International Journal of Statistics and Applied Mathematics 2018; 3(1): 384-388

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2018; 3(1): 384-388

www.mathsjournal.com
Received: 19-11-2017
Accepted: 20-12-2017

Rama
Research Scholar, Dept. of Mathematics, NIILM University, Kaithal, Haryana, India

Dr. Manjeet Jakhar
Assistant Professor, Dept. of Mathematics, NIILM University, Kaithal, Haryana, India

Correspondence
Rama
Research Scholar, Dept. of Mathematics, NIILM University, Kaithal, Haryana, India

Colombeau theory of generalized function with Leibniz's rule

Rama and Dr. Manjeet Jakhar

Abstract
To reveal some fundamental facts at the interaction among mathematics and the modern real world, putting in verification mathematical entities like "nonlinear generalized functions" that are required to model the modern era. In this present paper, some artefacts of distributions are illustrated. The repercussions are derived in Colombeau theory of nonlinear generalized functions, which is the quite pertinent algebraic construction for dealing with Schwartz distributions. Colombeau theory of nonlinear generalized function Ǥ(R) contains the space □(R) of Schwartz distributions as a subspace, and has a notion of mathematical 'association' that permits us to assess the outcomes in terms of distributions.

Keywords: Colombeau theory, Leibniz's rule

1. Introduction
The most important role of distributions in various mathematical domains as well as natural sciences has focused the requirement for resolving two main issues that distribution theory lies: first one is multiplication of distributions which means multiplication of two distributions are allowed and another one is differentiating the product of these distributions which does not follows Leibniz principle. Although, lots of attempts have been taken place to interpret the product of distributions or to enlarge the number of existing products by Fisher at the end of nineteenth century. At the couple of years later many researchers tried to include the distributions in differential algebras Oberguggenberger & Todorov in 1998. As per the fundamentals of distribution theory introduced in 1964 by Shilov and Zemanian in 1965 to distinguish two perspective: The first one is that distribution can be examined as a continuous function. This linear function f̧ lying on a smooth function Φ with dense support.
The test function Φ has a linear mapping like $\Phi \rightarrow \{f̧, \Phi\}$.
Another one is known as the sequential approach: taking a sequence of smooth functions (Φ_n) converging to the Dirac δ function, by means of this convolution product to achieve a family of regularization $(f̧_n)$ which converges less strongly to the distribution f̧.

$$f̧_n(x)=(f̧_* \Phi_n)(x)=< f̧ (y), \Phi_n (x-y)> \quad (1)$$

It is obtained all the sequences that converge less strongly to the same limit, and allow them as a similar class. The elements of every similar class are known as representatives of the appropriate distribution f̧. In this manner a sequential representation of distributions is illustrated. Few researchers use similar classes of nets of regularization,
Such that dirac-net $(\Phi_\varepsilon)_{\varepsilon>0}$ defined with $\Phi_\varepsilon = 1/\varepsilon * \Phi(x/\varepsilon)$.
The nonlinear structure of regularization process is vanish in a way by recognizing sequences with their limit. Therefore, overall operations have to be done on the smooth regularized functions as well as with the reverse process, the function is found from the regularization. Our focus is to achieve nonlinear fundamentals of generalized functions that's functioning depends on regularization. The optimum finding is illustrated by Colombeau in 1984 [3] w.r.t. the problems of Schwartz theory of distributions. Colombeau introduced an associative differential algebra theory of generalized functions Ǥ(R), which contains the space □(R) of distributions as subspace and the algebra of C^∞ functions as sub-algebra. This algebraic fundamentals of

generalized functions of Colombeau theory actually generalizes the concepts of Schwartz distributions. Colombeau generalized concepts can be differentiated in the similar manner of distributions which deals with, multiplication & nonlinear operations. The outcome of these nonlinear & multiplication operations always lies in Colombeau generalized function algebra [Jolevska in 2007]. The evolution of Colombeau algebra is solely unveiled the Schwartz distributions.

The opinion of 'association' in $\mathfrak{G}(R)$ is a true generalization of the equality of distributions. It also enables us to judge the findings in terms of distributions. So all these evidences make Colombeau theory to be an extensive application of engineering sciences, especially in domains where products of distributions with coinciding singularities are desired. These products include Dirac function (δ) which has singular point support. Couple of year back, Dirac function in the field of science as well as engineering for solving critical problems, was heuristically understood with properties given to coincide with experimental results. It was also well illustrated to demonstrate certain types of infinity concentrated at a single point in terms of charge density.

It means finite amount of charge is loaded into zero volume, so it is easy to say that the charge density must be infinite at that point and also Dirac function derivative was utilized to show a dipole of unit electric moment at the origin. Later the concept of distributions was developed in 1950 which represent the mathematical modelling of these fundamentals has been confirmed and Dirac delta function with the similar properties is examined as distribution. But, the problem still lies when product of two distributions in the Schwartz's space. Colombeau algebraic theory was established in a manner that lots of problems with productive distributions could be ignored. The application area of Colombeau algebra is illustrated by Aragona in 2014, Gsponer in 2009, [12] Ohkitani & Dowker in 2010 [11], Prusa & Rajagopal in 2016, Steinbauer & Vickers in 2006 [8], Capar in 2013, Nigsch & Samman in 2013 [14], Steinbauer in 1997, Alimohammady in 2014, Sojanovic in 2013, Farassat in 1994. So many researchers worked and unveiled the products of delta function as well as its derivatives with other distributions with singularities in the field of engineering.

2. Schwartz Impossibility Result

Let's assume Z be an algebraic term containing of all continuous functions on R as a sub-algebra is denoted by $C^0(R)$. Let's say the assumed function is the unit element in Z i.e. $1 \in C^0(R)$. Moreover it is suppose that there occurs a linear mapping ∂: Z→ Z expanding the derivative of continuously differentiable functions and fulfilling Leibniz's rule $\partial(xy) = \partial x.y + x.\partial y$, then $\partial^2(|p|) = 0$. The conclusive remark of this theorem is quite tough to establish an algebra Z containing □'(R) such that Leibniz's rule is satisfied.

Proof: Let's say

$$\partial(a|a|) = \partial a.|a| + a.\partial|a| = |a| + a.\partial|a|.$$

The above relation follows that

$$\partial^2(a|a|) = 2.\partial(|a|) + a.\partial^2(|a|).$$

The other part, in $C^1(R)$ which represent Z: $\partial(a|a|) = 2|a|$.
Moreover, $\partial^2(a|a|) = 2.\partial|a|$ gives $a.\partial^2(|a|) = 0$.
Thus, we will use the outcomes "In Z, if ax = 0, then x = 0",
so we conclude $\partial^2(|a|) = 0$. We just required to identify the last outcome:
It can be seen that $a(\log|a| - 1)$ and $a^2(\log|a| - 1) \in C^1(R)$ by putting 0 as the value of these required functions at zero.
By means of Leibniz's rule in Z, we find
$\partial\{a(\log|a| - 1)a\} = \partial\{a(\log|a| - 1)\}.a + a(\log|a| - 1)$.
So
$\partial^2\{a(\log|a| - 1)a\} = \partial^2\{a(\log|a| - 1)\}.a + 2.\partial\{a(\log|a| - 1)\}$.
Thus,
$\partial^2\{a(\log|a| - 1)\}.a = \partial^2\{a2(\log|a| - 1)\} - 2\partial\{a(\log|a| - 1)\}$.
But, since ∂ coexists with the ordinary derivative operator on $C^1(R)$-functions, and $a^2(\log|a| - 1) \in C^1(R)$, we receives
$\partial\{a^2(\log|a| - 1)\} = 2a(\log|a| - 1) + a$.
Moreover, in Z we have
$\partial^2\{a^2(\log|a| - 1)]\} = 2.\partial\{a(\log|a| - 1)\} + 1$.
So,
$\partial^2\{a(\log|a| - 1)\}.a = 1$.
To simplify the algebraic notation it is said that
$y = \partial^2 a(\log|a| - 1)\}$,
then b.a = 1; thus $a.x = 0 \Rightarrow y.(a.x) = 0 \Rightarrow 1.x = 0 \Rightarrow x = 0$.
This is the conclusive remark of the proof.
If the algebraic term Z containing $□^0(R)$ in which Leibniz's rule conditions is meeted, thus δ = 0, since $\partial^2(|a|) = 2\delta$ in $□^0(R)$.That is risibility which explains how this result called Schwartz impossibility result.

3. Colombeau Algebra

The Colombeau generalized functions indicate by €[R^n] the set of all the Algebraic functions
$R : Z1 \times R^n \rightarrow C$, where $(\Phi, a) \rightarrow R(\Phi, a)$
Which are C^∞ functions in 'a' for every fixed Φ. It can be see that €[R^n] is a pointwise algebraic operators.

Definition 1

It is easy to say that an element R ∈ €$[R^n]$ is generalized if for every miniaturized set U of R^n and each operator ∂^α (This derivative operator is valid even for order zero as well as in case of identity operator),

there is an N ∈ и such that for all Φ ∈ $Z_и$, we have

$(\partial^\alpha R)(\Phi\epsilon, a) = Ő(\epsilon^{-N})$

as ϵ decreases to zero uniformly on U. We denote by $€_M$ R^n the set of all ordinay elements of €$[R^n]$.

It is notable that N = N(α, U). If this relation exists for some N, then we can replace N by any N'such that N'> N.

It is concluded that if E1 and E2 are elements of €$[R^n]$,then we have $\partial^{\alpha 1}E1\partial^{\alpha 2}E2$ and $\partial^{\alpha 2}E2\partial^{\alpha 1}E1$ are both $Ő(\epsilon^{-N})$ for some N'.

After applying the Leibniz's rule we can find that the sum of finitely many elements of orders $Ő(\epsilon^{-N})$ is also of such order, we conclude that $€_M$ R^n is a sub-algebra of €$[R^n]$.

Definition 2

The Colombeau algebra theory of generalized function is denoted by Ꞡ(R^n), is the quotient algebra $€_M$ R^n /Ĭ.

It is concluded that F represent a generalized function in Ꞡ(R^n) if and only if F = F+ Ĭ,

Where, ᶂ ∈ $€_M$ R^n. It can also be say that ᶂ = ᶃ in Ꞡ(R^n) if and only if ᶂ-ᶃ ∈ Ĭ, where ᶂ, ᶃ belongs to F,G respectively. With the help of these conclusive remarks it can be seen that Ꞡ(R^n) is an associative as well as commutative algebra.

It is obvious that $\partial^\alpha €_M[R^n] \subset €_M[R^n]$and ∂^αĬ ⊂ Ĭ, ∀α.

Moreover, it can be define as:

∂^α : Ꞡ(R^n) → Ꞡ(R^n)

ᶂ→ ∂^αᶂ, where ∂^αᶂ = ∂^αF+Ĭ

The above relation describes that ∂^α is linear, and fulfill the conditions of Leibniz's rule. Further, it follows that generalized functions such as $C^\infty(R^n)$, $C^0(R^n)$ are continuous functions over R^n, also the distributions are elements of Ꞡ(R^n) given by Schwartz.

4. Non Linear Properties Of Ꞡ(R^n)

A nonlinear relation ᶂ : R^n → C is called slowly increasing function at infinity if c > 0, N ∈ и in such a manner that |ᶂ(a)| ≤ c(1 + |a|) N, ∀ x ∈ R^n. The complete set of functions $C^\infty(R^n)$ and its derivatives are gradually decreasing functions which can be represented by $Ő_M[R^n]$.

Theorem

If ᶂ ∈ $Ő_M[R^{2n}]$, where $R^{2n} \approx C^n$; if Ꞡ1,Ꞡ2,..., Ꞡn are nonlinear generalized functions in Ꞡ(Ş), where Ş is an subset of R^m; and R1, R2,..., Rn are respective representatives of Ꞡ1,Ꞡ2,..., Ꞡn, then ᶂ(R1, R2,..., Rn) is a general element in $€_M$[Ş].

Although, ᶂ(R1, R2,..., Rn) is an illustrative of a generalized function in G(Ş), represented by ᶂ(Ꞡ1,Ꞡ2,..., Ꞡn).

Definition

A distribution D ∈ □'(R^n) is called as an associative nonlinear generalized function ᶂ ∈ Ꞡ(R^n) which is represented by ᶂ ||- D, thus, for every β ∈ □(R^n), it would be get

$$\int_{R^n} (\beta.f)(a)da \,|- < D, \beta > .$$

This above relation shows such that β ∈ □(R^n), ∃r ∈ N, ∀τ ∈ A_q,

$$\int_{R^n} \beta(x) f(\tau_\in, a)da \,|- < D, \beta >$$

As ϵ decreases to zero. Also the two nonlinear generalized functions ᶂ1, ᶂ2 ∈ Ꞡ(R^n) are connected, illustrated by ᶂ1 ≈ ᶂ2, if ᶂ1−ᶂ2 (difference of these two generalized functions) is linked with 0 ∈ □'(R^n), It means the difference of these two generalized functions ᶂ1−ᶂ2 ||-0.

The above illustrated definition have some noticeable point.

1. Some relations like ~, ≈ are generally shows equivalency among given terms on Ꞡ(R^n) but
2. ||- ⊂ Ꞡ(R^n) × □'(R^n) is non symmetric as well as non-reflective.
3. Nonlinear generalized functions ᶂ1, ᶂ2 ∈ Ꞡ(R^n) and ᶂ1 ~ ᶂ2, thus ᶂ1 ≈ ᶂ2.
4. If distribution D ∈ □'(R^n) and D~0 or D≈0 then we can say D=0.

5. $L_1(R)$ Functions Issues

$L_1(R)$ functions belong to the class of generalized functions as a subset. It is seen in literature that $L_1(R^n)$ ⊂ □'(R^n) in the Schwartz distribution fundamental concepts & $L_1(R^n)$ functions are tempered distributions.

So,

ᶂ ∈ $L_1(R^n)$, it will be analogous element in Ꞡ(R^n) represented by ᶂ'+Ĭ (Here ᶂ' ∈ $€_M[R^n]$).

Moreover this ᶂ' can be represented as

$$f^{'}(\beta, a) = \langle f(b), \beta(b-a) \rangle = \int_{R^n} f(b)\beta(b-a)db$$, since ᶂ ∈ $L_1(R^n)$.

Thus, ᶂ'+Ĭ as the analogous element of ᶂ ∈ $L_1(R)$. Although, it is quite though to conclude from the above relation that ᶂ'+Ĭ ∈ Ꞡ(R^n).

Since ʄ ∈ $L_1(R)$, a function h is continuous in R which can be represented as

$$h(a) = \int_{-\infty}^{a} f(t)dt$$
, where 'a' is element of R.

$$|h(a)| = \left|\int_{-\infty}^{a} f(t)dt\right| \leq \int_{-\infty}^{a} |f(t)|dt \leq \int_R |f(t)|dt = \|f\|_{L_1}$$
, for all a ϵ R.

So, h ϵ $C_\tau(R)$. Now it follows that h analogous to $\mathcal{G}_\tau(R)$. Again in $\mathcal{G}_\tau(R)$, h is allocated with h'+Ǐ,

Where $\bar{h}(\beta, a) = \int_R h(a+b)\beta(b)db$

It inferred that ʄ associate to $\mathcal{G}_\tau(R)$, and appointed as $\partial \bar{h} + I_\tau$.
So, f is appointed with the element

$$\partial_a (\int_R (\int_{-\infty}^{a+b} f(t)dt\beta(b)db + I_\tau$$

It has been observed that β ϵ A_1 and ʄ ∈ $L_1(R)$, we find

$$\int_{-\infty}^{\infty} (\int_{-\infty}^{a+b} f(t)dt\beta(b)db = \int_{-\infty}^{\infty} (\int_{-\infty}^{a} f(t+b)dt)\beta(b)db$$
$$= \int_{-\infty}^{a} (\int_{-\infty}^{\infty} \beta(b) f(t+b)db)dt$$

and the inside integration is a function of t which belongs in $L_1(R) \cap C^\infty(R)$.
It also shows that

$$\partial_a (\int_R (\int_{-\infty}^{a+b} f(t)dt\beta(b)db = \int_{-\infty}^{\infty} \beta(b) f(a+b)db$$

Such that, f is associated with the element $\int_R f(a+b)\beta(b)db + I_\tau$ in $\mathcal{G}_\tau(R)$.
It also represent that in $\mathcal{G}(R)$, the function ʄ ∈ $L_1(R)$ is analogous to the element

$$\int_R f(a+b)\beta(b)db + I_\tau.$$

Thus, we will use these outcomes above to determine the relationship among the integration of ʄ ∈ $L_1(R)$ in the ordinary sense as well as the sense of tempered generalized function.

6. Conclusion

We have assessed few multiplications of generalized functions, which involve derivatives of the Dirac delta function δ, in Colombeau algebraic theory in terms of analogous distributions. This is quite relevant just because multiplications of this kinds are usually not only used in the physical science but also in engineering too as per literature. The fundamental points of Colombeau algebraic theory of generalized functions have been discussed. It can be concluded that in $\mathcal{G}(R)$ one can calculate product of two randomly generalized functions and prove the Leibniz's rule authenticity. Furthermore, the above product term is commutative as well as associative, and also it embeds as a subset of $\mathcal{G}(R)$ with the Schwartz distributions in a manner which is compatible with its derivatives.

7. References

1. Biagioni HA. A Nonlinear Theory of Generalized Functions", Lecture Notes in Math. 1421. Springer, 1990.
2. Christyakov VV. The Colombeau generalized nonlinear analysis and the Schwartz linear distibution theory", Journal of mathematics Science. 1999; 93:3-40.
3. Colombeau JF. New Generalized Functions and Multiplication of Distributions, North-Holland-Elsevier, 1984.
4. Oberguggenberger M. Multiplication of Distributions and Applications to Partial Differential Equations, Longman Sci. & Tech, 1992.
5. Nedeljkov M, Pilipovic S, Scarpalezos D. The Linear Theory of Colombeau Generalized Functions, Pitman Research Notes in Math, 1998.
6. Oberguggenberger M. Generalized functions in nonlinear models- a survey, Nonlinear Analysis. 2001; 47:5029-5040.

7. Aleksik J, Colombeau JF, Oberguggenberger M, Pilipovic S. Approximate generalized solutions and measure valued solutions to conservation laws, Integral Transform and Special Functions. 2009; 20(3, 4):163-170.
8. nbauer R, Vickers JA. The use of generalized functions and distributions in general relativity. Class Quantum Gravity", 2006.
9. Aragona J, Fernandez R, Juriaans SO, Oberguggenberger M. Differential calculus and integration of generalized functions over membranes, Monatshefte Math. 2012; 166:1-18.
10. Aragona J, Garcia ARG, Juriaans SO. Generalized solutions of a nonlinear parabolic equation with generalized functions as initial data", Nonlinear analysis. 2009; 71:5187-5207.
11. itani K, Dowker M. Burges equation with a passive scalar: dissipation anomaly and Colombeau calculus. J Math Phys, 2010.
12. Gsponer A. A concise introduction to Colombeau generalized functions and their applications in classical electrodynamics, Europian Journal of Physics. 2009; 30(1):109-126. doi: 10.1088/0143-0807/30/1/011.
13. Mirkov R, Pilipovic S, Selesi D. Generalized Stochastic Processes in algebras of generalized functions", Journal math. anal. appl. 2009; 353(1):260-270.
14. Nigsch E, Samann C. Global algebras of nonlinear generalized functions with applications in general relativity", In this volume and Arxiv. org, 2013.
15. eva M, Jolevska TB, Atanasova PT. On products of distributions in Colombeau algebra, Math Probl Eng, 2014.
16. Jolevska TB, Atanasova PT, Further results on Colombeau product of distributions", Int J Math Math Sci. doi:10.1155/2013/918905, 2013.
17. Miteva M, Jolevska TB, Atanasova PT. On products of distributions in Colombeau algebra", Math Probl Eng. doi:10.1155/2014/910510, 2014.