International Journal of Statistics and Applied Mathematics 2018; 3(2): 206-211

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2018; 3(2): 206-211

www.mathsjournal.com
Received: 28-01-2018
Accepted: 29-02-2018

Aarju
Assistant Professor, Govt. College, Bherian, Pehowa, Haryana, India

Correspondence
Aarju
Assistant Professor, Govt. College, Bherian, Pehowa, Haryana, India

Differential algebraic interpretation with dense singularities of the order completion method

Aarju

Abstract
In this paper we demonstrate the spaces of generalized function in terms of chain of algebras. In this work, we focus on generalized algebraic functions with nowhere dense singularities as mentioned in algebraic notation of Rosinger. This Rosinger's dense algebra distributions are embedded & ensures its consistency with partial differential equations. Furthermore, in most of the cases this embedding phenomena maintains the products of smooth functions. Vernaeve introduced the issues of embedding derivatives in to generalized algebraic function of nonlinear partial differential equations. In this paper it is observed that the embedding of the spaces are used to conserve algebraic as well as differential structures. These findings indicates the extent to which chains (differential/algebraic) are capable of handling the singularities on closed dense set. Lastly we revealed that almost all the properties of Rosinger's algebra have the ability to solve nonlinear partial differential equations.

Keywords: Algebraic interpretation, dense singularitie, order completion method

Introduction

The generalized algebraic functions with the help of sequential approach has been discussed in this paper. We will choose elements ranges 0 to 1. Let us consider $\overset{''}{O}$ as an open subset of R^n, the space of sequences of smooth maps $\overset{''}{O} \to C$ which is infinitely differentiable can be represented as $\xi(\overset{''}{O}) = (\xi^{\infty}(\overset{''}{O}))^{(0,1]}$. In the literature Rosinger focused nowhere-dense ideal $B_{nd}(\overset{''}{O})$ illustrated within the above mentioned approach as the set of whole sequences $(\vartheta_{\varepsilon})_{\varepsilon \in (0,1]} \in \xi(\overset{''}{O})$ for which there is a nowhere dense, closed subset $\Gamma \subset \overset{''}{O}$ in such the following manner:

$$\forall a \in \overset{''}{O} \setminus \Gamma : \exists \varepsilon_0 \in (0,1] : \varepsilon < \varepsilon_0 : \vartheta_{\varepsilon}(a) = 0$$

Further it can be explored as the nowhere-dense quotient algebra $Q_{nd}(\overset{''}{O}) := \xi(\overset{''}{O}) / B_{nd}(\overset{''}{O})$ in terms of nowhere-dense singularities of outcomes of partial differential equations. In this paper, we gives a general description of any classical solution of an arbitrary analytic partial differential equations off a nowhere-dense closed set gives increment to a global outcome in $Q_{nd}(\overset{''}{O})$.

As discussed in literature, it is quite known if an embedding of distributions into $Q_{nd}(\overset{''}{O})$ occurs that conserves derivatives of distributions.It can be easily said that the impression of distributions as well as generalized functions with nowhere dense singularities are little bit hard to make it compatible. Thus, it's our interest to explore that one can achieve such kind of embedding fordistributions into a different manner for algebra of generalized functions, without modification in the above described property related to the existence of outcomes of nonlinear partial differential equations. Moreover, we demonstrate that this construction is said to be compatible with Colombeau's. In this regard, we can further classified an ideal analogous to Colombeau's, in a way to embed $\xi^{\infty}(O)$ which is denoted as a sub-algebra (we can also say that it is used to preserve the product of ξ^{∞}-functions). The amalgamation of the constructed algebras & distributions which contain good embedding-properties should definitely diminishes the false suggestion of non-compatibility between distributions as well as generalized functions with dense singularities.

Algebraic Properties & Definitions

Here we find a new algebra instead of Rosinger's formulations $Q_{nd}(O)$. Let us suppose that $\mathrm{H}_0(O)$ be a subset of $\{\Gamma \subset O : \Gamma$ is nowhere dense as well as closed$\}$ having a property that a finite union of all the elements of $\mathrm{H}_0(O)$ is again in $\mathrm{H}_0(O)$. For a particular instance, $\mathrm{H}_0(O)$ could be treated as the set of whole nowhere dense as well as closed $\Gamma \subset O$ itself, however it can also be the combination of all closed $\Gamma \subset O$ of (Lebesgue-)measure zero.

Again let us say $\xi_{ae}(O)$ (almost everywhere) be the group of all sequences $(\vartheta_{\varepsilon})_{\varepsilon \in (0,1]}$ of $O \to C$ -maps for which there occurs $\Gamma \in \mathrm{H}_0(O)$ such that $\forall \varepsilon \in (0,1] : \varepsilon < \varepsilon_0 : \vartheta_{\varepsilon} \in \xi^{\infty}(O \setminus \Gamma)$. Now it is quite easy to see that $\xi_{ae}(O)$ is an algebra over specified maps with the operations ('+', '.') component wise.

By $S \subset\subset O$, we define a compact subset S of O. Let us express

$B_E(O) := \{(\vartheta_{\varepsilon}) \in \xi_{ae}(O) : \forall S \subset\subset O : \exists \varepsilon \in (0,1] : \forall \varepsilon < \varepsilon_0 : \vartheta_{\varepsilon} = 0$ on S

$= \{(\vartheta_{\varepsilon}) \in \xi_{ae}(O) : \forall a \in O : \exists v$ Neighborhood of a $: \exists \varepsilon_0 \in (0,1]$

$\forall \varepsilon < \varepsilon_0 : \vartheta_{\varepsilon} = 0$ on v

and

$B_{ae}(O) := \{(\vartheta_{\varepsilon}) \in \xi_{ae}(O) : \exists \Gamma \in H_0(O) : \forall \varepsilon \in (0,1] : \vartheta_{\varepsilon} = 0$ on $O \setminus \Gamma\}$

Since $B_E(O)$ and $B_{ae}(O)$ are ideals in $\xi_{ae}(O)$, thus we can describes the algebra $Q_{ae}(O) := \xi_{ae}(O) / \left(B_{ae}(O) + B_E(O) \right)$.

So, we consider a sequences of various functions which are nothing but smooth outside a fixed $\Gamma \in \mathrm{H}_0(O)$ and also identify sequences if they vary according to $\Gamma' \in \mathrm{H}_0(O)$. We can also give characterizations of the ideal $B_{ae}(O) + B_E(O)$ which are usually necessary for some technical reasons.

The definition (partial derivatives) does not lie on the representatives. It is clearly described that these partial derivatives on $Q_{ae}(O)$ are said to be mutually commutative linear maps which fulfill the Leibniz findings, in different way, we can say that this is a differential algebra. We can also mention that $\xi^{\infty}(O)$ is embedded in terms of sub-algebra in $Q_{ae}(O)$ by taking the family of the constant embedding $\sigma : \xi^{\infty}(O) \to \xi_{ae}(O) : \sigma(v) = (\vartheta_{\varepsilon})$ with $\vartheta_{\varepsilon} = v$.

Rosinger's Algebra & Embedding Distributions in $Q_{ae}(O)$

We reveal that a sequence $(\chi_{\varepsilon}) \in (G(O))^{(0,1]}$ is known as unit net on O if and only if for all $S \subset\subset O$ there exists ε_0 in such a manner $(\chi_{\varepsilon}) = 1$ over S for every value of $\varepsilon < \varepsilon_0$. We can also say that if and only if for all "a" belongs to O there is neighborhood V of a as well as ε_0 such that $(\chi_{\varepsilon}) = 1$ over the given neighborhood for $\varepsilon < \varepsilon_0$.

Lemma 1:

$(B_{ae}(O) + B_E(O)) \cap \xi(O) = B_E(O) \cap \xi(O)$

Proof: Let us assume $(\vartheta_\varepsilon) \in (B_{ae}''(O) + B_E''(O)) \cap \xi''(O)$. Also assume that "a" belongs to O, there occurs an open neighborhood V of "a" in such a manner for small ε, $\vartheta_\varepsilon = 0$ over $V \setminus \Gamma$ for few $\Gamma \in \mathrm{H}_0''(O)$. As we know that $V \setminus \Gamma$ is dense in the given open set V as well as $\vartheta_\varepsilon \in \xi^{\infty''}(O)$, $\vartheta_\varepsilon = 0$ over neighborhood. So $(\vartheta_\varepsilon) \in B_E''(O) \cap \xi''(O)$.

Theorem 1: There occurs a natural surjective morphism of differential algebras from $Q_{ae}''(O)$ into Rosinger's algebra $Q_{nd}''(O)$.

Proof: we describe the morphism $\Theta : Q_{ae}''(O) \to Q_{nd}''(O)$ as follows on respective representatives. If $\vartheta_\varepsilon \in \xi^{\infty''}(O \setminus \Gamma), \forall \in (with\ \Gamma \in \mathrm{H}_0''(O))$, assume (χ_ε) is nothing but unit net over the open set $O \setminus \Gamma$. Then

$$\Theta((\vartheta_\varepsilon) + B_{ae}''(O) + B_E''(O)) := B_{nd}''(O) + (\vartheta_\varepsilon \chi_\varepsilon)$$

Now, in case of a nowhere dense as well as closed set, we can see that for unit nets (χ_ε) and (ϑ_ε) over $O \setminus \Gamma$, $(\chi_\varepsilon - \lambda_\varepsilon) \in B_{nd}''(O)$, this is the only reason that the definition does not depend on the chosen unit net. Furthermore we can say that it is independent of the representatives:

$(\vartheta_\varepsilon) \in (B_{ae}''(O) + B_E''(O))$, then $(\vartheta_\varepsilon \chi_\varepsilon) \in (B_{ae}''(O) + B_E''(O)) \cap \xi''(O) = B_E''(O) \cap \xi''(O)$.

Again for any specified unit net (χ_ε) over $O \setminus \Gamma$ $(with\ \Gamma \in \mathrm{H}_0''(O))$, $(1 - \chi_\varepsilon) \in B_{nd}''(O)$, thus for every $(\vartheta_\varepsilon) \in \xi''(O)$, $\Theta\left(\bar{\vartheta}_\varepsilon\right) = (\vartheta_\varepsilon) + B_{nd}''(O)$. This represents that Θ is surjective. Moreover, it is obviously a linear morphism. It can be concluded that the product of two unit nets is nothing but a unit net, it can also belongs to a morphism of algebras. At the end, we can also check that Θ commutes with partial derivatives.

In case of Rosinger's algebra $B_{nd}''(O)$, generally convolution method is used to embed $G'''(O)$ in an algebra. This convolution result does not yield an injective map. But in other way it may do so in $B_{ae}''(O)$, thus the less sequences are observed than in $B_{nd}''(O)$. First of all we prove another embedding result.

Theorem 2: There occurs a natural embedding of differential algebras from $\xi''(O) / B_E''(O) \cap \xi''(O)$ into $B_{ae}''(O)$.

Proof: As earlier we described that $\xi''(O) / B_E''(O) \cap \xi''(O)$ is nothing but differential algebra with the point wise definitions of dot as well plus sign (+, '.') and σ_i.

Now consider the map $\xi''(O) / B_E''(O) \cap \xi''(O) \to B_{ae}''(O)$:

$$(\vartheta_\varepsilon) + (B_E''(O) \cap \xi''(O)) \mapsto (\vartheta_\varepsilon) + B_E''(O) + B_{ae}''(O)$$

As we say $\xi''(O) \subset \xi_{ae}''(O)$ & $(B_E''(O) \cap \xi''(O) \subset B_E''(O) + B_{ae}''(O)$ so this map is absolutely defined. Thus, it can be easily demonstrate a morphism of differential algebras.

However, some embedding findings $\xi''(O) / B_E''(O) \cap \xi''(O)$ just follow $B_{ae}''(O)$. Now reveal some definitions and say a sequence $(\chi_\varepsilon) \in (G(R^n))^{(0,1]}$ is known as strict delta net if and only if

supp $(\Psi_\varepsilon) \to 0$ as $(\in) \to 0$

$\int (\psi_\varepsilon) = 1, \forall \varepsilon \in (0,1]$

$\int |(\psi_\varepsilon)|$ is bounded without depends on ε.

A linear map with relation $G^{'}(\overset{''}{O}) \to \xi^{\infty}(\overset{''}{O})$ is known as smooth part map if and only if it commutes will every partial derivatives and leave all elements of $\xi^{\infty}(\overset{''}{O})$ invariant. We observe that the existence of the above mentioned part map is non-trivial but it can be said that it is guaranteed for convex $(\overset{''}{O})$.

Theorem 3:(i) Assume (ψ_{ε}) is said to be strict delta net and (χ_{ε}) is nothing but a unit net on $\overset{''}{O}$. The map $\tau : G^{'}(\overset{''}{O}) \to Q_{ae}(\overset{''}{O})$:

$\tau(T) := \left((T\chi_{\in})^{-} * \psi_{\in}\right)$

is a linear embedding of $G^{'}(\overset{''}{O})$ into $Q_{ae}(\overset{''}{O})$ that commutes with partial derivatives.

(ii) Let us say S be a smooth part map. The map $\tau : G^{'}(\overset{''}{O}) \to Q_{ae}(\overset{''}{O})$ represented as

$\tau(T) := \overline{(S(T) + [(T - S(T))\chi_{\in}] * \psi_{\in})}$

Is said to be linear embedding of $G^{'}(\overset{''}{O})$ into $Q_{ae}(\overset{''}{O})$ that commutes with partial derivatives and it is also represent in a manner which embeds $\xi^{\infty}(\overset{''}{O})$ as a sub-algebra.

A Colombeau Algebra with Non-Smooth Representatives& its PDEs solution

In order to provide a sense to the fundamental results in $Q_{ae}(\overset{''}{O})$ for a partial differential equations with given data, we can also explore some functionality over elements of $Q_{ae}(\overset{''}{O})$. First of all, we describe the composition of a complex-valued map $S \in \xi^{\infty}(\overset{''}{O} \times C^{p})$ with a p-tuple of elements of $Q_{ae}(\overset{''}{O})$ .With the given relation $\xi^{\infty}(C^{p})$, we want to illustrate that the space of functions are nothing but smooth & it is a function of 2p real variables. By means of chain rule for differentiation, it is considered that, for this relation $\Gamma_i \in H_0(\overset{''}{O})$, the composition map is given by the following

$S_0 : \xi^{\infty}(\overset{''}{O} \setminus \Gamma_1) \times .... \times \xi^{\infty}(\overset{''}{O} \setminus \Gamma_p) \to \xi^{\infty}(\overset{''}{O} \setminus (\Gamma_1 \cup \Gamma_2 \cup .... \cup \Gamma_p)) :$

$S \circ : (\vartheta_1, ..., \vartheta_p)(a) := S(a, \vartheta_1(a), ..., \vartheta_p(a))$ is well defined and therefore it can be described as

$S \circ : \xi_{ae}(\overset{''}{O})^{p} \to \xi_{ae}(\overset{''}{O}) :$

$S \circ : ((\vartheta_{1,\in})_{\varepsilon \in (0,1]}, ..., (\vartheta_{p,\in})_{\varepsilon \in (0,1]}) := (S \circ (\vartheta_{1,\in}, ..., \vartheta_{p,\in}))_{\varepsilon \in (0,1]}$

Lemma 2: The definition of $S \circ$ put more emphasis on a well-defined map $\xi_{ae}(\overset{''}{O})^{p} \to \xi_{ae}(\overset{''}{O})$.

Proof: First of all we have to find that it does not depends on the representatives, or we can say, that $(S \circ (\vartheta_{j,\in}) - S \circ (\mu_{j,\in}))_{\varepsilon \in (0,1]} \in B_{ae}(\overset{''}{O}) + B_{E}(\overset{''}{O})$ if $((\vartheta_{j,\in}) - (\mu_{j,\in}))_{\varepsilon \in (0,1]} \in B_{ae}(\overset{''}{O}) + B_{E}(\overset{''}{O})$. This can be follows from the characterization discussed in literature combined with the trivial fact that, if for some fixed values of ε and 'a' belongs to $\overset{''}{O}$, $(\vartheta_{j,\in}) = (\mu_{j,\in}), \forall j$. It can also be defined as $S(a, \vartheta_{j,\in}(a) = S(a, \mu_{j,\in}(a)$.

Further, we can describe point wise values of elements belongs to $Q_{ae}(\overset{''}{O})$. For $\vartheta \in \xi^{\infty}(\overset{''}{O} \setminus \Gamma)$ $(\Gamma \in H_0(\overset{''}{O}))$. However, we can illustrate the value of ϑ for all 'a' belongs to $\overset{''}{O}$ by the following relation:

$$val(\vartheta)(x) := \lim_{\substack{a \to x \\ a \in \overset{''}{o} \setminus \Delta}} \vartheta(a)$$

The above relation described that this limit exists for some $(\Delta \in H_0(\overset{''}{O}))$. If it occurs, it is clearly mentioned that this does not depends on the choice of Δ, since for $\Delta_1, \Delta_2 \in H_0(\overset{''}{O}), \overset{''}{O} \setminus (\Delta_1 \cup \Delta_2)$ is dense in $\overset{''}{O}$, so it can be determines the limit unambiguously. It can be observed that $val(\vartheta)$ is defined over $\ddot{O} \setminus \Gamma$. With the same scenario, if $\mu = \vartheta$ over $\ddot{O} \setminus \Gamma'$, then we can conclude that the values of μ, ϑ are similar. However, assume $(\vartheta_\varepsilon) \in (B_{ae}(\ddot{O})$. We express $B(C)$ as the group of sequences of complex numbers which are identified if they eventually coincide. Then we can describe the value of ϑ_ε at 'a' belongs to $\ddot{O}$ as $((val(\vartheta_\varepsilon)(a))_\varepsilon \in \xi(C)$, if $val(\vartheta_\varepsilon)(a)$ occurs for each sufficiently small ε.

The global findings of partial differential equations in $Q_{nd}(\ddot{O})$ give a classical results off nowhere dense sets. It can also follows that we observed enough sequences $Q_{ae}(\ddot{O})$ as well, to implement this lemma directly and also to achieve the same notion of global result.

Again we focus that the p-th order (p≥1) nonlinear analytic partial differential equation which is assumed to be in standard form

$$K_t^n W(t,b) = L(t,b,\ldots., K_t^o \partial_b^\beta W(t,b),\ldots..) \quad (1)$$

Where a=(t,b) belongs to $\ddot{O}$, t is element of R, likewise b is element of R^{m-1}, o belongs to N, o is less than n, β belongs to N^{m-1}, $o + |\beta| \le n$, L is analytic for every variables & $W: \ddot{O} \to C$ is nothing but the unknown function.

Moreover, on the non-characteristic analytic hyper surface in $\ddot{O}$, represented for a certain t_0 belongs to R

$$T := \{a = (t,b) \in \ddot{O} : t = t_0\}$$

Now we focus on the analytic Cauchy data described by the following equation

$$K_t^o W(t_0, b) = g_o(b), o < n, (t_0, b) \in T \quad (2)$$

Thus, we will illustrate an algebra with same kind of $Q_{ae}(\ddot{O})$ & it also has an extra identification same as construction of the Colombeau algebras. The important role for this is to achieve the strong embedding's of $G'(\ddot{O})$ under consideration of generalized algebraic functions with nowhere dense singularities also for non-convex $\ddot{O}$.

Theorem 4: There occurs a general embedding phenomena of differential algebras from the Colombeau-algebra $\zeta_s(\ddot{O})$ into $\zeta_{ae}(\ddot{O})$.

Proof: As per the fundamentals discussed earlier $\zeta_s(\ddot{O})$ is said to be a differential algebra with the point wise definitions of plus as well as dot (+, '.'), σ_i. Thus, the map $\zeta_s(\ddot{O}) \to \zeta_{ae}(\ddot{O})$:

$$(\vartheta_\varepsilon) + \Im(\ddot{O}) \mapsto (\vartheta_\varepsilon) + \Im_{ae}(\ddot{O})$$

As we know that $\xi_n(\ddot{O}) \subset \xi_{n,ae}(\ddot{O})$ and $\Im(\ddot{O}) \subseteq \Im_{ae}(\ddot{O})$, this map is well defined. Now we can clearly observe a morphism of differential algebras.

Theorem 5: There also occurs a linear embedding of $G'(\ddot{O})$ into $\zeta_{ae}(\ddot{O})$ which can commutes with partial derivatives and it can also embeds $\xi^\infty(\ddot{O})$.

Now we conclude that, any Colombeau algebra, the combination of a $\xi^\infty(\ddot{O})$ - function S with elements of $\zeta_{ae}(\ddot{O})$ is obviously well-defined over the complete set of $\zeta_{ae}(\ddot{O})$, if Sprogress slowly. In order of our application to analytic partial differential equations, this may be a severe restriction. Again, if we consider ourselves with a composition map that is only illustrated over a subset of $\zeta_{ae}(\ddot{O})$, then we can achieve the existence result.

Conclusion

The proposed work given an idea that the choice of $Q_{ae}(\ddot{O})$ as an algebra of generalized function which are occurs generally in nonstandard analysis. By means of different characterization of the nowhere dense $B_{nd}(\ddot{O})$ is quite similar to the group of all sequences $(\vartheta_{\in})_{\varepsilon \in (0,1]} \in \xi(O)$. In this paper we identified smooth nonstandard maps which differs only on the point infinitely close to dense close subset. The solution of analytic partial differential equation has been discussed.

References

1. Kovalevskaia S. ZurTheorie der partiellendifferentialgleichung, Journal fur die reine und angewandte Mathematik, 1875; 80:1-32.
2. Colombeau JF. New Generalized Functions and Multiplication of Distributions, vol. 84, North-Holland, Amsterdam, The Netherlands, 1984.
3. Rosinger EE. Global version of the Cauchy-Kovalevskaia theorem for nonlinear PDEs, Acta Applicandae Mathematicae, 1990; 21(3):331-343.
4. Oberguggenberger MB, Rosinger EE. Solution of Continuous Nonlinear PDEs through Order Completion, North Holland, Amsterdam, The Netherlands, 1994, 181.
5. Evans LC. Partial Differential Equations, of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, USA, 1998, 19
6. Baire R, Lecons Sur Les Fonctions Discontinues, Collection Borel, Paris, France, 1905.
7. Arnold VI. Lectures on Partial Differential Equations, Springer, Berlin, Germany, 2004.
8. van der Walt JH. The order completion method for systems of nonlinear PDEs revisited, ActaApplicandaeMathematicae, 2009; 106(2):149-176,
9. Beattie R, Butzmann HP. Convergence Structures and Applications to Functional Analysis, Kluwer Academic Publishers, Dordrecht, The Netherlands, 2002.
10. Anguelov R. Dedekind order completion of C(X) by Hausdorff continuous functions. Quaest. Math. 2004; 27:153-170.
11. Anguelov R, Rosinger EE. Solving large classes of nonlinear systems of PDE's. Comput. Math. Appl. 2007; 53:491-507.
12. Anguelov R, van der Walt JH. Order convergence structure on C (X). Quaest. Math. 2005; 28:425-457.
13. Egorov YV. A contribution to the theory of generalized functions, Russian Math Surveys 45: 1–49 (translated from Russian), 1990
14. Lindstr€om T. An invitation to Nonstandard Analysis. In: Cutland N (ed) Nonstandard Analysis and its Applications, Cambridge: Univer Press, 1988, 1-105.
15. Oberguggenberger M. Multiplication of Distributions and Applications to Partial Differential Equations, Essex: Longman, 1992
16. Oberguggenberger M, Todorov T. An Embedding of Schwartz Distributions in the Algebra of Asymptotic Functions, Internat J Math & Math Sci. 1998; 21:417-428.
17. Robinson A. Non-Standard Analysis, Amsterdam: North-Holland, 1966.
18. Rosinger EE. Nonlinear Partial Differential Equations: Sequential and Weak Solutions. Amsterdam: North-Holland, 1980.
19. Rosinger EE. Generalized Solutions of Nonlinear Partial Differential Equations. Amsterdam: North-Holland, 1987.
20. Rosinger EE. Nonlinear Partial Differential Equations, An Algebraic View of Generalized Solutions, Amsterdam: North-Holland, 1990.