium (0, 0, 0)of (3) is unstable for all $\tau \geq 0$.

Now we are interested to analyze the bifurcation phenomenon considering synaptic weight a between two neurons as bifurcating parameter. As the parameters are varied the stability may be lost if a real root of the characteristic equation (4) passes through the origin or by a pair of complex conjugate roots passing through imaginary axis. The former occurs if $\lambda = 0$ in equation (4) implying $k = (\beta + a)$or $k = (\beta + 2a)$, which can lead to the static bifurcation of the equilibrium points change as the bifurcation parameter very. The later deals with the Hopf-bifurcation such that the dynamical behavior of the system changes from a static stable to a periodic motion or vice-versa.

Let us assume $k \neq (\beta + a)$ or$k \neq (\beta + 2a)$. In this section we are interested in Hopf-bifurcation of trivial solution considering the synaptic strength an as bifurcating parameter. Let at a=a_0 characteristic equation (4) has a simple pair of purely imaginary root$\pm i\omega_0$, that is $\Delta_i(i\omega_0)$=0. This yields the following equations: $k - (\beta + 2a_0)cos\omega_0\tau$=0 and $\omega_0 + (\beta + 2a_0)sin\omega_0\tau$=0. Substituting the values of $cos\omega_0\tau$ and $sin\omega_0\tau$ obtaining above in $\grave{\Delta_1}(i\omega_0)$ and $\grave{\Delta_2}(i\omega_0)$ it can be shown that at a=a_0, $i\omega_0$ is a simple root of the characteristic equation. Now differentiating both side of the characteristic equation with respect to a and substituting λ= $i\omega_0$ for $a = a_0$ it is obtained as

Re($\frac{d\lambda}{da}$ $at\ a = a_0$)= $\frac{2k+2r(k^2+\omega_0)}{(\beta+2a_0)[(1+kr)^2+{\omega_0}^2]} \neq 0$.Therefore we have the following theorem:

Theorem 2

For given k, τ, β if at $a = a_0$ all the rots of the characteristic (4) have negative real parts except the pair of purely imaginary ones $\pm i\omega_0$ then the system (3) undergoes Hopf-bifurcation with respect to synaptic weight at $a = a_0$ where a_0 lies on the surface: $k - (\beta + 2a_0)cos\omega_0\tau$=0 and $\omega_0 + (\beta + 2a_0)sin\omega_0\tau$=0, $a_0 \neq -\frac{\beta}{2}$.

It is possible to show that the presence of points (bifurcating parameter) at which the characteristic equation has two pairs of purely imaginary roots $\pm i\omega_2, \pm i\omega_1$. As such points are commonly occurring where curves of Hopf-bifurcation cross, these points are referred to as points of resonant double Hopf-bifurcation. If $\omega_1{:}\,\omega_2$=$k_1{:}\,k_2$ then a possible double Hopf-bifurcation point appears with frequencies in the ratio $k_1{:}\,k_2$. If $k_1, k_2 \in Z^+$ and $k_1 \leq k_2$ then such a point is called $k_1{:}\,k_2$ resonant double Hopf-bifurcation point. To study the resonant double Hopf-bifurcation let us consider a special three neurons ring network with discrete delay describe by the following system of equations:

$$\frac{dx_1}{dt} = -kx_1(t) + \beta \tanh[x_1(t-\tau)] + a \tanh[x_2(t-\tau)] + a\ \tanh[x_3(t-\tau)]$$
$$\frac{dx_2}{dt} = -kx_2(t) + a \tanh[x_1(t-\tau)] + \beta \tanh[x_2(t-\tau)] + a \tanh[x_3(t-\tau)]$$
$$\frac{dx_3}{dt} = -kx_3(t) + a \tanh[x_1(t-\tau)] + a \tanh\ [x_2(t-\tau)] + \beta \tanh[x_3(t-\tau)]$$

…………………. (5)

Obviously (0, 0, 0) is the trivial steady state of the above equation. Linear zing above equation about (0, 0, 0) it gives

$$\frac{dy_1}{dt} = -ky_1(t) + \beta[y_1(t-\tau)] + a[y_2(t-\tau)] + a[y_3(t-\tau)]$$
$$\frac{dy_1}{dt} = -ky_1(t) + a[y_1(t-\tau)] + \beta[y_2(t-\tau)] + a[y_3(t-\tau)]$$
$$\frac{dy_1}{dt} = -ky_3(t) + a[y_1(t-\tau)] + a[y_2(t-\tau)] + \beta[y_3(t-\tau)]$$

……………….. (6)

The corresponding characteristic equation is

$(-k - \lambda + \beta e^{-\lambda\tau})^3 - a^2 e^{-2\lambda\tau}(-k - \lambda + \beta e^{-\lambda\tau}) = 0, \Rightarrow A_1(\lambda)A_2(\lambda)A_3(\lambda) = 0$ where
$A_1(\lambda)$=$(-k - \lambda + \beta e^{-\lambda\tau})$
$A_2(\lambda)$=$(-k - \lambda + (\beta + a)e^{-\lambda\tau})$
$A_3(\lambda)$=$(-k - \lambda + (\beta - a)e^{-\lambda\tau})$.
Therefore either $(-k - \lambda + \beta e^{-\lambda\tau}) = 0$ or $(-k - \lambda + (\beta \pm a)e^{-\lambda\tau}) = 0$.
Now we consider the following two cases:

Case 1

Let $(-k - \lambda + \beta e^{-\lambda\tau}) = 0$. For $\tau = 0$ the root of the above equation is given by $\lambda = \beta - k$ which is negative if $\beta < k$. Now let us find out when $A_1(\lambda)$=0 has a pair of purely imaginary root $\pm i\omega_s$. Now separating real and imaginary parts of the equation $A_1(\lambda)$=0 we get $k - \beta cos\omega_s\tau$=0 and $\omega_s + \beta sin\omega_s\tau$=0. Which implies $\omega_s^2 = \beta^2 - k^2$. Therefore the equation ($-k - \lambda + \beta e^{-\lambda\tau}) = 0$ has imaginary root $\pm i\omega_s$ if and only if $|\beta| > k$. Substituting the values of $cos\omega_s\tau$ and $sin\omega_s\tau$ obtained above in $A_1(i\omega_s)$ and $A_2(i\omega_s)$ it can be shown that $i\omega_s$ is a simple root of the characteristic equation. This root $i\omega_s$ occurs when $\tau = \tau_{n,s} = \frac{1}{\omega_s} arc \tan\left(-\frac{\omega_s}{k}\right) + \frac{n\pi}{\omega_s}$; n=0, 1, 2, 3, 4,…………..

Differentiating both sides of the characteristic equation with respect to τ and substituting $\lambda = i\omega_s$ for $\tau = \tau_{n,s}$ we get,
Re($\left|\frac{d\lambda}{d\tau}\right|$ $at\ \tau = \tau_{n,s}$)= $\frac{\omega_s^2}{(1+k\tau_{n,s})^2}$>0.

Case 2

Let $\left(-k - \lambda + (\beta \pm a)e^{-\lambda\tau}\right) = 0$. For $\tau = 0$ we have $\lambda = (\beta \pm a) - k$ which is negative if max$\{|\beta + a|, |\beta - a|\}$<k. Now let us find the equation has a pair of purely imaginary root$\pm i\omega_s$. Separating real and imaginary parts of $(-k - i\omega + (\beta \pm a)e^{-i\omega\tau}) = 0$, we get $k - (\beta \pm a)cos\omega\tau = 0$ and $\omega + (\beta \pm a)sin\omega\tau = 0$ which implies $\omega_s^2 = (\beta \pm a)^2 - k^2$. Therefore the equation $\left(-k - \lambda + (\beta \pm a)e^{-\lambda\tau}\right) = 0$ has imaginary roots $\pm i\omega_s$ if and only if max$\{|\beta + a|, |\beta - a|\}$>k. Substituting the values of $cos\omega_s\tau$ and $sin\omega_s\tau$ obtained above in $A_1'(i\omega)$, $A_2'(i\omega)$ and $A_3'(i\omega)$ it can be shown that characteristic equation $\left(-k - \lambda + (\beta \pm a)e^{-\lambda\tau}\right) = 0$ have two pair of imaginary roots $\pm i\omega_1, \pm i\omega_2$ where $\omega_1 = \sqrt{(\beta - a)^2 - k^2}$ and $\omega_1 = \sqrt{(\beta + a)^2 - k^2}$.

This root $\pm\omega_{1,2}$ occurs when $= \tau_n^{1,2} = \frac{1}{\omega_{1,2}} arc\tan\left(-\frac{\omega_{1,2}}{k}\right) + \frac{n\pi}{\omega_{1,2}}$; n=0, 1, 2, 3, 4, …………

Differentiating both sides of the characteristic equation with respect to τ and substituting $\lambda = i\omega_{1,2}$ for $\tau = \tau_n^{1,2}$ we get, Re($\left|\frac{d\lambda}{d\tau}\right|$ $at\ \tau = \tau_n^{1,2}$)= $\frac{\omega_{1,2}^2}{(1+k\tau_n^{1,2})^2+\omega_{1,2}^2(\tau_n^{1,2})^2}$>0. Therefore from the above discussion it can be stated as the following

Theorem 3

(i) If $\beta < -k$, then the system (5) undergoes a single Hopf-bifurcation with respect to delay parameter near origin at $\tau = \tau_{n,s}$ where $\tau = \tau_{n,s} = \frac{1}{\omega_s} arc\tan\left(-\frac{\omega_s}{k}\right) + \frac{n\pi}{\omega_s}$; n=0, 1, 2, 3, 4,………….. and $\omega_s = \sqrt{\beta^2 - k^2}$.

(ii) If max $\{|\beta + a|, |\beta - a|\}$<K<max$\{|\beta + a|, |\beta - a|\}$, then the system (5) undergoes a resonant co-dimension two Hopf-bifurcation near origin with respect to delay parameter at $\tau = \tau_n^{1,2}$, where $\tau_n^{1,2} = \frac{1}{\omega_{1,2}} arc\tan\left(-\frac{\omega_{1,2}}{k}\right) + \frac{n\pi}{\omega_{1,2}}$; n=0, 1, 2, 3, 4, ………….. and $\omega_{1,2} = \sqrt{(\beta \pm a)^2 - k^2}$ that is if max$\{|\beta + a|, |\beta - a|\}$<K< max$\{|\beta + a|, |\beta - a|\}$, then there exists two families of surfaces satisfying $\omega = \sqrt{(\beta - a)^2 - k^2}$ and then where $\tau = \frac{1}{\omega} arc\tan\left(-\frac{\omega}{k}\right) + \frac{n\pi}{\omega}$; n=0, 1, 2, 3, 4,………….. and $\omega = \sqrt{(\beta + a)^2 - k^2}$ where $\tau = \frac{1}{\omega} arc\tan\left(-\frac{\omega}{k}\right) + \frac{n\pi}{\omega}$; n=0, 1, 2, 3, 4…………

The curves of Hopf-bifurcation in this case are given by the above relation. The points of intersection of these curves (double Hopf-points) occurs when τ has the same value on the above curves. That is

$\frac{\sqrt{(\beta-a)^2-k^2}}{\sqrt{(\beta+a)^2-k^2}} = \frac{n}{m}$ Which implies that $= \beta\frac{n^2+m^2}{n^2-m^2} \pm \sqrt{1 + 2\frac{\beta^2m^2}{n^2-m^2}}$. So we have following theorem:

Theorem 4: Every point of double Hopf-bifurcation in the equation $\left(-k - \lambda + (\beta \pm a)e^{-\lambda\tau}\right) = 0$ is resonant when $a = \beta\frac{n^2+m^2}{n^2-m^2} \pm \sqrt{1 + 2\frac{\beta^2m^2}{n^2-m^2}}$ and this equation possesses two pairs of imaginary roots $i\omega_1$ and $i\omega_2$. The frequencies of those roots are in the ratio $\omega_1 : \omega_2 = n : m$ for $m, n \in Z$.

Conclusion

In this paper we have analyzed a neural network model composed of three identical neurons with discrete time delay. We first obtained in the result (1) the criteria under which no change in stability and instability occur for all the values of delay. We have also focused on investigating the conditions for single Hopf-Bifurcation with respect to synaptic weight and also the conditions for resonant co-dimension two bifurcations with respect to time delay. In the result (2) condition for single Hopf-Bifurcation with respect to synaptic weight (a). Again we have studied the periodic and stability of such Hopf-Bifurcating periodic solution when the value of the synaptic weight passes a critical values a_0. Lastly the parameter space is explicitly found, where the resonant double Hopf-Bifurcation with time delay as bifurcating parameter occurs.

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