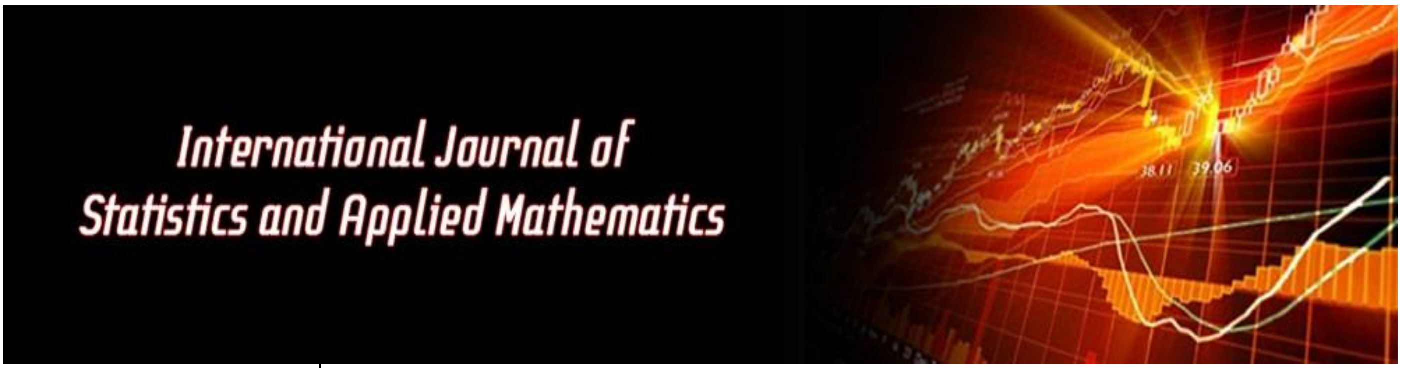

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Suzanne A Allam
Institute of Statistical Studies and Research, Cairo University, Egypt

Soha M Othman
Institute of Statistical Studies and Research, Cairo University, Egypt

Hesham M Reyad
College of Business and Economics, Qassim University, Saudi Arabia

Ahmed M Hashish
College of Business and Economics, Qassim University, Saudi Arabia

Correspondence
Suzanne A Allam
Institute of Statistical Studies and Research, Cairo University, Egypt

Parameter estimation for the Exponentiated Weibull distribution: Comparative study

Suzanne A Allam, Soha M Othman, Hesham M Reyad and Ahmed M Hashish

Abstract
For the estimation of probability distribution parameters, Haktanir (1997) [4] introduced the method of self-determined probability weighted moments as a refinement on the original method of probability weighted moments defined by Greenwood *et al* (1979) [2]. In this study, simulation experiments were conducted to investigate the performance of the self- determined probability weighted moments method versus the performances of the methods of maximum likelihood and moments in estimating the parameters of the Exponentiated Weibull distribution.

Keywords: Self-determined probability weighted moments, Exponentiated Weibull distribution, the method of maximum likelihood, the method of moments

1. Introduction

In most statistical studies the population parameters are unknown and must be estimated from a sample. The maximum likelihood (ML) method is an important method of estimation, whose justification is based on large sample theory; whereas the method of moments (MM) is mostly used because of its relative ease of application. Since the ML method does not always work well in small samples, other methods of estimation were developed as alternatives. Among these is the probability weighted moments (PWM) method which was devised by Greenwood *et al* (1979) [2]. This method works very well with small samples. In addition, it was found to posses some desirable statistical properties, it is also computationally simple and robust.

Another method of estimation is the method of self-determined probability weighted moments (SDPWM) which was proposed by Haktanir (1997) [4] as a refinement on the original method of PWM. It has many advantages over the method of PWM, such as the ability to directly consider outliers and the ability to detect whether a particular distribution is appropriate for describing a sample.

The Exponentiated Weibull distribution was introduced by Mudholkar and Srivastava (1993) as an extension of the Weibull distribution by adding a second shape parameter. The cumulative distribution function for the Exponentiated Weibull distribution is:

$$F(x;\alpha,\lambda,\mathrm{k}) = \left[1 - e^{-\left(x/\lambda\right)^{k}}\right]^{\alpha}; \quad \alpha,\lambda,\mathrm{k} > 0 \tag{1.1}$$

for $x > 0$ and 0 otherwise. The corresponding density function is:

$$f(x;\alpha,\lambda,k) = \frac{\alpha k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\left(1 - e^{-\left(x/\lambda\right)^{k}}\right)^{\alpha-1} e^{-\left(x/\lambda\right)^{k}}; \ \alpha,\lambda,\mathrm{k} > 0 \tag{1.2}$$

for $x > 0$ and 0 otherwise.

Here α and k are shape parameters and λ is a scale parameter. If the shape parameter $\alpha = 1$, then the Exponentiated Weibull distribution coincides with the Weibull distribution with a scale parameter λ. Also, when the shape parameter k=1 then it will be the Exponentiated exponential distribution.

The main aim of this paper is to study the efficiency of the SDPWM method in estimating the parameters of the Exponentiated Weibull distribution through comparing it with the more conventional methods of estimation, which are the methods of ML and MM. In the three methods of estimation, determination of parameter estimates requires solving non-linear equations. Therefore, all estimation algorithms are implemented using the Mathcad (version 14.0) software, which contains iterative procedures that are useful for solving the aforementioned non-linear equations.
The organization of this article is as follows. A theoretical overview of the method of SDPWM is provided in section 2. In section 3, the SDPWM estimation of the Exponentiated Weibull parameters is discussed. Then, sections 4 and 5 explain the estimation of the Exponentiated Weibull parameters by the methods of ML and MM, respectively. Section 6 contains simulation results and discussions. Finally, conclusions are included in section 7. Results tables are included in the appendix.

2. Overview

The method of self-determined probability weighted moments was proposed as a modification of the original method of PWM. The probability weighted moments of a random variable *X* with cumulative distribution function $F(x) = P(X \le x)$ and inverse distribution function $x = x(F)$ are the quantities:

$$M_{p,r,s} = E[X^p F^r (1-F)^s] = \int_0^1 [x(F)]^p F^r [1-F]^s \, dF \quad , \tag{2.1}$$

where *p*, *r*, and *s* are real numbers.

The most commonly used PWM are $M_{1,0,s}$ $(s = 0,1,2,...)$ and $M_{1,r,0}$ $(r = 0,1,2,...)$; for convenience, they are re-expressed as follows:

$$\alpha_s \equiv M_{1,0,s} = \int_0^1 x(F)(1-F)^s \, dF \quad . \tag{2.2}$$

$$\beta_r \equiv M_{1,r,0} = \int_0^1 x(F) F^r \, dF \quad . \tag{2.3}$$

The two PWM sets, α_s and β_r, are linear combinations of each other. Therefore, any one of them can be used; whichever is possible; for parameter estimation without loss of generality.
To estimate the parameters of a distribution, a sample estimator of α_s or β_r is needed. Landwehr *et al* (1979a) [7] introduced unbiased estimators of α_s and β_r, where *s* and *r* are nonnegative integers, which are based on the ordered sample $x_{(1)}, x_{(2)}, ..., x_{(n)}$ from the distribution F. They are given by:

$$\hat{\alpha}_s = \frac{1}{n}\sum_{i=1}^{n} x_{(i)} \binom{n-i}{s} \Big/ \binom{n-1}{s}, \tag{2.4}$$

and

$$\hat{\beta}_r = \frac{1}{n}\sum_{i=1}^{n} x_{(i)} \binom{i-1}{r} \Big/ \binom{n-1}{r}, \tag{2.5}$$

where $\binom{n-i}{s} \Big/ \binom{n-1}{s}$ and $\binom{i-1}{r} \Big/ \binom{n-1}{r}$ are estimates for the exceedance $[1 - F(x)]$ and non-exceedance $[F(x)]$ probabilities, respectively; and $x_{(i)}$ is the *i*th observation in the ordered sample. These estimates are not based on the assumed distribution, but are based solely on the position of $x_{(i)}$ within the ordered sample.
The method of SDPWM deviates from the method of PWM in that it is assumed that the ordered sample follows a particular distribution. Accordingly, the non-exceedance probabilities are assigned using the corresponding cumulative distribution function of the assumed distribution. Therefore, the SDPWM sample estimators $\tilde{\alpha}_s$ and $\tilde{\beta}_r$ for α_s and β_r, respectively, are defined as follows:

$$\tilde{\alpha}_s = \frac{1}{n}\sum_{i=1}^{n} [1 - F(x_{(i)}; \theta_1, \theta_2, ..., \theta_k)]^s \, x_{(i)} \tag{2.6}$$

and

$$\tilde{\beta}_r = \frac{1}{n}\sum_{i=1}^{n} [F(x_{(i)}; \theta_1, \theta_2, ..., \theta_k)]^r \, x_{(i)} \ , \tag{2.7}$$

Where F is the distribution function of a given distribution; $\theta_1, \theta_2, ..., \theta_k$ are the unknown parameters of this distribution; and $x_{(i)}$ is the ith observation in the ordered sample $x_{(1)} \le x_{(2)} \le ... \le x_{(n)}$.

3. Self-Determined Probability Weighted Moments Estimators

For the Exponentiated Weibull distribution, it is more convenient to work with the PWM of the form β_r which is given by:

$$\beta_r = \int_0^1 \lambda \left[-\ln\left(1 - F^{1/\alpha}\right)\right]^{1/k} F^r dF \tag{3.1}$$

Since the Exponentiated Weibull distribution is a three-parameter distribution, then only the first three PWM; β_0, β_1 and β_2 are needed. They are given as follows:

$$\beta_0 = \int_0^1 \lambda \left[-\ln\left(1 - F^{1/\alpha}\right)\right]^{1/k} dF \quad , \tag{3.2}$$

$$\beta_1 = \int_0^1 \lambda \left[-\ln\left(1 - F^{1/\alpha}\right)\right]^{1/k} F \; dF \tag{3.3}$$

and $$\beta_2 = \int_0^1 \lambda \left[-\ln\left(1 - F^{1/\alpha}\right)\right]^{1/k} F^2 \; dF \tag{3.4}$$

In the original method of PWM, the population PWMs ($\beta_0, \beta_1, \beta_2$) are replaced with the PWM sample estimators calculated as in equation (2.5). The procedure of SDPWM estimation of the Exponentiated Weibull parameters is the same as the procedure of PWM estimation. The difference is that in the case of SDPWM estimation, the population PWMs ($\beta_0, \beta_1, \beta_2$) in equations (3.2),(3.3) and (3.4) will be replaced by the SDPWM sample estimators, $\tilde{\beta}_0$, $\tilde{\beta}_1$ and $\tilde{\beta}_2$, where:

$$\tilde{\beta}_0 = \frac{1}{n}\sum_{i=1}^{n} x_{(i)} = \bar{x} \; , \tag{3.5}$$

$$\tilde{\beta}_1 = \frac{1}{n}\sum_{i=1}^{n} \left[1 - e^{-(x/\lambda)^k}\right]^{\alpha} x_{(i)} \; , \tag{3.6}$$

and $$\tilde{\beta}_2 = \frac{1}{n}\sum_{i=1}^{n} \left[1 - e^{-(x/\lambda)^k}\right]^{2\alpha} x_{(i)} \tag{3.7}$$

Therefore, the SDPWM parameter estimators, $\hat{\alpha}_{SDPWM}$, $\hat{k}_{SDPWM}$ and $\hat{\lambda}_{SDPWM}$, can be obtained by replacing the population PWMs (β_0, β_1, β_2) in equations (3.2), (3.3) and (3.4) by the SDPWM sample estimators ($\tilde{\beta}_0$, $\tilde{\beta}_1, \tilde{\beta}_2$). Then, this will make three new equations that are:

$$\bar{x} = \int_0^1 \lambda \left[-\ln\left(1 - F^{1/\alpha}\right)\right]^{1/k} dF \tag{3.8}$$

$$\frac{1}{n}\sum_{i=1}^{n} \left[1 - e^{-(x/\lambda)^k}\right]^{\alpha} x_{(i)} = \int_0^1 \lambda \left[-\ln\left(1 - F^{1/\alpha}\right)\right]^{1/k} F \; dF \tag{3.9}$$

$$\frac{1}{n}\sum_{i=1}^{n} \left[1 - e^{-(x/\lambda)^k}\right]^{2\alpha} x_{(i)} = \int_0^1 \lambda \left[-\ln\left(1 - F^{1/\alpha}\right)\right]^{1/k} F^2 \; dF \tag{3.10}$$

Which can be solved simultaneously by numerical simulation to obtain the estimates.

4. Maximum Likelihood Estimators

The method of maximum likelihood is one of the best known, most widely used, and most important of the methods of estimation. In this section, the procedure of the ML estimation of the parameters of the Exponentiated Weibull distribution is discussed as follows.

For a random sample, $x_1, x_2, ..., x_n$, from the Exponentiated Weibull distribution; the likelihood function $L(\alpha, \mathrm{k}, \lambda)$ is given by:

$$L(\alpha,k,\lambda)=\left(\frac{\alpha k}{\lambda^{k}}\right)^{n}\left(\prod_{i=1}^{n}x_i\right)^{k-1}\left(\prod_{i=1}^{n}\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{k}}\right)\right)^{\alpha-1}e^{-\frac{1}{\lambda^{k}}\sum_{i=1}^{n}x_i} \quad (4.1)$$

The ML estimators, $\hat{\alpha}_{ML}$, $\hat{k}_{ML}$ and $\hat{\lambda}_{ML}$ are the values which maximize $L(\alpha,\mathrm{k},\lambda)$. Since the maximum value of $L(\alpha,\mathrm{k},\lambda)$ occurs at the same points as the maximum value of $\ln\left[L(\alpha,\mathrm{k},\lambda)\right]$, then it is easier to work with the logarithm of the likelihood function. It is given by:

$$\ln\left[L(\alpha,k,\lambda)\right]=n(\ln\alpha+\ln k-k\ln\lambda)+(k-1)\sum_{i=1}^{n}\ln(x_i)+(\alpha-1)\sum_{i=1}^{n}\ln\left(1-e^{-\left(\frac{x_i}{\lambda}\right)^{k}}\right)-\frac{1}{\lambda^{k}}\sum_{i=1}^{n}x_i \quad (4.2)$$

Then the ML estimators; $\hat{\alpha}_{ML}$, $\hat{k}_{ML}$ and $\hat{\lambda}_{ML}$;can be obtained by simultaneously solving the following three equations by numerical simulation:

$$\frac{\partial\ln\left[L(\alpha,k,\lambda)\right]}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\ln\left(1-e^{-\left(x_i/\lambda\right)^{k}}\right)=0, \quad (4.3)$$

$$\frac{\partial\ln\left[L(\alpha,k,\lambda)\right]}{\partial k}=\frac{n}{k}-n\ln\lambda+\sum_{i=1}^{n}\ln x_i+(\alpha-1)\sum_{i=1}^{n}\left[\frac{e^{-\left(\frac{x_i}{\lambda}\right)^{k}}\left(x_i/\lambda\right)^{k}\ln\left(x_i/\lambda\right)}{1-e^{-\left(\frac{x_i}{\lambda}\right)^{k}}}\right]+\frac{\ln\lambda}{\lambda^{k}}\sum_{i=1}^{n}x_i=0 \quad (4.4)$$

$$\frac{\partial\ln\left[L(\alpha,k,\lambda)\right]}{\partial\lambda}=\frac{-nk}{\lambda}+(\alpha-1)\sum_{i=1}^{n}\left(\frac{-k\,e^{-\left(\frac{x_i}{\lambda}\right)^{k}}\left(\frac{x_i^{k}}{\lambda^{k+1}}\right)}{1-e^{-\left(\frac{x_i}{\lambda}\right)^{k}}}\right)-\frac{k}{\lambda^{k+1}}\sum_{i=1}^{n}x_i=0 \quad (4.5)$$

5. Moments Estimators

The method of moments (MM) is one of the oldest methods used for generating estimates for the unknown parameters from sample data. It was first proposed by Pearson (1894) and was used extensively for many years. In this section, the procedure of the MM estimation of the parameters of the Exponentiated Weibull distribution is illustrated as follows.

Since there are three parameters to be estimated then the first three moments; μ_1', μ_2' and μ_3'; are needed. They are given by:

$$\mu_1'=\int_0^{\infty}x\,\frac{\alpha k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\left(1-e^{-\left(x/\lambda\right)^{k}}\right)^{\alpha-1}e^{-\left(x/\lambda\right)^{k}}dx, \quad (5.1)$$

$$\mu_2'=\int_0^{\infty}x^{2}\,\frac{\alpha k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\left(1-e^{-\left(x/\lambda\right)^{k}}\right)^{\alpha-1}e^{-\left(x/\lambda\right)^{k}}dx \quad (5.2)$$

and $\mu_3'=\int_0^{\infty}x^{3}\,\frac{\alpha k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\left(1-e^{-\left(x/\lambda\right)^{k}}\right)^{\alpha-1}e^{-\left(x/\lambda\right)^{k}}dx$. (5.3)

Then find the first three sample moments that are:

$$m_1=\frac{\sum_{i=1}^{n}x_i}{n}=\bar{x}, \quad (5.4)$$

$$m_2=\frac{\sum_{i=1}^{n}x_i^{2}}{n} \quad (5.5)$$

and $m_3 = \dfrac{\sum_{i=1}^{n} x_i^3}{n}$. (5.6)

Then replace the population moments (μ_1', μ_2', μ_3') in the left hand sides of equations (5.1), (5.2) and (5.3) by the sample moments (m_1, m_2, m_3). This will yield the three following equations:

$$\bar{x} = \int_0^\infty x \frac{\alpha k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\left(1-e^{-\left(x/\lambda\right)^k}\right)^{\alpha-1} e^{-\left(x/\lambda\right)^k} dx \quad (5.7)$$

$$\frac{\sum_{i=1}^{n} x_i^2}{n} = \int_0^\infty x^2 \frac{\alpha k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\left(1-e^{-\left(x/\lambda\right)^k}\right)^{\alpha-1} e^{-\left(x/\lambda\right)^k} dx \quad (5.8)$$

$$\frac{\sum_{i=1}^{n} x_i^3}{n} = \int_0^\infty x^3 \frac{\alpha k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}\left(1-e^{-\left(x/\lambda\right)^k}\right)^{\alpha-1} e^{-\left(x/\lambda\right)^k} dx \quad (5.9)$$

The three equations (5.7), (5.8) and (5.9) can be solved simultaneously by numerical simulations to obtain the moment estimators $\hat{\alpha}_{MM}$, $\hat{k}_{MM}$ and $\hat{\lambda}_{MM}$.

6. Comparative Study

Since it is difficult to compare the performances of the different estimators (described in the previous sections) theoretically, extensive computer simulations are performed to conduct this comparison. The properties of the different estimators are compared mainly in terms of bias and mean square error (MSE), for different sample sizes and for different values of the shape parameter α. The simulation experiments are performed using the Mathcad (version 14.0) software. Different sample sizes are considered through the experiments that are n = 15, 20, 30, 50 and 100. In addition, different values for the shape parameter α are considered, which are α = 0.5, 1.5 and 2.5 with the two parameters k and λ taken to be 1 in all the experiments. For each combination of the sample size and α -value the experiment will be repeated 10000 times. In each experiment, the estimates of α, k and λ will be obtained by the different methods described in the previous sections. The biases and MSEs for the different estimators will be reported from these experiments.

The algorithm for obtaining the different estimates by the different methods of estimation can be described in details in the following steps:

Step (1): Generate a random sample of size *n*, $x_1, x_2, \ldots, x_n$, from the Exponentiated Weibull distribution. This can be achieved by firstly generating a random sample from the uniform (0,1) distribution, $u_1, u_2, \ldots, u_n$. Then the uniform random numbers can be transformed into Exponentiated Weibull random numbers using the following transformation:

$$x_i = \lambda\left[-\ln\left(1-u_i^{1/\alpha}\right)\right]^{1/k}, \qquad i = 1,2,\ldots,n \quad (6.1)$$

Step (2): Sort the random sample $x_1, x_2, \ldots, x_n$ to obtain the ordered complete sample $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$. Note that this step is required for SDPWM estimation. The ML and MM methods do not require the sample to be ordered for parameter estimation.

Step (3): Use the ordered sample with equations (3.8), (3.9) and (3.10); and solve them simultaneously by iteration to obtain $\hat{\alpha}_{SDPWM}$, $\hat{k}_{SDPWM}$ and $\hat{\lambda}_{SDPWM}$.

Step (4): Use the random sample obtained in Step (1) with equations (4.3), (4.4) and (4.5); and solve them simultaneously by iteration to obtain $\hat{\alpha}_{ML}$, $\hat{k}_{ML}$ and $\hat{\lambda}_{ML}$.

Step (5): Use the random sample obtained in Step (1) with equations (5.7), (5.8) and (5.9); and solve them simultaneously by iteration to obtain $\hat{\alpha}_{MM}$, $\hat{k}_{MM}$ and $\hat{\lambda}_{MM}$.

The biases and MSEs of the different estimators are reported in tables (1) and (2), respectively. From these tables many observations can be made on the performance of the methods of SDPWM versus the methods of ML and MM. These observations are summarized as follows:

(1) It is observed that the biases and MSEs of the different estimators depend on the value of the shape parameter α. For all the methods considered as α increases the biases and MSEs of the estimators of α and k increase. On the other hand, the biases and MSEs of the estimators of λ decrease as α increases for all the methods.

(2) For all the cases considered the biases and MSEs of the different estimators of α, k and λ decrease as the sample size increases. This indicates that all the methods provide consistent estimators for α, k and λ.

(3) Comparing the biases of the different estimators of α, k and λ, it is clear that the method of SDPWM yields the minimum bias in almost all the cases considered.

(4) Considering the MSEs of the different estimators of α, it is clear from table (2) that the SDPWM estimator has the minimum MSE in almost all of the cases considered for estimating α. On the other hand, the MM method is the worst in terms of bias and MSE.

7. Conclusions

The comparative study revealed that the SDPWM works the best in almost all the cases considered with respect to MSE and Bias, especially for small sample sizes. The performance of the MM method is the worst in terms of bias and MSE. In addition, all the methods considered have provided consistent estimators for α, k and λ.

Appendix

Table 1: Biases of the Parameter Estimators Obtained by the Methods of ML, MM, and SDPWM

n	Method	Estimators of α			Estimators of λ			Estimators of k		
		α = 0.5	α = 1.5	α = 2.5	α = 0.5	α = 1.5	α = 2.5	α = 0.5	α = 1.5	α = 2.5
15	ML	0.091	0.422	0.896	0.257	0.151	0.132	0.186	0.239	0.656
	MM	0.2	0.591	1.148	0.384	0.181	0.146	0.244	0.381	0.875
	SDPWM	0.055	0.201	0.409	0.193	0.083	0.063	0.116	0.117	0.296
20	ML	0.064	0.288	0.601	0.192	0.114	0.099	0.13	0.166	0.442
	MM	0.15	0.427	0.812	0.297	0.142	0.115	0.177	0.284	0.628
	SDPWM	0.038	0.147	0.313	0.144	0.063	0.049	0.08	0.089	0.221
30	ML	0.016	0.173	0.365	0.117	0.07	0.063	0.085	0.103	0.266
	MM	0.067	0.279	0.532	0.193	0.093	0.078	0.121	0.188	0.407
	SDPWM	0.013	0.095	0.223	0.084	0.034	0.029	0.049	0.058	0.154
50	ML	0.023	0.098	0.205	0.068	0.041	0.036	0.048	0.059	0.151
	MM	0.065	0.169	0.314	0.117	0.057	0.046	0.072	0.115	0.241
	SDPWM	0.014	0.06	0.146	0.046	0.018	0.013	0.023	0.033	0.097
100	ML	0.012	0.051	0.099	0.034	0.021	0.019	0.024	0.03	0.075
	MM	0.035	0.09	0.155	0.061	0.03	0.024	0.037	0.059	0.121
	SDPWM	0.008	0.036	0.1	0.019	0.005	0.005	0.007	0.018	0.063

Table 2: MSEs of the Parameter Estimators Obtained by the Methods of ML, MM, and SDPWM

n	Method	Estimators of α			Estimators of λ			Estimators of *k*		
		α = 0.5	α = 1.5	α = 2.5	α = 0.5	α = 1.5	α = 2.5	α = 0.5	α = 1.5	α = 2.5
15	ML	0.059	1.205	5.692	0.415	0.161	0.125	0.231	0.408	3.396
	MM	0.168	2.125	9.413	0.677	0.217	0.162	0.329	0.759	4.825
	SDPWM	0.032	0.378	1.314	0.334	0.106	0.075	0.161	0.147	0.784
20	ML	0.035	0.639	2.791	0.27	0.111	0.087	0.141	0.21	1.416
	MM	0.104	1.158	4.548	0.444	0.156	0.118	0.21	0.47	2.766
	SDPWM	0.022	0.259	0.909	0.225	0.078	0.056	0.105	0.102	0.522
30	ML	0.002	0.274	1.14	0.14	0.062	0.05	0.08	0.101	0.6
	MM	0.016	0.585	2.133	0.24	0.094	0.073	0.126	0.242	1.2
	SDPWM	0.003	0.151	0.541	0.124	0.048	0.035	0.063	0.06	0.305
50	ML	0.009	0.131	0.493	0.07	0.034	0.027	0.042	0.049	0.278
	MM	0.033	0.296	1.007	0.124	0.054	0.042	0.069	0.124	0.588
	SDPWM	0.008	0.084	0.301	0.063	0.027	0.02	0.035	0.033	0.172
100	ML	0.004	0.055	0.195	0.03	0.015	0.012	0.019	0.02	0.112
	MM	0.015	0.131	0.414	0.055	0.025	0.02	0.032	0.055	0.247
	SDPWM	0.004	0.039	0.14	0.028	0.012	0.01	0.016	0.016	0.08

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