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Mathematical model of divorce epidemic in Ghana

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Abstract
A non-linear MSD mathematical model was used to study the dynamics of divorce epidemic. The existence and stability of the divorce free and endemic equilibria was discussed. The divorce free equilibrium was locally asymptotically stable if $R_0 \prec 1$ and unstable if $R_0 \succ 1$. Global stability of divorce free and endemic equilibria were also considered in the model, using Lassalle's invariance principle of Lyapunov functions. Numerical simulations were conducted to confirm our analytic results. Our findings was that, reducing the contact rate between the marriaged and divorce, increasing the number of marriage that go into separation and educating separators to refrain from divorce can be useful in combating the divorce epidemic.

Keywords: Divorce free, mathematical model, Lyapunov stability, reproductive number, equilibrium points

1. Introduction
Divorce rates in most countries have changed in recent years, although the pace and direction of change has differed across regions of the world (Cherlin 1992; Heaton, Cammack, and Young 2001). Curiously, the same factors associated with modernization (e. g. increasing female labour force participation and ideational change) that ushered in the rise in divorce rates in Western countries also contributed to the falling divorce trends in Southeast Asian Islamic countries, although the mechanisms involved were different (1997). The factors associated with modernization and economic development that are usually invoked to explain changes in divorce trends in both developed and developing countries are varied, but those that consistently emerge include increasing employment and educational opportunities for women, and increasing age at marriage (Guest 1992; Carmichael, Webster, and McDonald 1997; Jones 1997; Heaton, Cammack, and Young 2001; Hirschman and Teerawichitchainan 2003; Teerawichitchainan 2004). These factors have also been found to be significant correlates of marital dissolution in most countries, along with factors such as type of union, urban-rural residence during childhood, ethnicity, and religion.
Marriage in the Ghanaian certain is the union between a man and a woman. Marriage as an institution is protected by rites and laws. Once marriage is contracted it is presumed to be free from any impediments, which means, divorce was not a part of the original plan in marriage in the Ghanaian societies.
Divorce is the dissolution of marriage between two partness. A consequence of divorce incidence is the attacks against the family and attempts at re-defining marriage in contemporary society. The prevalence of divorce has led to concern about the children who experience divorce. Many of these children are assumed to be users of illicit substances (Simons, Lin, Gordon, Conger, & Lorenz, 1999) [1], have low academic achievement, be more likely to have a teen pregnancy, suffer from social isolation, and exhibit externalizing and internalizing behavior problems (Hipke, Wolchik, Sandler, & Braver, 2002) [2]. Additionally, there are assumptions of children with divorced parents being more likely to drop out of school or fall victim to adult mental health problems (Hipke *et al.* 2002) [2] or be angry, demanding, noncompliant, lack self-regulation, have low social responsibility and diminished achievement (Hetherington & Stanley-Hagan, 1999; Nair & Murray, 2005) [3, 4].
In this paper, we use SIR model to model divorce as an epidemic. The paper is organized as follows: In section 2, we present the model description and assumptions.

Stability analysis of the divorce- free and endemic equilibria is discussed in section 3. In Section 4, we use the numerical example to show the dynamical behavior of our results. In Section 5, we performed the sensitivity analysis of the basic reproductive number of the model. Section 6 is made up of discussion of our results. We ended the paper with a conclusion.

2. Materials and Methods

2.1 Model description

We formulate a mathematical model and divide the population into three compartments: marriage (M), separated (S), Divorced (D). The interaction between the three states are shown in the schematic diagram in Fig. 1.

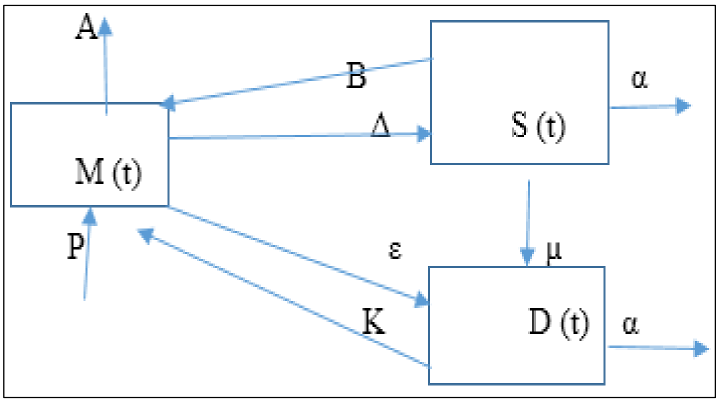

Fig 1: Schematic diagram of the three classes in the model

2.2 Model assumptions

The following assumptions were made in the model:

- The divorce epidemic occurs in a closed environment.
- Sex, race and social status do not affect the probability of being divorced.
- Members mix homogeneously (have the same interaction to the same degree)

The divorce epidemic is modelled using the system of Differential Equations.

$$\frac{dM}{dt} = P - \alpha M - \delta MS - \varepsilon MD + \beta S + \kappa D \quad (1)$$

$$\frac{dS}{dt} = \delta MS - \beta S - \mu S - \alpha S \quad (2)$$

$$\frac{dD}{dt} = \mu S + \varepsilon MD - \kappa D - \alpha D \quad (3)$$

With the initial conditions $M(0) \geq 0$, $D(0) \geq 0$ and $S(0) \geq 0$, where

p=rate of entering M
α= rate of leaving any of the three compartments through other means
δ=transmission rate from M to S
ε=transmission rate from M to D
β= transmission rate from S to M
κ= transmission rate from D to M
μ= transmission rate from S to D

We assume that the system of differential equations (1)-(3) has positive initial conditions, then every solution $(M(t), S(t), D(t))$ of (1)-(3) has the positive properties, that is $M(0) \geq 0$, $D(0) \geq 0$ and $S(0) \geq 0$. Hence the feasible region $\Omega = \left\{ (M, S, D) \in R^3 : M + S + D \leq \frac{p}{\alpha} \right\}$, is positively invariant set for the system (1)-(3).

This implies that: $N(t) = M(t) + D(t) + S(t)$ (4)

Also

$$\frac{dN}{dt} = \frac{dM}{dt} + \frac{dD}{dt} + \frac{dS}{dt}$$

$$\frac{dN}{dt} = p - \alpha M - \alpha D - \alpha S \leq p - \alpha N \qquad (5)$$

From (5), it follows that:

$$\lim_{t \to \infty} \sup N(t) \leq \frac{p}{\alpha}$$

Which means, the feasible region of the system (1)-(3) is given by the set Ω.

2.3 Model Analysis
2.3.1 Divorce-free equilibrium and basic reproductive number

In this section, we consider the divorce -free equilibrium $E_0 = \left(\frac{p}{\alpha}, 0, 0\right)$. That is a situation where there is no divorce problem.

We analyze the stability of the divorce-free equilibrium by considering the linearized system of ODE's (1)-(3), taking the Jacobian matrix we obtain

$$J(M,S,D) = \begin{bmatrix} -\alpha - \delta S - \varepsilon D & -\delta M + \beta & -\varepsilon M + \kappa \\ \delta S & \delta M - \beta - \mu - \alpha & 0 \\ \varepsilon D & \mu & \varepsilon M - \kappa - \alpha \end{bmatrix} \qquad (6)$$

The local stability of the equilibrium may be determined from the Jacobian matrix (6). This implies that the Jacobian matrix for the divorce-free equilibrium is given by

$$J(E_0) = \begin{bmatrix} -\alpha & -\delta + \beta & -\varepsilon + \kappa \\ 0 & \delta - \beta - \mu - \alpha & 0 \\ 0 & \mu & \varepsilon - \kappa - \alpha \end{bmatrix} \qquad (7)$$

From the characteristic equation of $J(M,0,0)$, the following eigenvalues were obtained: $\lambda_1 = -\alpha$, $\lambda_2 = \delta - \beta - \mu - \alpha$ and $\lambda_3 = \varepsilon - \kappa - \alpha$. It can be seen that λ_1 is real and negative. But we also know that, $R_0 \prec 1$, this means that $\delta \prec \beta + \mu + \alpha$ and $\varepsilon \prec \kappa + \alpha$, hence λ_2, λ_3 are also real and negative. This implies that the system (1) to (3) is asymptotically stable.

The basic reproductive number R_0, is given by $R_0 = \frac{\varepsilon}{\kappa + \alpha}$ (8)

Theorem 1: The divorce-free equilibrium $E_0 = \left(\frac{p}{\alpha}, 0, 0\right)$ of the system (1)-(3) is asymptotically stable if $R_0 \prec 1$ and unstable if $R_0 \succ 1$.

2.3.2 Endemic equilibrium

Evaluating the equilibrium points of the ODE (1)-(3) by setting the right -hand side of equation (1)-(3) to zero and then solve for M^*, S^* and D^*. We obtained:

$$M^* = \frac{\beta + \mu + \alpha}{\delta}, \qquad (9)$$

$$S^* = \frac{\left[\delta\kappa + \alpha\delta - \varepsilon(\beta + \mu + \alpha)\right]\left[\alpha\mu(\beta + \mu + \alpha) - \mu\delta\rho\right]}{\delta\mu\left[\delta\mu\kappa - \varepsilon\mu(\beta + \mu + \alpha) - \kappa\delta(\beta + \mu + \alpha) - \alpha\delta(\beta + \mu + \alpha) + \varepsilon(\beta + \mu + \alpha)^2 + \kappa\beta\delta + \alpha\beta\delta - \varepsilon\beta(\beta + \mu + \alpha)\right]}$$

and

$$D^* = \frac{\alpha\mu(\beta + \mu + \alpha) - \mu\delta\rho}{\delta\mu\kappa - \varepsilon\mu(\beta + \mu + \alpha) - \kappa\delta(\beta + \mu + \alpha) - \alpha\delta(\beta + \mu + \alpha) + \varepsilon(\beta + \mu + \alpha)^2 + \kappa\beta\delta + \alpha\beta\delta - \varepsilon\beta(\beta + \mu + \alpha)}$$

We now consider the case when $R_0 \succ 1$. At the endemic equilibrium, all the three states are present in the population. The steady states consider conditions under which all three states can coexist in the equilibrium. We represent $E^* = \left(M^*, S^*, D^*\right)$ as endemic equilibrium of the system (1)-(3) and $\left(M^* \neq 0, S^* \neq 0, D^* \neq 0\right)$. We Substitute the equilibrium points in (9) into equation (6), and get

$$J\left(E^*\right) = \begin{bmatrix} -\alpha - \dfrac{[\delta\kappa + \alpha\delta - \varepsilon q][\alpha\mu q - \mu\delta\rho]}{\mu\left[\delta\mu\kappa - \varepsilon\mu q - \kappa\delta q - \alpha\delta q + \varepsilon q^2 + \kappa\beta\delta + \alpha\beta\delta - \varepsilon\beta q\right]} - \varepsilon\dfrac{\alpha\mu q - \mu\delta\rho}{\delta\mu\kappa - \varepsilon\mu q - \kappa\delta q - \alpha\delta q + \varepsilon q^2 + \kappa\beta\delta + \alpha\beta\delta - \varepsilon\beta q} & -\mu - \alpha & -\varepsilon\dfrac{q}{\delta} + \kappa \\ \dfrac{[\delta\kappa + \alpha\delta - \varepsilon q][\alpha\mu q - \mu\delta\rho]}{\mu\left[\delta\mu\kappa - \varepsilon\mu q - \kappa\delta q - \alpha\delta q + \varepsilon q^2 + \kappa\beta\delta + \alpha\beta\delta - \varepsilon\beta q\right]} & 0 & 0 \\ \varepsilon\dfrac{\alpha\mu q - \mu\delta\rho}{\delta\mu\kappa - \varepsilon\mu q - \kappa\delta q - \alpha\delta q + \varepsilon q^2 + \kappa\beta\delta + \alpha\beta\delta - \varepsilon\beta q} & \mu & \varepsilon\dfrac{q}{\delta} - \kappa - \alpha \end{bmatrix} \quad (10)$$

Where $q = \beta + \mu + \alpha$

Let

$$A_{11} = -\alpha - \frac{[\delta\kappa + \alpha\delta - \varepsilon q][\alpha\mu q - \mu\delta\rho]}{\mu\left[\delta\mu\kappa - \varepsilon\mu q - \kappa\delta q - \alpha\delta q + \varepsilon q^2 + \kappa\beta\delta + \alpha\beta\delta - \varepsilon\beta q\right]} - \varepsilon\frac{\alpha\mu q - \mu\delta\rho}{\delta\mu\kappa - \varepsilon\mu q - \kappa\delta q - \alpha\delta q + \varepsilon q^2 + \kappa\beta\delta + \alpha\beta\delta - \varepsilon\beta q}$$

$$A_{12} = -\mu - \alpha$$

$$A_{13} = -\varepsilon\frac{q}{\delta} + \kappa$$

$$A_{21} = \frac{[\delta\kappa + \alpha\delta - \varepsilon q][\alpha\mu q - \mu\delta\rho]}{\mu\left[\delta\mu\kappa - \varepsilon\mu q - \kappa\delta q - \alpha\delta q + \varepsilon q^2 + \kappa\beta\delta + \alpha\beta\delta - \varepsilon\beta q\right]}$$

$$A_{22} = 0$$

$$A_{23} = 0$$

$$A_{31} = \varepsilon\frac{\alpha\mu q - \mu\delta\rho}{\delta\mu\kappa - \varepsilon\mu q - \kappa\delta q - \alpha\delta q + \varepsilon q^2 + \kappa\beta\delta + \alpha\beta\delta - \varepsilon\beta q}$$

$$A_{32} = \mu$$

$$A_{33} = \varepsilon\frac{q}{\delta} - \kappa - \alpha$$

Substituting A_{ij} into (10), we get

$$J\left(E^*\right) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \quad (11)$$

The characteristic equation of (12) can be computed as follows

$$\left|J\left(E^*\right) - \lambda I\right| = \begin{vmatrix} A_{11} - \lambda & A_{12} & A_{13} \\ A_{21} & A_{22} - \lambda & A_{23} \\ A_{31} & A_{32} & A_{33} - \lambda \end{vmatrix} = 0$$

$$\lambda^3 - (A_{11} + A_{22} + A_{33})\lambda^2$$

$$+ (A_{11}A_{22} - A_{12}A_{21} + A_{22}A_{32} - A_{13}A_{31} - A_{23}A_{32} + A_{11}A_{33})\lambda$$

$$+ (A_{12}A_{21}A_{33} - A_{12}A_{23}A_{31} - A_{13}A_{21}A_{32} + A_{13}A_{22}A_{31} - A_{11}A_{22}A_{33} + A_{11}A_{23}A_{32}) = 0$$

Letting

$$C_1 = A_{11} + A_{22} + A_{33}$$
$$C_2 = A_{11}A_{22} - A_{12}A_{21} + A_{22}A_{32} - A_{13}A_{31} - A_{23}A_{32} + A_{11}A_{33}$$
$$C_3 = A_{12}A_{21}A_{33} - A_{12}A_{23}A_{31} - A_{13}A_{21}A_{32} + A_{13}A_{22}A_{31} - A_{11}A_{22}A_{33} + A_{11}A_{23}A_{32}$$

We can write the characteristic equation above as:

$$\lambda^3 - C_1\lambda^2 + C_2\lambda + C_3 = 0 \quad (12)$$

Using the Routh –Hurwitz criterion. It can be seen that all eigenvalues of the characteristic equation (12) has negative real part if and only if: $C_1 \succ 0, C_3 \succ 0, C_1C_2 - C_3 \succ 0$ (13)

Theorem 2: E^* is asymptotically stable if and only if inequalities (13) is satisfied.

2.3.3 Global stability of the equilibrium points

2.3.3.1 Global stability of the divorce free equilibrium

We prove the global stability when $\varepsilon \le \alpha$

Theorem 3: The global stability E_0 is asymptotically stable in the region

$$\Omega = \left\{ (M, S, D) \in R^3 : M + S + D \le \frac{p}{\alpha} \right\} \text{ if } \varepsilon \le \alpha \text{ (note that } \varepsilon \le \alpha \text{ implies } R_0 \prec 1 \text{).}$$

Proof: It should be noted that $M \prec 1$ in Ω for time $(t) \succ 1$. Consider the Lyaponov function:

$$L = S + D$$

$$\frac{dL}{dt} = (\delta M - \beta - \alpha)S + (\varepsilon M - \kappa - \alpha)D$$

$$\le (\delta - \beta - \alpha)S + (\varepsilon - \kappa - \alpha)D$$

$\frac{dL}{dt} \prec 0$ For $\varepsilon \le \alpha$ and $\frac{dL}{dt} = 0$ only if $S = 0$ and $D = 0$. Therefore, the only trajectory of the system in which $\frac{dL}{dt} = 0$ is E_0. Hence, Lasalle's invariance principle, E_0 is globally asymptotically stable in Ω [Busenberg *et al*. (1990), Alkhudhari *et al.* (2014), Hirsch *et al.* (1974) and Cai *et al*. (2009)] [5, 6, 8, 7].

2.3.3.2 Global stability of the endemic equilibrium (E^*)

We determine the global stability of the endemic equilibrium in this section, by using the first and third equations of the system (1)-(3) that is:

$$\frac{dM}{dt} = P - \alpha M - \delta MS - \varepsilon MD + \beta S + \kappa D$$

$$\frac{dD}{dt} = \mu S + \varepsilon MD - \kappa D - \alpha D \tag{14}$$

In the region $\Omega^* = \{(M, D) \in R^2 : M + D \leq 1, M \succ 0, D \succ 0\}$, Ω^* is positively invariant, i.e. every solution of the model (14), with initial conditions in Ω^* remains there for time $(t \succ 0)$. We also consider

$\Omega^{**} = \left\{(M, D) : M + D\left(\frac{\alpha + \kappa}{\alpha}\right) = 1, M \succ 0, D \succ 0\right\}$ where $\Omega^{**} \subset \Omega^*$, Ω^{**} is positively invariant, $E^* \in \Omega^*$ and $p = \varepsilon$.

Theorem 4: The endemic equilibrium point E^* of model (14) is globally asymptotically stable if $R_0 \succ 1$ (This means that $\kappa \leq \varepsilon$).

Proof: From **theorem 1**, if $R_0 \succ 1$ in Ω^{**}, then E_0 is unstable. Also Ω^{**} is positively invariant subset of Ω^* and the ω -limit set of each solution of model (13) is a single point in Ω^{**} since there is no periodic solutions, homoclinic loops and oriented phase polygons inside Ω^{**} if. $\kappa \leq \varepsilon$ Therefore E^* is globally asymptotically stable [Alkhudhari *et al*. (2014) and Cai *et al.* (2009)] [6, 7].

3 Results

3.1 Numerical Example

In this section, we use numerical simulations to show the dynamical behavior of our results, by assuming that our total population(M,S,D) is 100% and M =0.55, D =0.30 and S =0.15. The other parameters that would be used in this section are displayed in Table 1 and Table 2 respectively.

Table 1: Model parameters at divorce free equilibrium

parameter	p	ε	κ	α	β	δ	μ
value	0.4	0.5	0.4	0.25	0.09	0.4	0.2

Table 2: Model parameters at endemic equilibrium

parameter	p	ε	κ	α	β	δ	μ
value	0.4	0.6	0.2	0.25	0.1	0.5	0.025

3.2 Sensitivity Analysis of the Basic Reproductive Numbers

We investigate the nature of the model by conducting sensitivity analysis of the reproductive number (R_0).

(a) At the divorce –free equilibrium $\varepsilon = 0.5$, $\kappa = 0.4$ and $\alpha = 0.25$, R_0 = 0.76923<1.

If the value of ε is increased to any figure greater than 0.65 and the values of κ and α are maintained R_0 >1.

(b) At the endemic equilibrium, $\varepsilon = 0.6, \kappa = 0.2$ and $\alpha = 0.25$, $R_0 = 1.333 \succ 1$

If ε is reduced to 0.44 and α, κ are maintained the same, $R_0 \prec 1$.

4. Discussion of Results

We use MSD model to study the dynamics of divorce as an epidemic. We discussed the existence and stability of divorce free and endemic equilibria and performed the sensitivity analysis of the reproductive numbers. Based on the data in Table 1, the basic reproductive number of the divorce free equilibrium is estimated to be R_0 = 0.76923<1. This implies that only marriage population is present and the separated and divorced populations reduces to zero (S= 0, D= 0).

This means that the model is asymptotically stable at R_0 <1 and therefore satisfies Theorem1. This has been verified numerically in Fig. 2. In the stability analysis of the divorce free equilibrium, the eigenvalues are $\lambda_1 = -0.25$, $\lambda_2 = -0.05$ and $\lambda_3 = -0.15$. This also indicates that the divorce free equilibrium is asymptotically stable.

Considering the situation when $R_0 > 1$, the reproductive number of the endemic equilibrium is estimated to be

R_0 =1.333 >1 using the data in Table 2. This shows the situation in which the marriage, separated and divorced coexist in the population $(M^*, S^*, D^*) = (0.75, 0.204, 0.85)$. This indicates the existence of divorce problem in the population. People with divorce problem will continue to transform more Marriaged into divorced and the divorced free equilibrium becomes unstable at $R_0 \succ 1$. This is in line with our analytical results and has also been verified numerically.

5. Conclusion

Our model shows that, divorce epidemic cannot only be controlled by reducing the contact rate between the marriage and divorce but also increasing the number of marriages that go into separation and educating separated to refrain from divorce can be useful in combating the epidemic.

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