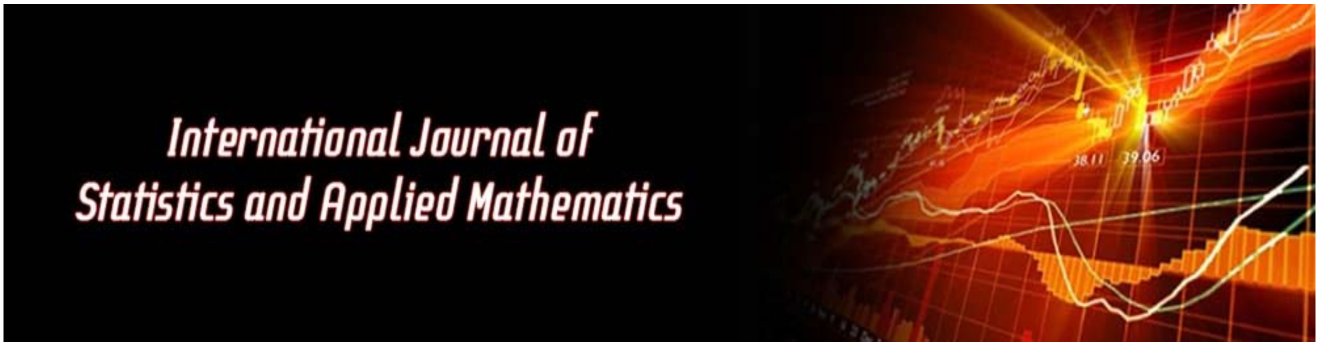

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A generalized Esscher transformed Laplace distribution and its application

Rimsha H and Dais George

Abstract
In this article, we introduce a new family of distribution namely, Marshall-Olkin generalized Esscher transformed Laplace distribution which is a generalization of the three parameter Esscher transformed is also Laplace distribution. We obtain explicit forms for their density, distribution function and hazard function. Properties of the distribution are studied and the parameters are estimated. A real application of the distribution considered.

Keywords: Esscher transformed Laplace distribution, Marshall-Olkin generalized Esscher transformed Laplace distribution, Maximum likelihood method of estimation, Moment method of estimation

Introduction

Many generalizations of the various probability distributions that can be used to better reflect the distribution of the original data sets from diverse fields such as biomedical sciences, climatology, environmental, financial, image processing, signal processing and telecommunications are introduced so far and one among them is that introduced by Marshall and Olkin (1997) [7]. It is a method of including an extra shape parameter to a given baseline model so that we obtain the Marshall-Olkin generalization of that distribution. The recently introduced univariate istributions belonging to the Marshall-Olkin family of distributions are Marshall-Olkin Weibull Ghitany *et al*. (2005) [3], Marshall-Olkin q-Weibull Jose *et al*. (2010) [4] bivariate Marshall-Olkin Weibull Jose *et al*. (2011) [5], Marshall-Olkin ETL Dais George and Sebastian George (2013) [2], Marshall-Olkin Morgenstern Weibull Jose *et al*. (2013), Marshall-Olkin extended Weibull family Manoel Santos-Neto *et al*. (2014) [8] and MarshallOlkin bivariate Weibull distribution Chin Diew Lai *et al*. (2017) [1].

In this paper, we introduce Marshall-Olkin generalized Esscher transformed Laplace distribution which is a generalization of the three parameter Esscher transformed Laplace distribution denoted by ETL (θ,μ,σ). Three parameter Esscher transformed Laplace distribution is the location scale family of the one parameter Esscher transformed Laplace distribution introduced in S. George and D. George (2012). The probability density function and distribution function of the three parameter Esscher transformed Laplace distribution are

$$f(x;\theta,\mu,\boldsymbol{\sigma}) = \begin{cases} \frac{(1-\theta^2)}{2\sigma} e^{[(\frac{x-\mu}{\sigma})(1+\theta)]}, x < \mu \\ \frac{(1-\theta^2)}{2\sigma} e^{[(\frac{\mu-x}{\sigma})(1-\theta)]}, x \geq \mu, \end{cases} \quad |\theta| < 1, \sigma > 0, (1) \text{ and}$$

$$F(x;\theta,\mu,\boldsymbol{\sigma}) = \begin{cases} \frac{(1-\theta)}{2} e^{[(\frac{x-\mu}{\sigma})(1+\theta)]}, x < \mu \\ 1 - \frac{(1+\theta)}{2} e^{[(\frac{\mu-x}{\sigma})(1-\theta)]}, x \geq \mu \end{cases} \quad |\theta| < 1, \sigma > 0 \text{ . } (2)$$

The characteristic function of the distribution is

$$\Phi_X(t) = \frac{e^{it\mu}}{1 - \frac{2it\theta\sigma^2}{(1-\theta^2)} + \frac{t^2\sigma^2}{(1-\theta^2)}} .$$

Being heavy-tailed, this distribution is an alternative to various heavy-tailed distributions.

It is infinite divisible, geometric infinite divisible and self-decomposable.

The rest of the paper is organized as follows. Section 2 presents the cdf and pdf of the proposed distribution and its main statistical properties. In Section 3, we estimate the parameters of the distribution by the method of maximum likelihood and by method of moments. A real application of the distribution in remission times of bladder cancer patients is discussed in Section 4. Finally we conclude the paper by Section 5.

Marshall-Olkin Generalized Esscher Transformed Laplace Distribution

According to the method proposed by Marshall and Olkin (1997), if Φ(t) is a characteristic function of some arbitrary distribution, then $\Psi(t) = \frac{\beta\Phi(t)}{1-(1-\beta)\Phi(t)}$, $\beta > 0$ (3) forms another class of characteristic functions. So if X is a random variable with characteristic function $\Phi_X(t) = \frac{e^{it\mu}}{1-\frac{2it\theta\sigma^2}{(1-\theta^2)}+\frac{t^2\sigma^2}{(1-\theta^2)}}$ we get a new family of distributions using (3) which we shall refer to Marshall-Olkin generalized Esscher transformed Laplace [MOGETL] distribution.

The characteristic function of the distribution is, $\Psi(t) = \frac{1}{1+\frac{1}{\beta e^{it\mu}}\left[1-\frac{2it\theta\sigma}{1-\theta^2}+\frac{t^2\sigma^2}{1-\theta^2}-e^{it\mu}\right]}$ and a random variable X with this characteristic function is distributed as MOGETL (β,θ,μ,σ). The probability density function and cumulative distribution of MOGETL distributions are respectively, $f(x;\theta,\mu,\boldsymbol{\sigma}) = \frac{\sqrt{\beta e^{i\mu t}}}{2\sigma^2}\begin{cases}\frac{(1-\theta^2)}{2\sigma}e^{[(\frac{x-\mu}{\sigma^2})(1+\theta)\sqrt{\beta e^{i\mu t}}]}, x<\mu\\ \frac{(1-\theta^2)}{2\sigma}e^{[(\frac{\mu-x}{\sigma^2})(1-\theta)\sqrt{\beta e^{i\mu t}}]}, x\ge\mu,\end{cases}$ $|\theta|<1, \sigma>0,$ (4) and

$$F(x;\theta,\mu,\boldsymbol{\sigma}) = \begin{cases}\frac{(1-\theta)}{2}e^{[(\frac{x-\mu}{\sigma^2})(1+\theta)\sqrt{\beta e^{i\mu t}}]}, x<\mu\\ 1-\frac{(1+\theta)}{2}e^{[(\frac{\mu-x}{\sigma^2})(1-\theta)\sqrt{\beta e^{i\mu t}}]}, x\ge\mu,\end{cases} \quad |\theta|<1, \sigma>0, (5)$$

where $\theta = \frac{2\sigma\sqrt{\beta e^{i\mu t}}}{\mu+\sqrt{4\sigma^2\beta e^{it\mu}+\mu^2}}$.

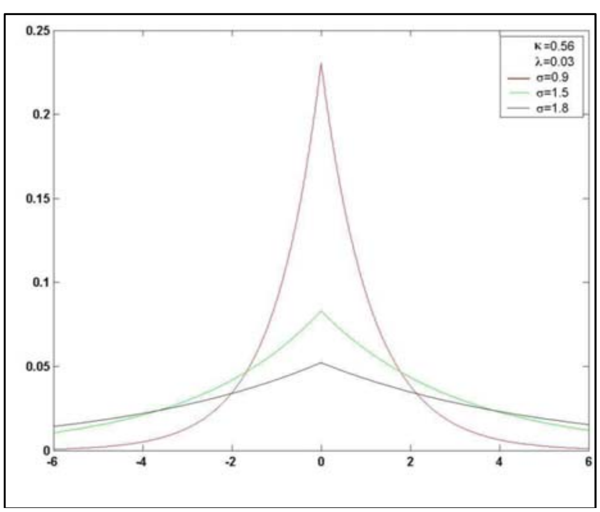
Now we consider the Marshall-Olkin generalized Esscher transformed Laplace distribution with location at origin. Then the pdf and d.f of the MOGETL (β,θ,σ) distribution are $f(x;\beta,\theta,\boldsymbol{\sigma}) = \frac{\sqrt{\beta}}{2\sigma^2}(1-\theta^2)\begin{cases}e^{[(\frac{x}{\sigma}(1+\theta)\sqrt{\beta})]}, x<0\\ e^{[(\frac{-x}{\sigma^2})(1-\theta)\sqrt{\beta}]}, x\ge 0,\end{cases}$ $|\theta|<1, \sigma>0,$ (6) and

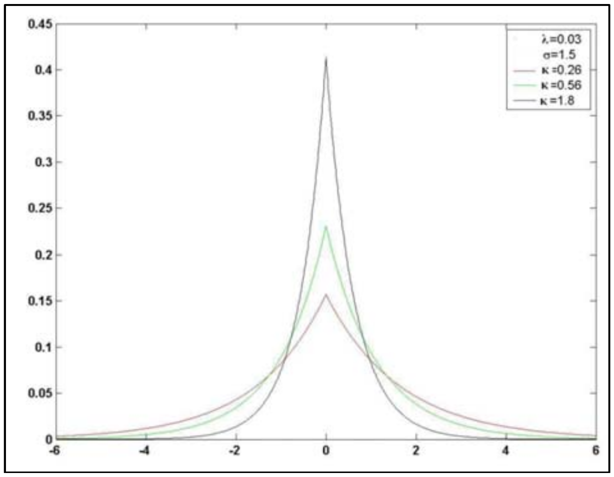
$$F(x;\beta,\theta,\boldsymbol{\sigma}) = \begin{cases}\frac{(1-\theta)}{2}e^{[(\frac{x}{\sigma})(1+\theta)\sqrt{\beta}]}, x<0\\ 1-\frac{(1+\theta)}{2}e^{[(\frac{-x}{\sigma})(1-\theta)\sqrt{\beta}]}, x\ge 0,\end{cases} \quad |\theta|<1, \sigma>0,$$

We re-parameterize the distribution given in (6), by putting $\sqrt{\beta(1-\theta^2)} = \lambda$ and $\frac{\sqrt{1-\theta}}{\sqrt{1+\theta}} = k$ so that the re-parameterized model MOGET(κ,λ,σ) is given by, $f(x) = \frac{\lambda}{2\sigma^2}\frac{\kappa}{1+\kappa^2}\begin{cases}e^{\frac{x\lambda}{\kappa\sigma^2}}, x<0\\ e^{\frac{-x\lambda\kappa}{\sigma^2}}, x\ge 0,\end{cases}$ $|\theta|<1, \sigma>0,$ (7) and

$$F(x) = \begin{cases}\frac{\kappa^2}{1+\kappa^2}e^{\frac{x\lambda}{\kappa\sigma^2}}, x<0\\ 1-\frac{1}{1+\kappa^2}e^{\frac{-x\kappa\lambda}{\sigma^2}}, x\ge 0,\end{cases} \quad |\theta|<1, >0 .$$

The density plots of this distribution are given in Figure 2.2.





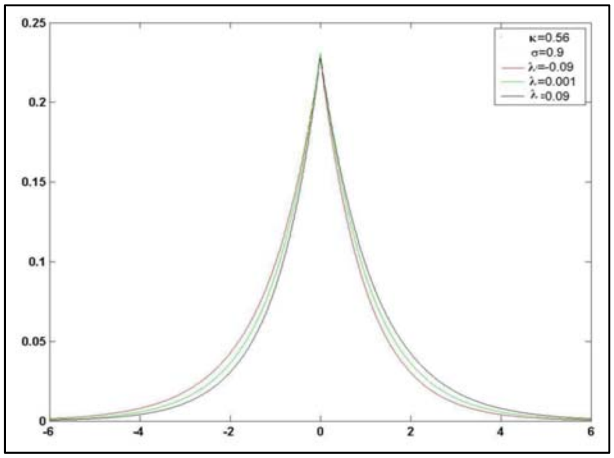


Fig 2.1: Density Plots of Marshall Olkin Generalized Esscher Transformed Laplace Distribution

Densities of Marshall Olkin Generalized Esscher transformed Laplace distribution for(a) κ = 0.56;λ = 0.03 and σ= 0.9, 1.5, 1.8 (b) λ=-0.03, σ = 1.5 and κ = 0.26, 0.56, 1.8 (c) κ = 0.56, λ = 1.5 and σ = -0.09, 0.001, 0.09.

It is clear that the distribution is unimodal with mode equal to zero and we can notice characteristic peakedness of the density at zero. The nth arbitrary moment of MOGETL distribution is given by E(x^n) =n! $\left(\frac{\sigma^2}{\kappa\lambda}\right)^n \frac{1+(-1)^n\ \kappa^{2(n+1)}}{1+\kappa^2}$.

The mean and variance are Mean = $\frac{(1+\kappa^2)\sigma^2}{\kappa\lambda}$ and Variance = $\frac{(1+\kappa^4)(\sigma^2)^2}{\kappa^2\lambda^2}$.

Estimation of Parameters
In this section, for estimating the parameters, we use the method of maximum likelihood and method of moments.

Maximum Likelihood Estimation
For the easiness of the estimation process, we re-parameterize the distribution given in (7), by putting $\kappa^2 = \frac{\eta}{\delta}$ and $\lambda^2 = \eta\delta$ so that the re-parameterized model, MOGETL (η, δ, σ) is given by f(x,η, δ, σ) =$\frac{1}{\sigma^2}\ \frac{\eta\delta}{\eta+\delta}\begin{cases} e^{\frac{x\delta}{\sigma^2}}, x < 0 \\ e^{\frac{-x\eta}{\sigma^2}}, x \geq 0, \end{cases}$ $|\theta| < 1, \sigma > 0,$ (8) where $\eta > 0$ and δ> 0 are the model parameters. Let D = (X_1,X_2,…,Xn) where X_i's are independently and identically distributed random variables following MOGETL(η, δ, σ) distribution given by (8). The log likelihood function is obtained as LL ($\eta, \delta, \sigma \backslash D$) = -nlog σ^2+nlog $\left(\frac{\eta\delta}{\eta+\delta}\right)$ +$\sum_{(x\epsilon D\backslash xi<0)}\left(\frac{\delta xi}{\sigma^2}\right)$+$\sum_{(x\epsilon D\backslash xi>0)}\left(\frac{\eta xi}{\sigma^2}\right)$, = -nlog σ^2+nlog $\left(\frac{\eta\delta}{\eta+\delta}\right)$ +$\frac{\delta}{\sigma^2}$ S_l+ $S_r\frac{\eta}{\sigma^2}$, where S_l =$\sum_{(x\epsilon D\backslash xi<0)} x_i$ and S_r =$\sum_{(x\epsilon D\backslash xi>0)} x_i$.First by fixing σ and solving the equations formed by equating the partial derivatives of the log likelihood function with respect to η and δ to zero, we obtain the maximum likelihood estimates of δ and η and then estimating σ by iteration.

The ML estimates of η and δ are $\hat{\delta}$ =$\frac{n\sigma^2}{-Sl+\sqrt{(-Sl)Sr}}$ and $\hat{\eta}$ =$\frac{n\sigma^2}{-Sr+\sqrt{(-Sl)Sr}}$.

Method of Moments
By equating the population and sample moments obtained from (6), we get the moment estimates of β, θ and σ are

$$\hat{\theta}=\frac{m_2'-(m_1')^2}{m_2'+(m_1')^2}, \hat{\beta}=\frac{4\theta^2\sigma^2}{(1-\theta^2)^{\frac{3}{2}}m_1'} \text{ and } \hat{\sigma}=\sqrt{\frac{\left(\frac{2\theta\sqrt{1-\theta}}{(1-\theta)^{\frac{3}{2}}}\right)m_2'}{\left(\frac{(1+\theta)^2-(1-\theta)^2}{(1+\theta)^2}\ m_2'\right)\ m_1'}}.$$

Data Analysis

Now we consider the real data application of MOGETL (β, θ, σ) distribution. In this study we use a secondary data, which is the ordered remission times (in months) of a random sample of 142 bladder cancer patients reported in Lee and Wang (2003). We draw the histogram of the standardized observed data and it is shown in Figure 4.1.

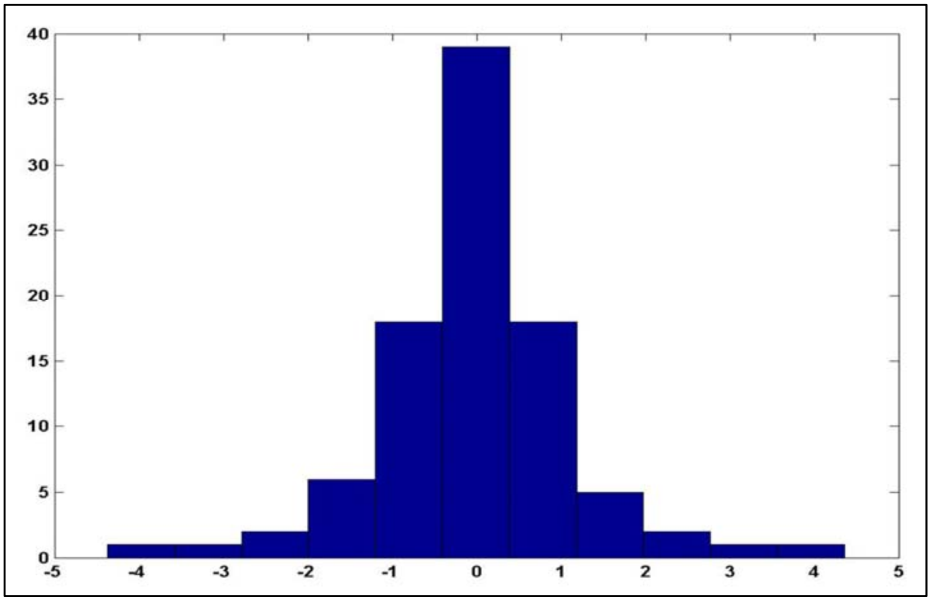


Fig 4.1: Histogram of the observed data

The figure resembles the shape of the graph of MOGETL distribution given in Figure 2.1. We estimate the parameters of the distribution from the data and it is obtained as $\hat{\beta}$= 1.68762, $\hat{\theta}$ = 0.79325 and $\hat{\sigma}$= 1.26501. The frequency curve of the distribution is superimposed in the histogram and is presented in Figure 4.2.

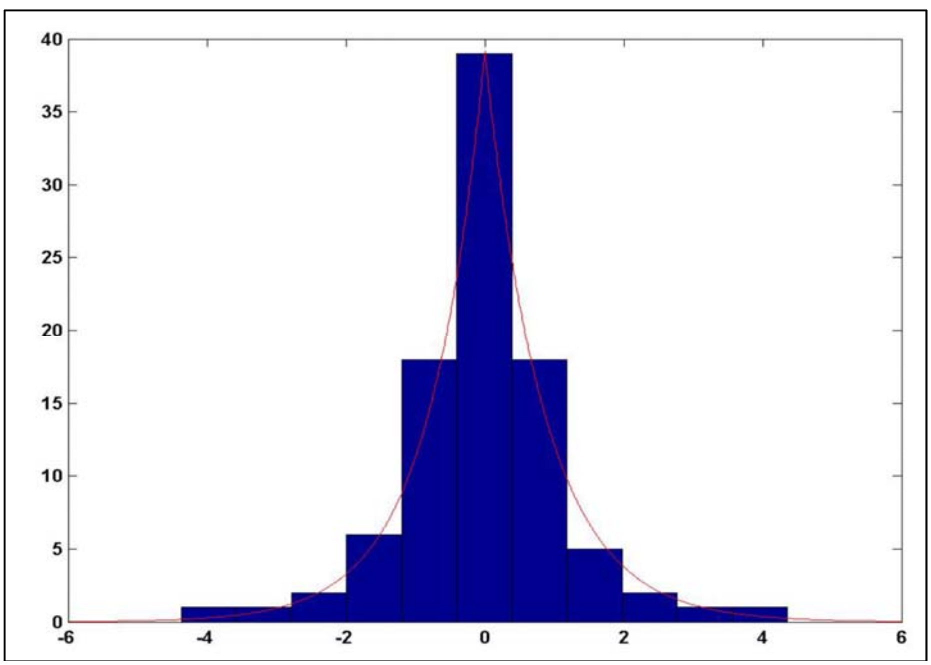


Fig 4.2: Embedded Frequency Polygon of the Observed Data

From the Figure it is clear that the Marshall-Olkin generalized Esscher transformed Laplace distribution is a good fit for this survival data. Also we calculate the value of K-S statistic and it is obtained as 0.049074. So we can conclude that MOGETL distribution is suitable for modeling the data on remission times of bladder cancer patients.

Conclusion

In this paper, we introduced the Marshall-Olkin generalized Esscher transformed Laplace distribution and studied its properties. We fit the distribution to the real data on remission times of bladder cancer patients and proved that it is good model. Moreover, since this model provides more over flexibility for various values of β, allowing for asymmetry, peakedness and tail heaviness it can be an alternative to various asymmetric and heavy-tailed distributions. Also by adjusting β, it can even be used for modeling left heavy-tailed and right heavy-tailed distributions.

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