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On the diophnatine equation $2^x + p^y = z^2$

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Abstract
In this paper, we find all the solutions of the titled equation where $p \equiv 3(mod\ 4)$ and x, y and z are non negative integers except $x = 1$.

Keywords: exponential Diophantine equation

1. Introduction
In 2011, A. Suvernamani [1], studied the titled equation and claimed to find all the solutions. But in 2013, Somchit Chotchaisthit [3], showed some misleading arguments in [1], and studied the equation for p = 11. In the same paper, author stated that to find all the solutions to the equation is still an open problem.
In this paper, we give all the solutions to the titled equation when $p \equiv 3(mod\ 4)$ and $\boldsymbol{x}, \boldsymbol{y}$ and $\boldsymbol{z}$ are non negative integers except the case when x = 1. We list some solutions to the equation for x = 1. We use Catalan's conjecture proved in [2].

2. Preliminaries
2.1 Proposition
(The Catalan's conjecture) (3,2,2,3) is a unique solution (a, b, x, y) of the Diophantine equation $a^x - b^y = 1$ where a, b, x and y are integers with $\min\{a, b, x, y\} > 1$.

Proof: See in [3]

2.2 Lemma
The equation $1 + p^y = z^2$, where p is odd prime, y and z are non negative integers has a unique solution $(p, y, z) = (3,1,2)$

Proof: Let p be prime, y and z are non negative integers such that $1 + p^y = z^2$. That is $p^y = (z+1)(z-1)$. Thus there are non negative integers α and β such that $p^\alpha = z + 1$ and $p^\beta = z - 1$, where $\alpha + \beta = y$ and $\alpha > \beta$. Then $p^\beta(p^{\alpha-\beta} - 1) = 2$. If $\beta \neq 0$ then taking this equation modulo p, we get $0 \equiv 2(mod\ p)$. But this is contradiction.
Let $\beta = 0$, then $p^\alpha = 3$ and thus $\alpha = 1$ and $p = 3$. Hence there is unique solution $(p, y, z) = (3,1,2)$.

3. Main Theorem
3.1 Theorem
The Diophantine equation $2^x + p^y = z^2$, where $p \equiv 3(mod\ 4)$, x, y and z are non negative integers, $x \neq 1$, then

i) for $p = 3$, the equation has the solutions (x, y, z) as $\{(3,0,3), (0,1,2), (4,2,5)\}$
ii) for every prime p, the equation has the solutions of the form $(x, y, z) = (3,0,3)$
iii) for primes of the form $p = 2^q - 1$, the solutions are given by $(p, x, y, z) = (2^q - 1, q + 2, 2, p + 2)$

Proof: Let $p \equiv 3(mod\ 4)$ and x, y and z be non negative integers such that

$$2^x + p^y = z^2$$

Here z must be greater than 1.

If $x = 0$, then by Lemma 2.2, we have unique solution $(p, x, y, z) = (3,0,1,2)$. So let $x > 1$.

If $y = 0$, then we have $z^2 - 2^x = 1$. By Catalan's conjecture we have $z = 3$ and $x = 3$. Hence for any prime p, we have the solutions $(x, y, z) = (3,0,3)$.

Observe that z^2 must be odd and $z^2 \equiv 1 (mod\ 4)$. Now $2^x \equiv 0 (mod\ 4)$ and Since $p \equiv 3 (mod\ 4)$, and $p^y \equiv 3^y (mod\ 4)$, we have y must be even.

Let $y = 2m$. Thus $2^x = (z + p^m)(z - p^m)$. Thus there are non negative integers α and β such that $2^\alpha = z + p^m$ and $2^\beta = z - p^m$, where $\alpha + \beta = x$ and $\alpha > \beta$. Then $2^\beta(2^{\alpha-\beta} - 1) = 2p^m$, i.e. $2^{\beta-1}(2^{\alpha-\beta} - 1) = p^m$. If $\beta - 1 \neq 0$, then taking this equation modulo 2, we get $p^m \equiv 0 (mod\ 2)$. But this is contradiction.

Let $\beta - 1 = 0$, *i.e.* $\beta = 1$. Then $2^{\alpha-1} - p^m = 1$. If $\min\{\alpha - 1, m\} > 1$, then by Catalan's conjecture, there is no solution. Again if $\alpha - 1 = 0$ or $\alpha - 1 = 1$, there is no solution.

Let $m = 1$, then $p = 2^{\alpha-1} - 1$, here p is of the form of Mersenne prime. Thus if $p = 2^q - 1$, then $x = q + 2, y = 2$ and $z = 2 + p$. Thus the solutions are given by $(p, x, y, z) = (2^q - 1, q + 2,2,2 + p)$ where p is Mersenne prime, $p = 2^q - 1$.

This complete the proof.

In this result, we exclude the case when $x = 1$, and we assert that the titled equation has many solutions in this case. We list some solutions in Table 1.

Table 1: Some solutions of Equation (1) when $x = 1$

SR NO	p	x	y	z
1	7	1	1	3
2	23	1	1	5
3	223	1	1	15
4	359	1	1	19
5	727	1	1	27
6	1087	1	1	33
7	1367	1	1	37
8	2207	1	1	47
9	3023	1	1	55
10	3967	1	1	63

4. Conclusion

In this paper, we found all the solutions of the titled equation for $p = 3(mod\ 4)$ except the case when $x = 1$. However we listed some solutions when $x = 1$. Nevertheless, to find all the solutions of titled equation when $x = 1$ is still open problem. Also to find all the solutions of $2^x + p^y = z^2$ for any prime p is still an open problem.

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