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Stochastic analysis of a two-unit redundant system with preventive maintenance

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Abstract
This manuscript deals with a new class of the stochastic model of a two-unit cold stand by system is analyzed. There is a single repair facility to repair the failed unit. A unit is repaired in two phases and each phase the repair is performed in two parts. Therefore, the repair time distribution in each phase is assumed to be two-stage Erlangian with parameters (2, λ) and (2, μ) in phase I and in phase II respectively. The repair facility repairs both the phases of repair and can repair one unit at a time. After phase II repair a units works as good as new. Using suitably chosen regeneration points the system model is analyzed probabilistically to obtain various useful reliability measures as in earlier chapters.
we consider a two-unit standby system with preventive maintenance of the operative unit. There is a transfer switch to connect online the standby unit. The transfer switch operates instantaneously at the time of need with some fixed probability. There is also a repair facility, which repairs the failed unit. The repair facility is also takes some random time in arriving at the system. Considering failure and appearance time of repair facility as exponential and all other distributions time dependent, the system model is analysed to obtain the various performance measures.

Keywords: Transition Probabilities, Reliability, Repair Facilities, Maintenance Facilities and Expected life time.

Introduction
Two-unit redundant system with preventive maintenance have been studied extensively. Osaki and Asakura (1970) have analysed a system model with preventive maintenance. Lau (1983) has explained the following three types of actions in preventive maintenance

(i) Regular care of the system such as lubrication, refuelling, cleaning, adjustments and depending on other requirements of the components.
(ii) Checking and repairing of failed components. It is a function of probabilities.
(iii) Replacing or overhauling near wearout components. It depends on wearout characteristics of the components. Gupta and Pandya (1998) have analysed a system model having a transfer switch which takes some random time for switch over. Gupta and Sammerawar (2000) have considered a system having a transfer switch working instantaneously, which connects a protection device to main unit.

In the present paper we consider a two-unit standby system with preventive maintenance of the operative unit. There is a transfer switch to connect online the standby unit. The transfer switch operates instantaneously at the time of need with some fixed probability. There is also a repair facility, which repairs the failed unit. The repair facility is also takes some random time in arriving at the system. Considering failure and appearance time of repair facility as exponential and all other distributions time dependent, the system model is analysed to obtain the various performance measures such as : transition probabilities and mean sojourn times, reliability and expected life time of the system, expect uptime of the system during (0,t), Expected busy period of the repair and maintenance facilities particular case with all exponential distributions is considered.

System Description and Assumptions
A two-unit cold standby system is considered. During operation a unit requires its preventive maintenance to prolong its life time and improve the efficiency. There is a maintenance facility

which performs the preventive maintenance at random epochs of time. The unit operates during its maintenance and may fail in that duration. As soon as the unit under preventive maintenance fails, its preventive maintenance is stopped immediately and the repair facility is called for the repair of the unit. The repair facility takes some random time before starting the repair of the failed unit. During the presence of repair facility the preventive maintenance of the other operative unit is not required. The repair facility repairs the units one by one. The first come first served rule is followed for repair. The preventive maintenance and repairs are perfect. The failure time distribution of the unit is exponential while the repair time distribution are time dependent. Inter-arrival time of maintenance facility and maintenance time distributions are also considered as arbitrary functions of time. The arrival of repair facility follows the exponential distribution.

Notations and States of the System

(a) We define the following symbols for the states of the system:

Mode symbols of the units

O : One good unit in operating mode.
S : One good unit is kept as cold standby.
F : Unit in failed mode or a failed unit.
P : Operative unit under preventive maintenance.

Suffix symbols

r : unit under repair
w : waiting for repair

Using the above symbols and inview of system description and assumptions of section 3.2, we define the following possible states of the system.

$S_0 \equiv (O, S)$: One unit operative and the other is standby.
$S_1 \equiv (F_w, O)$: One failed unit waiting for repair and the other unit operating.
$S_2 \equiv (F_r, O)$: One failed unit under repair and the other unit operating.
$S_3 \equiv (P, S)$: One operating unit under preventive maintenance and the other unit as standby.
$S_4 \equiv (P, F_w)$: One operating unit under preventive maintenance and the other unit waiting for repair.
$S_5 \equiv (F_w, F_w)$: Both the failed units waiting for repair.
$S_6 \equiv (F_r, F_w)$: One unit under repair and the other failed unit waiting for repair.

Therefore, the states may be classified as follows :

Up States : S_0, S_1, S_2, S_3, S_4
Failed States : S_5, S_6

(b) Other Notations

E : Set of regenerative states.
$\equiv \{ S_0, S_1, \ldots, S_6 \}$.
$\bar{E}$: Set of non-regenerative state.
$\equiv \{ S_6 \}$.
α : Constant failure rate of an operative unit.
θ : Constant arrival rate of the repair facility.
f(1),F(1) : pdf and cdf of repair time.
g(1),G(1) : pdf and cdf of maintenance time of an operative unit.
h(1),H(1) : pdf and cdf of inter-arrival time of maintenance facility.
n_1,n_2,n_3 : mean repair time, mean maintenance time and mean inter-arrival time of maintenance facility.

Transition Diagram

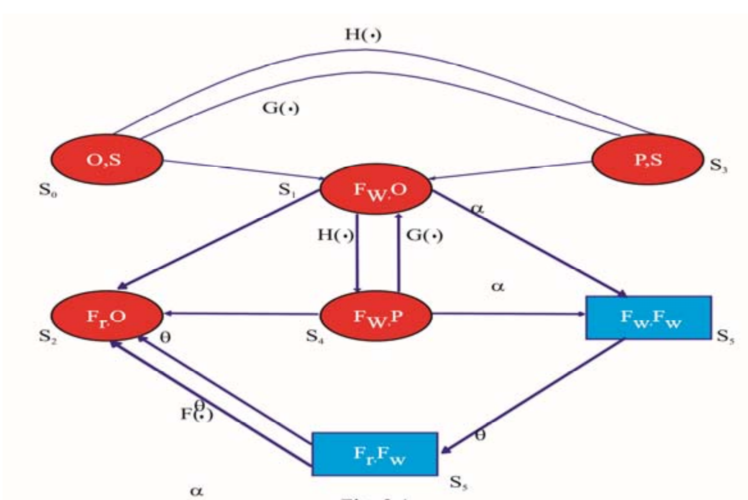
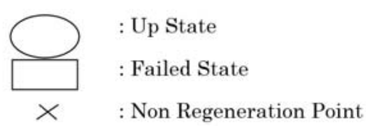


Fig. 3.1



The other notations not given here may be seen from the Glossary of general notations given at the starting of the thesis.
The transition diagram of the system model along with transition rates or transition time cdfs is shown in Fig.3.1. It may be observed that the epoch of transition S_2 to S_6 is non-regenerative, while entry from S_5 to S_6 is a regeneration point. Therefore, S_6 is non-regenerative as well as the regenerative state.

Transition Probabilities and Expected Sojourn Times

(a) As defined earlier $Q_{ij}(t)$ is the cdf of transition time from regenerative state Si to Sj. As the states defined in E are regenerative, by simple probabilistic arguments, the non-zero elements of the matrix Q= $(Q_{ij}(t))$ are as follows :

$Q_{01}(t)$ = Probability that the system transits from state S_0 to S_1 during time ≤ t

= $\int_0^t P$ [System transits from S_0 to S_1 durin (u,u+du) while it is not transited to S_3 up to time u]

$= \int_0^t \alpha\, e^{-\alpha u} du . \bar{H}(u)$ (1)

Similarly, we have

$Q_{03}(t) = \int_0^t e^{-\alpha u}\, dH(u)$

$Q_{12}(t) = \int_0^t \theta e^{-(\theta+\alpha)u}\, \bar{H}(u) du$

$Q_{14}(t) = \int_0^t e^{-(\alpha+\theta)u}\, dH(u)$

$Q_{15}(t) = \int_0^t \alpha e^{-(\theta+\alpha)u}\, \bar{H}(u) du$

$Q_{20}(t) = \int_0^t e^{-\alpha u}\, dF(u)$

$Q_{30}(t) = \int_0^t e^{-\alpha u}\, dG(u)$

$Q_{31}(t) = \int_0^t \alpha\, e^{-\alpha u} \bar{G}(u) du$

$Q_{41}(t) = \int_0^t e^{-(\theta+\alpha)u}\, dG(u)$

$Q_{42}(t) = \int_0^t \theta e^{-(\theta+\alpha)u}\, \bar{G}(u) du$

$Q_{45}(t) = \int_0^t \alpha e^{-(\theta+\alpha)u}\, \bar{G}(u) du$

$Q_{56}(t) = \int_0^t \theta e^{-\theta u}\, du = 1 - e^{-\theta t}$

$Q_{62}(t) = \int_0^t dF(u)$ = F(t) (2-13)

To write the expression for $Q_{22}^{(6)}(t)$, suppose that the system transits from state S2 to S6 during (u,u+du), u≤t ; the probability of this event is $\alpha e^{-\alpha u} du . \bar{F}(u)$. Further, suppose that the system passes from S6 to S2 during the interval $(v + v + dv)$ in (u,t) with probability

$$\frac{dF(v)}{\bar{F}(u)}$$

Therefore,

$$Q_{22}^{(6)}(t) = \int_0^t \alpha\, e^{-\alpha u}\, \bar{F}(u) du \int_u^t \frac{dF(v)}{\bar{F}(u)}$$

Changing the limits of integration, we have

$$Q_{22}^{(6)}(t) = \alpha \int_0^t d\, F(v) \int_0^v e^{-\alpha u}\, du$$
$$= \int_0^t (1 - e^{-\alpha v})\, dF(v)$$
$$= (1 - e^{-\alpha t})\text{F(t)} - \alpha \int_0^t e^{\alpha v}\, F(v)\, dv \quad (14)$$

The unconditional probability that the system will transit from state S_i to any state $\epsilon\ E$ on or before time t is

$Q_i(t) = \sum_j [Q_{ij}(t) \sum_k Q_{ij}^{(k)}(t)]$

Thus,

$Q_0(t) = 1 - e^{-\alpha t}\, \bar{H}(t)$

$Q_1(t) = 1 - e^{-(\theta+\alpha)t}\, \bar{H}(t)$

$Q_2(t) = F(t)$

$Q_3(t) = 1 - e^{-\alpha t}\, \bar{G}(t)$

$Q_4(t) = 1 - e^{-(\theta+\alpha)t}\, \bar{G}(t)$

$Q_5(t) = 1 - e^{-\theta t}$

$Q_6(t) = F(t)$

(b) The steady-state transition probabilities are given by

$p_{ij} = \lim_{t\to\infty} Q_{ij}(t)$

and

$p_{ij}^{(k)} = \lim_{t\to\infty} Q_{ij}^{(k)}(t)$

Therefore,

$p_{01} = 1 - \widetilde{H}(\alpha)$, $p_{03} = \widetilde{H}(\alpha)$

$p_{12} = \theta[1 - \widetilde{H}(\alpha+\theta)]/(\theta+\alpha)$, $p_{14} = \widetilde{H}(\theta+\alpha)$

$p_{15} = \alpha[1 - \widetilde{H}(\alpha+\theta)]/(\theta+\alpha)$, $p_{20} = \tilde{F}(\alpha)$

$p_{22}^{(6)} = 1 - \tilde{F}(\alpha)$, $p_{30} = \tilde{G}(\alpha)$

$p_{31} = 1 - \tilde{G}(\alpha)$, $p_{41} = \tilde{G}(\theta+\alpha)$

$p_{42} = \theta[1 - \tilde{G}(\theta+\alpha)]/(\theta+\alpha)$,

$p_{45} = \alpha[1 - \tilde{G}(\theta+\alpha)]/(\theta+\alpha)$

$p_{56} = p_{62} = 1$

We observe the following relationships

$p_{01} + p_{03} = 1$, $p_{12} + p_{14} + p_{15} = 1$

$p_{20} + p_{22}^{(6)} = 1$, $p_{30} + p_{31} = 1$

$p_{41} + p_{42} + p_{45} = 1$, $p_{56} = p_{62} = 1$ (15-20)

(a) Mean sojourn time μ_i in state Si is defined as the expected time for which the system stays in Si. In particular

$\mu_0 = \int_0^\infty P[T_0 > t]dt = \int_0^\infty e^{-\alpha t}.\overline{H}(t)dt$

$= [1 - \widetilde{H}(\alpha)]/\alpha$

$\mu_1 = \int_0^\infty e^{-(\theta+\alpha)t}.\overline{H}(t)dt$.

$= [1 - \widetilde{H}(\theta+\alpha)]/(\theta+\alpha)$

$\mu_2 = \int_0^\infty e^{-\alpha t}.\bar{F}(t)dt$.

$= [1 - \tilde{F}(\alpha)]/\alpha$

$\mu_3 = \int_0^\infty e^{-\alpha t}.\bar{G}(t)dt$.

$= [1 - \tilde{G}(\alpha)]/\alpha$

$\mu_4 = \int_0^\infty e^{-(\theta+\alpha)t}.\bar{G}(t)dt$.

$= [1 - \tilde{G}(\theta+\alpha)]/(\theta+\alpha)$

$\mu_5 = \int_0^\infty e^{-\theta t}dt$.

$= 1/\theta$

$\mu_6 = \int_0^\infty \bar{F}(t)dt = n_1$. (21-27)

To obtain m_{ij} , we have

$m_{ij} = \int_0^\infty t.dQ_{ij}(t) = \int_0^\infty t.q_{ij}(t)dt$.

Here, we have

$m_{01} = \alpha\int_0^\infty t.e^{-\alpha t}\overline{H}(t)dt$

$m_{03} = \int_0^\infty t.e^{-\alpha t}dH(t)$

$m_{12} = \theta\int_0^\infty t.e^{-(\theta+\alpha)t}\overline{H}(t)dt$.

$m_{14} = \int_0^\infty t.e^{-(\theta+\alpha)t}dH(t)$.

$m_{15} = \alpha\int_0^\infty t.e^{-(\theta+\alpha)t}\overline{H}(t)dt$.

$m_{20} = \int_0^\infty t.e^{-\alpha t}dF(t)$.

$m_{30} = \int_0^\infty t.e^{-\alpha t}dG(t)$.

$m_{31} = \alpha\int_0^\infty t.e^{-\alpha t}\bar{G}(t)dt$.

$m_{41} = \int_0^\infty t.e^{-(\theta+\alpha)t}dG(t)$.

$m_{42} = \theta\int_0^\infty t.e^{-(\theta+\alpha)t}\bar{G}(t)dt$.

$m_{45} = \alpha\int_0^\infty t.e^{-(\theta+\alpha)t}\bar{G}(t)dt$.

$m_{56} = 1/\theta$

$m_{62} = n_1$

$m_{22}^{(6)} = \int_0^\infty t.(1 - e^{-\alpha t})dF(t)$.

Obviously the following relations may be observed.

$m_{01} + m_{03} = [1 - \widetilde{H}(\alpha)]/\alpha = \mu_0$

$m_{12} + m_{14} + m_{15} = [1 - \widetilde{H}(\theta+\alpha)]/(\theta+\alpha) = \mu_1$

$m_{20} + m_{22}^{(6)} = n_1$
$m_{30} + m_{31} = [1 - \tilde{G}(\alpha)]/\alpha = \mu_3$
$m_{40} + m_{41} + m_{45} = [1 - \tilde{G}(\theta + \alpha)]/(\theta + \alpha) = \mu_4$
$m_{56} = \frac{1}{\theta} = \mu_5$
$m_{62} = n_1 = \mu_6$. (28-34)

Analysis of Reliability and MTSF

Assuming the failed states S_5 and S_6 as absorbing states and employing the arguments used in the theory of regenerative process, the following relations among $R_i(t)$ may be obtained:

$R_0(t) = Z_0(t) + q_{01}(t) © R_1(t) + q_{03}(t) © R_3(t)$
$R_1(t) = Z_1(t) + q_{12}(t) © R_2(t) + q_{14}(t) © R_4(t)$
$R_2(t) = Z_2(t) + q_{20}(t) © R_0(t)$
$R_3(t) = Z_3(t) + q_{30}(t) © R_0(t) + q_{31}(t) © R_1(t)$
$R_4(t) = Z_4(t) + q_{41}(t) © R_1(t) + q_{42}(t) © R_2(t)$ (1-5)

where
$Z_0(t) = e^{-\alpha t}\overline{H}(t)$, $Z_1(t) = e^{-(\theta+\alpha)t}\overline{H}(t)$,
$Z_2(t) = e^{-\alpha t}\overline{F}(t)$, $Z_3(t) = e^{-\alpha t}\overline{G}(t)$,
$Z_4(t) = e^{-(\theta+\alpha)t}\overline{G}(t)$.

For an illustration, the equation for $R_0(t)$is the sum of the following mutually exclusive contingencies:

(i) System sojourns in state S_0 upto time. The probability of this contingency is
$e^{-\alpha t}\overline{H}(t) = Z_0(t)$, say

(ii) The system transits from state S_0 to S_1, during (u,u+du), u ≤ t, and then starting from state S_1, it survives for the remaining time duration (t-u). The probability of this event is
$\int_0^t q_{01}(u)du\, R(t-u) = q_{01}(t) © R_1(t)$

(iii) The system transits from state S_0 to S_3 during (u,u+du), u ≤ t, and then starting from state S_3, it survives for the remaining time duration (t-u). The probability of this event is
$\int_0^t q_{03}(u)du\, R_3(t-u) = q_{03}(t) © R_3(t)$

Taking Laplace transform of the relations (1-5) of algebraic equations in matrix

The argument‘s’ has been omitted from $R_i^*(s)$, $q_{ij}^*(s)$ and $z_i^*(s)$ for brevity. Computing the above matrix equation for $R_0^*(s)$, we get

$R_0^*(s) = \frac{N_1(s)}{D_1(s)}$, (7)

where

$$N_1(s) = \begin{vmatrix} z_0^* & -q_{01}^* & 0 & -q_{03}^* & 0 \\ z_1^* & 1 & -q_{12}^* & 0 & -q_{24}^* \\ z_2^* & 0 & 1\ 0 & 0 & \\ z_3^* & -q_{31}^* & 0 & 1 & 0 \\ z_4^* & -q_{41}^* & -q_{42}^* & 0 & 1 \end{vmatrix}$$

$= z_0^* + (q_{01}^* + q_{03}^* q_{31}^*)[z_1^* + (q_{12}^* + q_{14}^* q_{42}^*)z_2^* + q_{14}^* z_4^*] + q_{03}^* z_3^*$

and

$$D_1(s) = \begin{vmatrix} 1 & -q_{01}^* & 0 & -q_{03}^* & 0 \\ 0 & 1 & -q_{12}^* & 0 & -q_{24}^* \\ -q_{20}^* & 0 & 1\ 0 & 0 & \\ -q_{30}^* & -q_{31}^* & 0 & 1 & 0 \\ 0 & -q_{41}^* & -q_{42}^* & 0 & 1 \end{vmatrix}$$

$= [1 - q_{03}^* q_{30}^* - (q_{01}^* + q_{03}^* q_{31}^*)(q_{12}^* + q_{14}^* q_{42}^*)q_{20}^*]$

Taking the inverse Laplace transform of expression (T), we may get the reliability of the system when system initially starts from state S_0. To obtain the MTSF, we use the following formula

$E[T_0] = \int_0^{\infty} R_0(t)dt = \lim_{s \to 0} R_0^*(s)$

Using $q_{ij}^*(0) = p_{ij}$ and $z_i^*(0) = \mu_i$, we get

$$E[T] = \frac{\mu_0 + (p_{01} + p_{03}p_{31})\mu_1 + (1 - p_{03}p_{30})[(p_{12} + p_{14}p_{42})\mu_2 + p_{14}\mu_4] + p_{03}\mu_3}{(p_{12} + p_{14}p_{42})p_{20}} \quad (8)$$

Expected Up Time Of The System During (0,t)

In order to find the expected up time of the system due to the operation of a unit, we define, $A_i(t)$ as the probability that the system is up (operative) at epoch t when initially the time starts from state $S_i \in E$. Using the similar probabilistic arguments as in section 2.6 of paper-II, pointwise availability $A_i(t)$ are seen to satisfy the following relations.

$$\begin{aligned}
A_0(t) &= z_0(t) + q_{01}(t) \copyright A_1(t) + q_{03}(t) \copyright A_3(t) \\
A_1(t) &= z_1(t) + q_{12}(t) \copyright A_2(t) + q_{14}(t) \copyright A_4(t) + q_{15}(t) \copyright A_5(t) \\
A_2(t) &= z_2(t) + q_{20}(t) \copyright A_0(t) + q_{22}^{(6)}(t) \copyright A_2(t) \\
A_3(t) &= z_3(t) + q_{30}(t) \copyright A_0(t) + q_{31}(t) \copyright A_1(t) \\
A_4(t) &= z_4(t) + q_{41}(t) \copyright A_1(t) + q_{42}(t) \copyright A_2(t) + q_{45}(t) \copyright A_5(t) \\
A_5(t) &= q_{56}(t) \copyright A_6(t) \\
A_6(t) &= q_{62}(t) \copyright A_2(t)
\end{aligned} \quad (1\text{-}7)$$

Taking Laplace transform of the set of relations (1-7), the solution for $A_i^*(s)$ can be written in the matrix form as follows :

$(A_0^*, A_1^*, A_2^*, A_3^*, A_4^*, A_5^*, A_6^*)' = (q_{ij}^*)' = (z_0^*, z_1^*, z_2^*, z_3^*, z_4^*, 0,0)'$, (8)

$$(q_{ij}^*) = \begin{pmatrix} 1 & -q_{01}^* & 0 & -q_{03}^* & 0 & 0 & 0 \\ 0 & 1 & -q_{12}^* & 0 & -q_{14}^* & -q_{15}^* & 0 \\ -q_{20}^* & 0 & 1-q_{22}^{(6)*} & 0 & 0 & 0 & 0 \\ -q_{30}^* & -q_{31}^* & 0 & 1 & 0 & 0 & 0 \\ 0 & -q_{41}^* & -q_{42}^* & 0 & 1 & -q_{45}^* & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -q_{56}^* \\ 0 & 0 & -q_{62}^* & 0 & 0 & 0 & 1 \end{pmatrix}$$

For brevity, the argument 's' is omitted from $-q_{ij}^*(s)$, $A_i^*(s)$ and $z_i^*(s)$. Solving the matrix equation (8) for $A_0^*(s)$, we get

$A_0^*(s) = N_2(s)/D_2(s)$, (9)

$[(q_{12}^* + q_{15}^* + q_{14}^*q_{45}^*)q_{56}^*q_{62}^* + q_{14}^*q_{42}^*]$ (10)

and

$D_2(s) = \ (q_{ij}^*)$ (11)

$= \left(1 - q_{22}^{(6)*}\right)(1 - q_{03}^*q_{30}^*)(1 - q_{14}^*q_{41}^*) - (q_{01}^* + q_{03}^*q_{31}^*)q_{20}^*$

$\{q_{12}^* + (q_{15}^* + q_{14}^*q_{45}^*)q_{56}^*q_{62}^* + q_{14}^*q_{42}^*\}$ (12)

Now to obtain the steady-state probabilities that the system will be operative, we proceed as follows:

$z_i^*(0) = \int_0^{\infty} z_i(t)dt = \mu_i$, $i = 0,1,2,3,4$.

and using the result $q_{ij}^*(0) = p_{ij}$, we have

$$D_2(0) = \begin{vmatrix} 1 & -p_{01} & 0 & -p_{03} & 0 & 0 & 0 \\ 0 & 1 & -p_{12} & 0 & -p_{14} & -p_{15} & 0 \\ -p_{20} & 0 & 1-p_{22}^{(6)} & 0 & 0 & 0 & 0 \\ -p_{30} & -p_{31} & 0 & 1 & 0 & 0 & 0 \\ 0 & -p_{41} & -p_{42} & 0 & 1 & -p_{45} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -p_{56} \\ 0 & 0 & -p_{62} & 0 & 0 & 0 & 1 \end{vmatrix}$$

Adding all the columns of the above determinant into first column and using the relations (15-20) of section 3.4, we observe that all the elements of first column of $D_2(0)$ become zero. Therefore,

$D_2(0) = 0$ (13)

Now the steady state probabilities that the system will be operative due to operation of a unit is given as follows:

$$A_0 = \lim_{s\to 0} s.A_0^*(s) = \lim_{s\to 0} s.\frac{N_2(s)}{D_2(s)}$$

$$= \frac{N_2(0)}{D_2'(0)}, (14)$$

where

$$N_2(0) = p_{20}(1-p_{14}p_{41})(\mu_0 + p_{03}\mu_3) + p_{20}(p_{01}+p_{03}p_{31})(\mu_1 + p_{14}\mu_4) + (p_{01}+p_{03}p_{31})(1-p_{14}p_{41})\mu_2 \quad (15)$$

To find $D_2'(0)$, we collect the coefficients of $m_{ij} = -q_{ij}^{*'}(0)$ for various values of i and j in $D_2'(s)\big|_{s=0}$ as follows:

Coefficient of m_{01}

$$=(-1)\begin{vmatrix} 0 & -p_{12} & 0 & -p_{14} & -p_{15} & 0 \\ -p_{20} & 1-p_{22}^{(6)} & 0 & 0 & 0 & 0 \\ -p_{30} & 0 & 1 & 0 & 0 & 0 \\ 0 & -p_{42} & 0 & 1 & -p_{45} & 0 \\ 0 & 0 & 0 & 0 & 1 & -p_{56} \\ 0 & -p_{62} & 0 & 0 & 0 & 1 \end{vmatrix}$$

Adding all the columns into first column, we get

$$=(-1)\begin{vmatrix} -1 & -p_{12} & 0 & -p_{14} & -p_{15} & 0 \\ 0 & 1-p_{22}^{(6)} & 0 & 0 & 0 & 0 \\ p_{31} & 0 & 1 & 0 & 0 & 0 \\ p_{41} & -p_{42} & 0 & 1 & -p_{45} & 0 \\ 0 & 0 & 0 & 0 & 1 & -p_{56} \\ 0 & -p_{62} & 0 & 0 & 0 & 1 \end{vmatrix}$$

(ii) coefficient of m_{03}

$= p_{20}(1-p_{14}p_{41})$

Similarly, we have

(iii) coefficient of $m_{12} = p_{20}(p_{01}+p_{03}p_{31}) = p_{20}(1-p_{03}p_{30})$

(iv) coefficient of $m_{14} = p_{20}(1-p_{03}p_{30})$

(v) coefficient of $m_{15} = p_{20}(p_{01}+p_{03}p_{31}) = p_{20}(1-p_{03}p_{30})$

(vi) coefficient of $m_{20} = (p_{01}+p_{03}p_{31})\{p_{12} + (p_{15}+p_{14}p_{45})p_{56}p_{62} + p_{14}p_{42}\}$

$= (p_{01}+p_{03}p_{31})\{1 - p_{14} + p_{14}p_{45} + p_{14}p_{42}$

$= (1-p_{03}p_{30})(1-p_{14}p_{41})$

(vii) coefficient of $m_{22}^{(6)} = (1-p_{03}p_{30})(1-p_{14}p_{34})$

(viii) coefficient of $m_{30} = p_{03}p_{20}(1-p_{14}p_{34})$

(ix) coefficient of $m_{31} = p_{20}p_{03}\{p_{12} + (p_{15}+p_{14}p_{45})p_{56}p_{62} + p_{14}p_{42}\}$

$= p_{03}p_{20}(1-p_{14}p_{41})$

(x) coefficient of $m_{41} = p_{20}(1-p_{03}p_{30})p_{14}$

(xi) coefficient of $m_{42} = (p_{01}+p_{03}p_{31})p_{20}p_{14}$

$= p_{20}p_{14}(1-p_{03}p_{30})$

(xii) coefficient of $m_{45} = p_{20}(p_{01}+p_{03}p_{31})p_{14}$

$= p_{20}p_{14}(1-p_{03}p_{30})$

(xiii) coefficient of $m_{56} = p_{20}(p_{01}+p_{03}p_{31})(p_{15}+p_{14}p_{45})p_{62}$

$= p_{20}(1-p_{03}p_{30})(p_{15}+p_{14}p_{45})$

(xiv) coefficient of $m_{62} = p_{20}(p_{01} + p_{03}p_{31})(p_{15} + p_{14}p_{45})p_{56}$
$= p_{20}(1 - p_{03}p_{30})(p_{15} + p_{14}p_{45})$

Therefore, we have

$$D_2'(0) = p_{20}(1 - p_{14}p_{41})(m_{01} + m_{03}) + p_{20}(1 - p_{03}p_{30})(m_{12} + m_{14}m_{15}) + (1 - p_{03}p_{30})(1 - p_{14}p_{41})(m_{20} + m_{22}^{(6)}) + p_{20}p_{03}(1 - p_{14}p_{41})(m_{30} + m_{31}) + p_{20}p_{14}(1 - p_{03}p_{30})(m_{41} + m_{42} + m_{45}) + p_{20}(1 - p_{03}p_{30})(p_{15} + p_{14}p_{45})m_{56} + p_{20}(1 - p_{03}p_{30})(p_{15} + p_{14}p_{45})m_{62}$$
$$= p_{20}(1 - p_{14}p_{41})(\mu_0 + p_{03}\mu_3) + p_{20}(1 - p_{03}p_{30})(\mu_1 + p_{14}\mu_4) p_{20}(1 - p_{03}p_{30})(p_{15} + p_{14}p_{45})(\mu_5 + \mu_6) + p_{20}(1 - p_{03}p_{30})(1 - p_{14}p_{41})\mu_6 \quad (16)$$

The expected uptime of the system due to the operations of a unit during (0,t) is

$$\mu_{up}(t) = \int_0^t A_0(u)du \quad (17)$$

so that

$$\mu_{up}^*(s) = A_0^*(s)/s \quad (18)$$

Expected Busy Period Of The Repair And Maintenance Facilities During (0,t)

Let $B_i^1(t)$ and $B_i^2(t)$ be the probabilities that the maintenance and repair facilities are busy respectively in maintenance and repair of operative/failed unit at time t when the system initially starts from regenerative state S_i. Using the similar probabilistic arguments as in section 2.7, in respect to the above definition of $B_i^1(t)$, we have the following recursive relations:

$$B_0^1(t) = q_{01}(t) © B_1^1(t) + q_{03}(t) © B_3^1(t)$$
$$B_1^1(t) = q_{12}(t) © B_2^1(t) + q_{14}(t) © B_4^1(t) + q_{15}(t) © B_5^1(t)$$
$$B_2^1(t) = q_{20}(t) © B_0^1(t) + q_{22}^{(6)}(t) © B_2^1(t)$$
$$B_3^1(t) = z_3(t) + q_{30}(t) © B_0^1(t) + q_{31}(t) © B_1^1(t)$$
$$B_4^1(t) = z_4(t) + q_{41}(t) © B_1^1(t) + q_{42}(t) © B_2^1(t) + q_{45}(t) © B_5^1(t)$$
$$B_5^1(t) = q_{56}(t) © B_6^1(t)$$
$$B_6^1(t) = q_{62}(t) © B_2^1(t) \quad (1\text{-}7)$$

For an illustration$B_3^1(t)$ is the sum of the following mutually exclusive contingencies:

(i) The maintenance facility remains busy in state S_3 upto time t in the maintenance of a unit. The probability of this event is $e^{-\alpha t}\bar{G}(t) = z_3(t)$

(ii) The system transits from state S_3 to S_0 during (u,u+du), u≤t and then the maintenance facility remains busy in maintenance at epoch t starting from state S_0 at epoch u. The probability of this contingency is

$\int_0^t q_{30}(u)du\, B_0^1(t-u)= q_{30}(t) © B_0^1(t)$

(iii) System transits from state S_3 to S_1 during (u,u+du), u≤t and then starting from state S_1 at epoch u, the maintenance facility may be observed to be busy in maintenance at epoch t. The probability of this contingency is

$\int_0^t q_{31}(u)du\, B_1^1(t-u)= q_{31}(t) © B_1^1(t)$

Taking Laplace transforms of relations (1-7) and solving the resulting set of algebraic equations for $B_0^{1*}(s)$, we get

$$B_0^{1*}(s) = \frac{N_3(s)}{D_2(s)},$$

where

$$N_3(s) = \begin{vmatrix} 0 & -q_{01}^* & 0 & -q_{03}^* & 0 & 0 & 0 \\ 0 & 1 & -q_{12}^* & 0 & -q_{14}^* & -q_{15}^* & 0 \\ 0 & 0 & 1-q_{22}^{(6)*} & 0 & 0 & 0 & 0 \\ z_3^* & -q_{31}^* & 0 & 1 & 0 & 0 & 0 \\ z_4^* & -q_{41}^* & 0 & 0 & 1 & -q_{45}^* & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -q_{56}^* \\ 0 & 0 & -q_{62}^* & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= (1 - q_{22}^{(6)*})[q_{03}^* z_3^* + (q_{01}^* + q_{03}^* q_{31}^*)q_{14}^* z_4^*] \quad (9)$$

and $D_2(s)$ is the same as (12) of section 3.6

Similarly, the recursive relations in $B_i^2(t)$ may be developed as follows:

$$B_0^2(t) = q_{01}(t) © B_1^2(t) + q_{03}(t) © B_3^2(t)$$

$B_1^2(t) = q_{12}(t) © B_2^2(t) + q_{14}(t) © B_4^2(t) + q_{15}(t) © B_5^2(t)$
$B_2^2(t) = z_2(t) + q_{20}(t) © B_0^2(t) + q_{22}^{(6)}(t) © B_2(t)$
$B_3^2(t) = q_{30}(t) © B_0^2(t) + q_{31}(t) © B_1^2(t)$
$B_4^2(t) = q_{41}(t) © B_1^2(t) + q_{42}(t) © B_2^2(t) + q_{45}(t) © B_5^2(t)$
$B_5^2(t) = q_{56}(t) © B_6^2(t)$
$B_6^2(t) = \bar{F}(t) + q_{62}(t) © B_2^2(t)$ (10-16)

Taking Laplace transforms of (10-16) and solving for $B_0^{2*}(s)$, one gets
$B_0^{2*}(s) = \frac{N_4(s)}{D_4(s)}$, where
$= (q_{01}^* + q_{03}^* q_{31}^*)[\{(q_{12}^* + q_{14}^* q_{42}^* + (q_{15}^* + q_{14}^* q_{45}^*) q_{56}^* q_{62}^*\} z_2^* + \bar{F}^* \, (q_{15}^* + q_{14}^* q_{45}^*) \, q_{56}^*$ (18)
and $D_2(s)$ is the same as (12) of section 3.6

Now to obtain the steady-state probabilities $B_0^1(t)$ and $B_0^2(t)$ the maintenance and repair facilities are busy in the long run, respectively, we use the results.
$q_{ij}^*(0) = p_{ij}$, $z_i^*(0) = \mu_i$
and

$$\bar{F}^*(0) = \frac{1-\tilde{F}(s)}{s} \parallel_{s=0} = \frac{1-f^*(s)}{s} \parallel_{s=0} = \int_0^t t dF(t) = n_1 = \mu_6$$

So that

$$B_0^1 = \frac{N_3(0)}{D_2'(0)}$$

and

$$B_0^2 = \frac{N_4(0)}{D_2'(0)}$$

(19-20)

where $D_2'(0)$ is given by (16) of section 3.6,
$N_3(0) = p_{20}[p_{03}\mu_3 + (p_{01} + p_{03}p_{31})p_{14}\mu_4]$
and
$N_4(0) = (1 - p_{03}p_{30})(1 - p_{14}p_{41})\mu_2 + (p_{15} + p_{14}p_{45})n_1$

Profit Function Analysis

We are now in a position to obtain the profit function by considering mean up time of the system during (0,t) and expected busy periods of maintenance and repair facilities during (0,t)
Let us suppose
K_0= Revenue earned per unit uptime due to the operation of the system
K_1= payment to maintenance facility per unit time when it is busy in preventive maintenance
K_2= payment to repair facility when it is busy in repair of a unit.
The net expected total profit per unit time in steady-state is

$$P = \lim_{t\to\infty} \frac{P(t)}{t} = \lim_{s\to 0} s^2 P^*(s)$$
$$= K_0 A_0 - K_1 B_0^1 - K_2 B_0^2 ,$$

is given by (16) of section 3.6 and

where A_0 , B_0^1 and B_0^2 are given by expressions (19-20) of section 3.7 respectively.

Particular Case

When repair time, maintenance time and inter-arrival time of maintenance facility are also exponential with parameters β, γ and δ respectively, then
$F(t) = 1 - e^{-\beta t}$, $G(t) = 1 - e^{-\gamma t}$ and $H(t) = 1 - e^{-\delta t}$
$n_1 = 1/\beta$, $n_2 = 1/\gamma$ and $n_3 = 1/\delta$

The results of sec. 3.4b will change and new values will be as follows:

$$p_{01} = \frac{\alpha}{\alpha+\delta}, p_{03} = \frac{\delta}{\alpha+\delta}$$
$$p_{12} = \theta/(\alpha+\theta+\delta), p_{14} = \frac{\delta}{\alpha+\theta+\delta}$$
$$p_{15} = \alpha/(\alpha+\theta+\delta), p_{20} = \frac{\beta}{(\alpha+\beta)}$$
$$p_{22}^{(6)} = \alpha/(\alpha+\beta), p_{30} = \gamma/(\alpha+\gamma)$$

$p_{31} = \alpha\ (\alpha + \gamma)\ , p_{41} = \gamma/(\alpha + \theta)$
$p_{42} = \theta/(\alpha + \theta + \gamma)\ , p_{45} = \alpha/(\alpha + \theta + \gamma)$

Similarly the results of section 3.4 c will be as follows:
$\mu_0 = 1/(\alpha + \delta)\ , \mu_1 = 1/(\alpha + \theta + \delta)$
$\mu_2 = 1/(\alpha + \beta)\ , \mu_3 = 1/(\alpha + \gamma),\ \mu_4 = 1/(\alpha + \theta + \gamma)$

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