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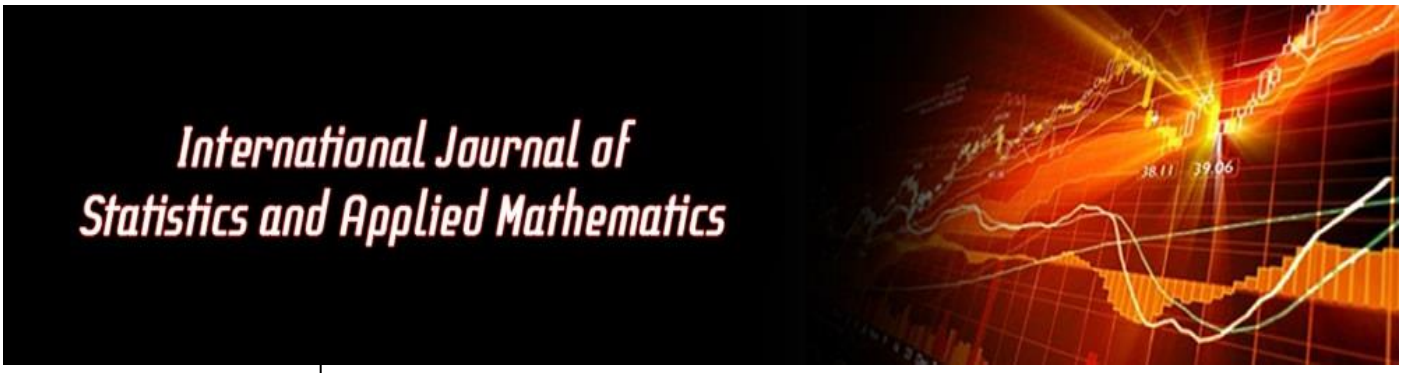

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An upper outlier detection procedure in a sample from a Johnson S_B distribution with known parameters

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Abstract
In this paper, a test statistics is developed for detection of an upper outlier in a sample from a Johnson S_B distribution when parameters are known. Performance of this statistic was studied using simulation technique. This statistic was applied to a real life data with a planted outlying observation to show the utility of the proposed test statistic.

Keywords: Johnson S_B distribution, outlier detection, contaminant observation, critical value, census 2011, simulation technique

1. Introduction
Johnson (1949) [5] derived a family of distributions using the method of translation and was called as Johnson family of distributions, which provides flexibility of covering wide varieties of distributional shapes. The advantage of Johnson system of distributions is that it approximates commonly used continuous distributions such as Normal, Log-normal, Gamma, Beta, Exponential distributions. Gerald & Samuel (1967) [4] have shown these distributions as special cases of Johnson system of distributions.
Due to its flexible nature, this distribution has wide applications. For example, the S_B distribution is used as a model for human exposure data; see Flynn (2004) [2], Mage (1980) [8] used it for ambient air pollution, Kotteguda (1987) [7] for rainfall distribution, Zang *et. al*, (2003) [9] for forestry and Konduru *et al.,* (2013) [6] for Infrared brightness temperature distribution of the deep convective clouds. This distribution is also very useful in complicated data set like microarray data analysis, see Florence George (2007) [3] *etc.*
An outlier is defined as an observation that deviates from the rest of the data in some sense. In this paper, a test statistic is developed from a statistic used in the case of a sample from Normal distribution for detection of an upper outlier from Johnson S_B distribution when all the parameters are known. For this, Johnson S_B distribution is firstly transformed into Standard normal distribution using some suitable transformation. Then the critical region of a test statistic of Normal distribution as given in Barnett and Lewis (1994) [1] is used for obtaining the critical region for Johnson S_B distribution.

2. Johnson S_B distribution and test statistic
Let $X_1, X_2, \cdots, X_n$ be the *n* observations from Johnson S_B distribution and $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be the corresponding order statistics of these *n* observations.
The *p.d.f* of Johnson S_B distribution is given by:

$$f(x) = \frac{\delta}{\sqrt{2\pi}} \frac{\lambda}{\{\lambda-(x-\xi)\}(x-\xi)} \exp\left[-\frac{1}{2}\left\{\gamma + \delta \ln\left(\frac{x-\xi}{\lambda-(x-\xi)}\right)\right\}^2\right], \quad (2.1)$$
$$\xi \le x \le \xi + \lambda, \delta > 0, -\infty < \gamma < \infty, \lambda > 0, -\infty < \xi < \infty,$$

where ξ is the location parameter, λ is the scale parameter and δ and γ are shape parameters.
By making a transformation

$$z = \gamma + \delta \ln\left(\frac{x-\xi}{\lambda-(x-\xi)}\right). \quad (2.2)$$

the density (2.1) transforms to a standard normal distribution *i.e.*

$f(z) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}z^2\right], -\infty < z < \infty.$

Let $z_1, z_2, \cdots, z_n$ be n observations of normal distribution corresponding to the n observations $X_1, X_2, \cdots, X_n$ and $z_{(1)}, z_{(2)}, \cdots, z_{(n)}$ be the corresponding order statistics of the n transformed observations.

The null hypothesis H_0 states that there is no outlying observation in the sample while the alternative hypothesis H_1 states that there is one outlying observation on the right side. The test statistic for detection of a single upper outlier of a sample from a normal distribution with σ^2 known is given in Barnett and Lewis (1994) [1] as

$$T = \frac{z_{(n)} - z_{(n-1)}}{\sigma}. \quad (2.3)$$

The α level critical region for this statistic is $T > T_\alpha$, where T_α is the critical value of T at α level of significance and its tabulated values are given by Barnett and Lewis (1994) [1] for different sample sizes at levels of significance 5 and 1 percent.

This test statistic when used for Johnson S_B distribution using transformation (2.2) becomes

$$T = z_{(n)} - z_{(n-1)} \text{ as } \sigma = 1. \quad (2.4)$$

As the transformation (2.2) is order preserving, $z_{(n)} = \gamma + \delta \ln\left(\frac{x_{(n)} - \xi}{\lambda - (x_{(n)} - \xi)}\right)$ and

$z_{(n-1)} = \gamma + \delta \ln\left(\frac{x_{(n-1)} - \xi}{\lambda - (x_{(n-1)} - \xi)}\right)$.

Thus, $T = \gamma + \delta \ln\left(\frac{x_{(n)} - \xi}{\lambda - (x_{(n)} - \xi)}\right) - \gamma - \delta \ln\left(\frac{x_{(n-1)} - \xi}{\lambda - (x_{(n-1)} - \xi)}\right)$

$= \delta\left[\ln\left\{\left(\frac{x_{(n)} - \xi}{x_{(n-1)} - \xi}\right)\left(\frac{\lambda - (x_{(n-1)} - \xi)}{\lambda - (x_{(n)} - \xi)}\right)\right\}\right]$.

Denoting $x_{(n)} - \xi$ by $y_{(n)}$,

$T = \delta\left[\ln\left\{\frac{y_{(n)}}{y_{(n-1)}}\left(\frac{\lambda - y_{(n-1)}}{\lambda - y_{(n)}}\right)\right\}\right]$

$= \left[\ln\left\{\frac{y_{(n)}}{y_{(n-1)}}\left(\frac{\lambda - y_{(n-1)}}{\lambda - y_{(n)}}\right)\right\}^{\delta}\right]$.

Let exponential of T be denoted by W, then

$W = e^T = \left\{\frac{y_{(n)}}{y_{(n-1)}}\left(\frac{\lambda - y_{(n-1)}}{\lambda - y_{(n)}}\right)\right\}^{\delta}, \quad 0 < T < \infty.$

$W = \left\{\left(\frac{x_{(n)} - \xi}{x_{(n-1)} - \xi}\right)\left(\frac{\lambda - (x_{(n-1)} - \xi)}{\lambda - (x_{(n)} - \xi)}\right)\right\}^{\delta}, 1 < W < \infty.$ (2.5)

Corresponding to the size α critical region $T > T_\alpha$ of the statistic T, the critical region of the statistic W is $W > W_\alpha$.

As W is a function of sample values from Johnson S_B distribution alone, it can be considered as a suitable test statistic for detection of an upper outlier for the case of Johnson S_B distribution. The critical values W_α are tabulated in Table 2.1. for 5 and 1 percent significance levels and for different values of n between 3 and 1000. As the derivation of the critical values W_α is independent of the parameters$(\xi, \lambda, \gamma, \delta)$, they can be used for any Johnson S_B distribution.

Table 2.1: Critical value of W_α for different sample sizes and at different levels of significance.

Wα					
n	**α=0.05**	**α=0.01**	**n**	**α=0.05**	**α=0.01**
3	8.758	18.1742	80	2.8292	4.4817
10	4.3059	7.6141	100	2.7732	4.3492
20	3.5966	6.0496	200	2.5857	3.9749
30	3.3201	5.4739	500	2.3869	3.5966
40	3.1268	5.1039	1000	2.2933	3.3872
60	2.9447	4.7588			

It can be observed from the table that the critical values of test statistic W decreases with an increase in sample size n.

3. Examples

For highlighting the utility of the statistic, the following 20 observations were taken from the Census data of India taken in the year 2011, (in, 000).

1028610, 1045547, 1062388, 1095722, 1112186, 1128521, 1160813, 1176742, 1192506, 1223581, 1238887, 1254019, 1283600, 1298041, 1312240, 1339741, 1352695, 1365302, 1388994, 1399838.

This is used in the following examples when the outlying observation is from another sample with a shift in one or all the parameters. The critical value at 5% level of significance for all the examples considered below is 3.5966. Then an outlying observation was introduced by deviating each of the parameters as follows.

Example.3.1: For introducing a contaminant observation with a shifted location parameter, another sample of Johnson S_B distribution has to be generated with a location parameter $\xi + a\lambda$, where the range of a should be such that $0 < a < 1$ for the new

observations to lie within the range of the original sample. Here for this example the value of a was taken as 0.5 and a sample with a shifted location parameter was generated. The largest observation of the first sample was replaced with the largest observation of the second sample. Test statistic W_U was calculated and was equal to 9.898013 which fall in the critical region at 5% level of significance. Hence, the largest observation of this sample was declared as a contaminant observation.

Example.3.2: For introducing an outlier with a shift in the scale parameter, another sample of Johnson S_B distribution with scale parameter b λ, with b>1, was generated. Here again for the new sample to be within the range of the old one, b should be in the range of $\xi/\lambda < b < 1+\xi/\lambda$. The largest observation of the first sample was replaced with the largest observation of the second sample. The calculated value of the test statistic W_L was equal to 9.152462. On comparison with critical value at 5% level of significance it was found that the calculated value is larger than the critical value. Hence, the null hypothesis gets rejected. This implies that the largest observation is an outlier.

Example.3.3: For studying the effect of a shift in the shape parameter, another sample with a shift c in shape parameter was generated. To get a sample with finite values, the value of c should be chosen so as to fall in the interval $\left(-(\xi+\lambda),(\xi+\lambda)\right)$, i.e. $-(\xi+\lambda) < c < (\xi+\lambda)$. In this example, c was taken as 0.5. The largest observation of the first sample was replaced with the largest observation of the second sample. On computation of the test statistic W_U it was found to be 3.743415, and thereby the largest observation was declared as a contaminant observation as it was larger than the corresponding critical value.

Example.3.4: Now for detection of outlying observation arising due to a shift in the shape parameter δ, the second sample was generated with a shift d (where d > 0, for this example d was taken as 0.5) in shape parameter. The largest observation of the first sample was replaced with the largest observation of the second sample. Test statistic W_U was calculated and found to be equal to 5.336801. This on comparison with the critical value at 5% level of significance, the null hypothesis gets rejected. This infers that the largest observation to be an outlying observation.

Example. 3.5: Now for considering the presence of outlier due to a shift in all the four parameters, a second sample was generated with a shift in all the four parameters, i.e. ξ + aλ, bλ, c γ and dδ, where $0 < a < 1$, $\xi/\lambda < b < 1+\xi/\lambda$, $-(\xi+\lambda) < c < (\xi+\lambda)$, $d > 0$ with a=0.5, b=0.75, c=0.5 and d=0.5 respectively. The largest observation of the first sample was replaced with the second sample. Test statistic W_U was calculated and it was equal to 5.820842. This on comparison with critical value was found to be lying in the critical region. This implies that largest observation is a contaminant observation.

4. Performance study

All the performance studies were done using simulation technique. The performance study was done for slippage alternatives for location, scale and both the shape parameters.

For a slippage alternative of the location parameter, a random sample of size n was generated using R software from a Johnson S_B distribution with location parameter ξ(=20), scale parameter λ(=10), with two shape parameters γ(=1) and δ(=2) (known). For introducing a contaminant observation another sample of Johnson S_B distribution with a shift, (aλ), where $0 < a < 1$ in the location parameter was generated. The largest observation of the original sample was replaced with the largest observation of the second sample. The value of the test statistic W_U was calculated at 5% level of significance. Simulation study was carried out for sample sizes n = 10(10)30, 60, 100, 200, 500, 1000 and different values of a. Here, the process was simulated 10,000 times and the number of times the test statistic falling in the critical region was noted. The probability of rejection of the null hypothesis is then the ratio of total number of times the null hypothesis was rejected to the total number of trials, i.e. 10,000. These probabilities for different sample sizes- n and different values of a between 0.2 and 1 are shown in the table 4.1.

Table 4.1: Probabilities of identification of the upper contaminant observation with a shift in the location parameter by an amount aλ.

n \ a	0.2	0.3	0.4	0.5	0.6	0.8	1
10	0.8600	0.9903	0.9999	1.0000	0.9998	0.9848	0.8872
20	0.9346	0.9988	0.9998	1.0000	0.9994	0.9581	0.7470
30	0.9580	0.9999	0.9997	1.0000	0.9981	0.9298	0.6255
60	0.9883	1.0000	0.9998	0.9999	0.9953	0.8156	0.3382
100	0.9955	1.0000	0.9999	0.9992	0.9845	0.6372	0.1276
200	0.9989	1.0000	1.0000	0.9965	0.9427	0.2962	0.0087
500	1.0000	1.0000	0.9995	0.9807	0.7384	0.0218	0.0000
1000	1.0000	1.0000	0.9989	0.9130	0.3977	0.0006	0.0000

From table 4.1, it can be seen that the performance of the test is reasonably good enough for a shift of aλ in the location parameter with values of a up to 0.5, beyond that it starts declining. This is due to the fact that larger values of the location parameter leading to a smaller value of the statistic W_U as observed from (2.5). Also for large values of n, W_U being a fraction of two very small quantities tends towards zero with an increase in the value of δ.

Proceeding in a similar manner, for a slippage alternative of the scale parameter, a sample was generated and for introducing an outlying observation, another sample was generated with a shift b, where $\xi/\lambda < b < 1+\xi/\lambda$ in the scale parameter. The largest observation of the first sample was replaced with that of the second sample. The value of the test statistic W_U was calculated at 5% level of significance. This process was repeated 10,000 times and in this process, the number of times the null hypothesis was

rejected was observed. The probabilities of rejection of null hypothesis for sample sizes n = 10(10)30, 60, 100, 200, 500, 1000 and for different value of b between 1.2 and 2 are shown in the table 4.2.

Table 4.2: Probabilities of identification of the upper contaminant observation with a shift in the scale parameter by b.

n \ b	1.2	1.3	1.4	1.5	1.6	1.8	2
10	0.5133	0.7156	0.8601	0.9431	0.9762	0.9978	0.9971
20	0.6184	0.8454	0.9529	0.9891	0.9976	0.9994	0.9940
30	0.6842	0.8994	0.9773	0.9957	0.9998	0.9987	0.9841
60	0.7988	0.9651	0.9975	0.9997	1.0000	0.9963	0.9501
100	0.8685	0.9868	0.9993	0.9998	0.9999	0.9890	0.8768
200	0.9396	0.9978	1.0000	1.0000	0.9986	0.9485	0.6480
500	0.9838	0.9998	1.0000	1.0000	0.9909	0.7593	0.1916
1000	0.9960	1.0000	1.0000	1.0000	0.9601	0.4141	0.0145

It can be observed from table 4.2 that, the test statistic is performing well for the detection of upper outlier for different sample sizes up to b = 1.5. For sample size up to 100 it increases up to b = 1.6, beyond that it start decreasing and for sample size 200 to 1000 it start decreasing beyond b = 1.5. This is because for large value of shift the denominator of (2.5) becomes very large which leads to a smaller value of the statistic W_U.

Again for studying the outlier arising due to a shift in the shape parameter (γ) a sample was generated with the original parameters. Then for introduction of an outlying observation, another sample was generated with a shift c in the shape parameter, where $-(\xi + \lambda) < c < (\xi + \lambda)$. The largest observation of the first sample was replaced with the second sample and a new sample was formed. Then the test statistic W_U was calculated at 5% level of significance. The Probability of rejection which is the probability of identification of the outlying observation is then the ratio of total number of outcomes to the total number of trials (i.e.10, 000). This process was repeated 10,000 times for different sample sizes and for different values of c. The probabilities of identification of outlying observation for sample sizes n = 10(10)30, 60, 100, 200, 500, 1000 and different values of shift c between -20 and 5 are shown in table 4.3.

Table 4.3: Probabilities of identification of the upper contaminant observation with a shift in the shape parameter (γ) by c.

n \ c	-0.95	-0.75	-0.50	-0.25	0.25	0.50	0.75	0.95
10	0.9194	0.8686	0.7798	0.6635	0.4003	0.2796	0.1821	0.1168
20	0.9663	0.9318	0.8544	0.7427	0.4375	0.2966	0.1847	0.1190
30	0.9756	0.9492	0.8884	0.7803	0.4717	0.3240	0.2007	0.1231
60	0.9912	0.9773	0.9318	0.8413	0.5267	0.3508	0.2050	0.1215
100	0.9941	0.9849	0.9520	0.8754	0.5473	0.3605	0.2092	0.1218
200	0.9978	0.9932	0.9730	0.9136	0.5931	0.3845	0.2128	0.1214
500	0.9993	0.9971	0.9842	0.9422	0.6475	0.4166	0.2217	0.1190
1000	0.9997	0.9986	0.9910	0.9614	0.6852	0.4368	0.2287	0.1170

It can be observed from this table, that for negative values of shift, the test statistic certainly identifies the contaminant observation, whereas for positive values of shift the performance of test statistic is extremely poor. For c=0 the performance is very good even for small sample sizes.

Now, for a slippage alternative of the shape parameter (δ), a sample was generated and for introducing a contaminant observation, another sample with a shift d in the shape parameter was generated. The largest observation of the first sample was replaced with the corresponding observation of the second sample and a new sample was formed. Then the test statistic W_U was calculated at 5% level of significance. This process was repeated 10,000 times for different sample sizes and for different values of d and the probability of identification of the contaminant observation was obtained. These probabilities for sample sizes n = 10(10)30, 60, 100, 200, 500, 1000 and for different values of d between 0.25 and 4 are shown in table 4.4.

Table 4.4: Probabilities of identification of the upper contaminant observation with a shift in the shape parameter (δ) by d.

n \ d	0.25	0.5	0.75	1.5	2	4
10	0.5936	0.3648	0.2012	0.0333	0.0125	0.0028
20	0.7982	0.4852	0.2386	0.0214	0.0046	0.0018
30	0.8920	0.5736	0.2780	0.0126	0.0026	0.0033
60	0.9806	0.7271	0.3315	0.0092	0.0034	0.0044
100	0.9964	0.8280	0.3742	0.0059	0.0028	0.0036
200	1.0000	0.9286	0.4472	0.0036	0.0029	0.0022
500	1.0000	0.9856	0.5480	0.0033	0.0027	0.0020
1000	1.0000	0.9974	0.6343	0.0029	0.0030	0.0030

It can be concluded from this table that the test statistic is performing good for the values of shift less than unity. For the values of shift greater than unity the performance of the test statistic decreases and for higher values of shift it gets stabilized.

5. Conclusion

It can be concluded that the test statistic is performing well for different sample sizes. On the other hand, for large shift and large sample sizes it may not perform well. This was happen due to its masking effect. This test statistic is good enough for population data as well as when data comes from Johnson S_B distribution.

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