International Journal of Statistics and Applied Mathematics 2018; 3(2): 37-40

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2018; 3(2): 37-40

www.mathsjournal.com
Received: 12-01-2018
Accepted: 15-02-2018

S Devibala
Assistant Professor, Department of Mathematics, Sri Meenakshi Govt. Arts College for Women (A), Madurai, Tamil Nadu, India.

MA Gopalan
Professor, Department of Mathematics, SIGC, Trichy, Tamil Nadu, India

V Sivaranjani
M. Phil Scholar, Department of Mathematics, SIGC, Trichy, Tamil Nadu, India

Correspondence
S Devibala
Assistant Professor, Department of Mathematics, Sri Meenakshi Govt. Arts College for Women (A), Madurai, Tamil Nadu, India.

On the negative pell equation $y^2 = 3x^2 - 2$

S Devibala, MA Gopalan and V Sivaranjani

Abstract

The binary quadratic equation represented by the negative pellian $y^2 = 3x^2 - 2$ is analyzed for its distinct integer solutions. A few interesting relations among the solutions are also given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbola, parabola and special Pythagorean triangle.

Keywords: Binary quadratic, hyperbola, parabola, integral solutions, Pell equations. 2010 mathematics subject classification: 11D09.

Introduction

Diophantine equation of the form $y^2 = Dx^2 + 1$, where D is a given positive square – free integer is known as Pell equation and is one of the oldest Diophantine equation that has interesting mathematicians all over the world, since antiquity, J.L. Lagrange proved that the positive Pell equation $y^2 = Dx^2 + 1$ has infinitely many distinct integer solutions whereas the negative Pell equation $y^2 = Dx^2 - 1$ does not always have a solution. In [1], an elementary proof of a criterium for the solvability of the Pell equation $X^2 - Dy^2 = -1$ where D is any positive non - square integer has been presented. For examples the equations $y^2 = 3x^2 - 1$, $y^2 = 7x^2 - 4$ have no integer solution where as $y^2 = 65x^2 - 1$, $y^2 = 202x^2 - 1$ have integer solutions. In this context, one may refer [2-17]. More specifically, one may refer "The on – line encyclopedia of integer sequences" (A031396, A130226, A031398) for values of D for which the negative Pell equation $y^2 = Dx^2 - 1$ is solvable or not. In this communication, the negative Pell equation given by $y^2 = 3x^2 - 2$ is considered and infinitely many integer solutions are obtained. A few interesting relations among the solutions are presented.

Method of Analysis

The negative Pell equation representing hyperbola under consideration is

$$y^2 = 3x^2 - 2 \quad (1)$$

whose smallest positive integer solution is $x_0 = 1,\ y_0 = 1$.

To obtain the other solutions of (1), consider the Pell equation $y^2 = 3x^2 + 1$, whose solution is given by,

$$\tilde{y}_n = \frac{1}{2} f_n, \quad \tilde{x}_n = \frac{1}{2\sqrt{3}} g_n$$

where,

$$\mathrm{f}_{\mathrm{n}} = \left(2+\sqrt{3}\right)^{n+1} + \left(2-\sqrt{3}\right)^{n+1}$$

$$\mathrm{g}_{\mathrm{n}} = \left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}$$

Applying Brahamagupta Lemma between (x_0, y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = \frac{1}{2}\mathrm{f}_{\mathrm{n}} + \frac{1}{2\sqrt{3}}\mathrm{g}_{\mathrm{n}}$$

$$y_{n+1} = \frac{1}{2}\mathrm{f}_{\mathrm{n}} + \frac{\sqrt{3}}{2}\mathrm{g}_{\mathrm{n}}$$

The recurrence relations satisfied by the solutions *x* and *y* are given by,

$$x_{n+1} - 4x_{n+2} + x_{n+3} = 0$$

$$y_{n+1} - 4y_{n+2} + y_{n+3} = 0$$

Some numerical examples of *x* & *y* satisfying (1) are given in the table (1) below.

Table 1: Examples

n	x_n	y_n
0	1	1
1	3	5
2	11	19
3	41	71
4	153	265

From the above table, we observe some interesting relations among the solutions which are presented below.

1. x_n and y_n both are always odd.
2. Each of the following expressions is a nasty number:

- $30x_{2n+2} - 6x_{2n+3} + 12$
- $27x_{2n+2} - 3y_{2n+3} + 12$
- $114x_{2n+3} - 30x_{2n+4} + 12$
- $9x_{2n+3} - 15y_{2n+2} + 12$
- $54x_{2n+3} - 30y_{2n+3} + 12$
- $99x_{2n+3} - 15y_{2n+4} + 12$
- $27x_{2n+4} - 57y_{2n+3} + 12$
- $198x_{2n+4} - 114y_{2n+4} + 12$
- $6y_{2n+3} - 18y_{2n+2} + 12$
- $18y_{2n+4} - 66y_{2n+3} + 12$
- $\dfrac{(57x_{2n+2} - 3x_{2n+4} + 24)}{2}$
- $\dfrac{(198x_{2n+2} - 6y_{2n+4} + 84)}{7}$
- $\dfrac{(18x_{2n+4} - 114y_{2n+2} + 84)}{7}$
- $\dfrac{(3y_{2n+4} - 33y_{2n+2} + 24)}{2}$

3. Each of the following expressions is a cubical integer:

- $(5x_{3n+3} - x_{3n+4}) + 3(5x_{n+1} - x_{n+2})$
- $48(19x_{n+1} - x_{n+3}) + 16(19x_{3n+3} - x_{3n+5})$
- $12(9x_{n+1} - y_{n+2}) + 4(9x_{3n+3} - y_{3n+4})$
- $147(33x_{n+1} - y_{n+3}) + 49(33x_{3n+3} - y_{3n+5})$
- $3(19x_{n+2} - 5x_{n+3}) + (19x_{3n+4} - 5x_{3n+5})$
- $12(3x_{n+2} - 5y_{n+1}) + 4(3x_{3n+4} - 5y_{3n+3})$
- $3(9x_{n+2} - 5y_{n+2}) + (9x_{3n+4} - 5y_{3n+4})$
- $12(33x_{n+2} - 5y_{n+3}) + 4(33x_{3n+4} - 5y_{3n+5})$
- $147(3x_{n+3} - 19y_{n+1}) + 49(3x_{3n+5} - 19y_{3n+3})$
- $12(9x_{n+3} - 19y_{n+2}) + 4(9x_{3n+5} - 19y_{3n+4})$
- $3(33x_{n+3} - 19y_{n+3}) + (33x_{3n+5} - 19y_{3n+5})$
- $3(y_{n+2} - 3y_{n+1}) + (y_{3n+4} - 3y_{3n+3})$
- $48(y_{n+3} - 11y_{n+1}) + 16(y_{3n+5} - 11y_{3n+3})$
- $3(3y_{n+3} - 11y_{n+2}) + (3y_{3n+5} - 11y_{3n+4})$

4. Relations among the solutions:

- $y_{n+1} = x_{n+2} - 2x_{n+1}$
- $y_{n+2} = 2x_{n+2} - x_{n+1}$
- $y_{n+3} = 7x_{n+2} - 2x_{n+1}$
- $4y_{n+1} = x_{n+3} - 7x_{n+1}$
- $2y_{n+2} = x_{n+3} - x_{n+1}$
- $4y_{n+3} = 7x_{n+3} - x_{n+1}$
- $2y_{n+1} = y_{n+2} - 3x_{n+1}$
- $2y_{n+3} = 3x_{n+1} - 7y_{n+2}$
- $7y_{n+1} = y_{n+3} - 12x_{n+1}$
- $y_{n+1} = 2x_{n+3} - 7x_{n+2}$
- $y_{n+2} = x_{n+3} - 2x_{n+2}$
- $y_{n+3} = 2x_{n+3} - x_{n+2}$
- $2y_{n+2} = 3x_{n+2} + y_{n+1}$
- $y_{n+3} = 6x_{n+2} + y_{n+1}$
- $y_{n+3} = 3x_{n+2} + 2y_{n+2}$
- $7y_{n+2} = 3x_{n+3} + 2y_{n+1}$
- $7y_{n+3} = 12x_{n+3} + y_{n+1}$
- $2y_{n+3} = 3x_{n+3} + y_{n+2}$

Remarkable Observations:

I. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the

Table 2 below.

Table 2: Hyperbolas

S. No.	(X, Y)	Hyperbolas
1	$(x_{n+2}-3x_{n+1}, 5x_{n+1}-x_{n+2})$	$Y^2-3X^2=4$
2	$(x_{n+3}-11x_{n+1}, 19x_{n+1}-x_{n+3})$	$Y^2-3X^2=64$
3	$(y_{n+2}-5x_{n+1}, 9x_{n+1}-y_{n+2})$	$Y^2-3X^2=16$
4	$(y_{n+3}-19x_{n+1}, 33x_{n+1}-y_{n+3})$	$Y^2-3X^2=196$
5	$(3x_{n+3}-11x_{n+2}, 19x_{n+2}-5x_{n+3})$	$Y^2-3X^2=4$
6	$(3y_{n+1}-x_{n+2}, 3x_{n+2}-5y_{n+1})$	$Y^2-3X^2=16$
7	$(3y_{n+2}-5x_{n+2}, 9x_{n+2}-5y_{n+2})$	$Y^2-3X^2=4$
8	$(3y_{n+3}-19x_{n+2}, 33x_{n+2}-5y_{n+3})$	$Y^2-3X^2=16$
9	$(11y_{n+1}-x_{n+3}, 3x_{n+3}-19y_{n+1})$	$Y^2-3X^2=196$
10	$(11y_{n+2}-5x_{n+3}, 9x_{n+3}-19y_{n+2})$	$Y^2-3X^2=16$
11	$(11y_{n+3}-19x_{n+3}, 33x_{n+3}-19y_{n+3})$	$Y^2-3X^2=4$
12	$(5y_{n+1}-y_{n+2}, y_{n+2}-3y_{n+1})$	$3Y^2-X^2=12$
13	$(19y_{n+1}-y_{n+3}, y_{n+3}-11y_{n+1})$	$3Y^2-X^2=192$
14	$(19y_{n+2}-5y_{n+3}, 3y_{n+3}-11y_{n+2})$	$3Y^2-X^2=12$

II. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the Table 3 below.

Table 3: Parabolas

S.N	(X, Y)	Parabolas
1	$(x_{n+2}-3x_{n+1}, 5x_{2n+2}-x_{2n+3}+2)$	$3X^2=Y-4$
2	$(x_{n+3}-11x_{n+1}, 19x_{2n+2}-x_{2n+4}+8)$	$3X^2=4Y-64$
3	$(y_{n+2}-5x_{n+1}, 9x_{2n+2}-y_{2n+3}+4)$	$3X^2=2Y-16$
4	$(y_{n+3}-19x_{n+1}, 33x_{2n+2}-y_{2n+4}+14)$	$3X^2=7Y-196$
5	$(3x_{n+3}-11x_{n+2}, 19x_{2n+3}-5x_{2n+4}+2)$	$3X^2=Y-4$
6	$(3y_{n+1}-x_{n+2}, 3x_{2n+3}-5y_{2n+2}+4)$	$3X^2=2Y-16$
7	$(3y_{n+2}-5x_{n+2}, 9x_{2n+3}-5y_{2n+3}+2)$	$3X^2=Y-4$
8	$(3y_{n+3}-19x_{n+2}, 33x_{2n+3}-5y_{2n+4}+4)$	$3X^2=2Y-16$
9	$(11y_{n+1}-x_{n+3}, 3x_{2n+4}-19y_{2n+2}+14)$	$3X^2=7Y-196$
10	$(11y_{n+2}-5x_{n+3}, 9x_{2n+4}-19y_{2n+3}+4)$	$3X^2=2Y-16$
11	$(11y_{n+3}-19x_{n+3}, 33x_{2n+4}-19y_{2n+4}+2)$	$3X^2=Y-4$
12	$(5y_{n+1}-y_{n+2}, y_{2n+3}-3y_{2n+2}+2)$	$X^2=3Y-12$
13	$(19y_{n+1}-y_{n+3}, y_{2n+4}-11y_{2n+2}+8)$	$X^2=12Y-192$
14	$(19y_{n+2}-5y_{n+3}, 3y_{2n+4}-11y_{2n+3}+2)$	$X^2=3Y-12$

III. Consider p = $x + y$, q = x. Observe that p > q > 0. Treat p, q as the generators of the Pythagorean triangle $T(X, Y, Z)$,

where $X = 2pq, Y = p^2 - q^2, Z = p^2 + q^2$.

Let A and P be denote the area and perimeter of the Pythagorean triangle.

Then the following interesting relations are observed:

a) $2X - 3Y + Z = 4$

b) $2Z - 5X + \frac{12A}{P} = -4$

c) $xy = \frac{2A}{P}$

Conclusion

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the negative Pell equation $y^2 = 3x^2 - 2.$ As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of negative Pell equations and determine their integer solutions along with suitable properties.

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