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On multivariate Bayesian frailty models

SG Parekh and SR Patel

Abstract
Defining the multivariate (mostly bi-variate) frailty models such as correlated gamma, correlated compound Poisson, correlated log normal frailty models through survival functions, we obtained the estimators of frailty parameters in this article. Further the estimation of Bayesian frailty parameters of multivariate normal and compound Poisson have been obtained for some priors with quadratic loss function.

Keywords: Bayesian estimation, Correlated compound distribution, Frailty distribution, Multivariate models, Survival function

1. Introduction
Amongst many authors Clayton (1978) [3], Hougaard (1986a, 1986b) [6, 7], McGilchrist and Aisbett (1991) [13], Sahu *et al*. (1997) [17], Parekh *et al*. (2015) [14], Hanagal (2011) [5] have defined bivariate frailty distributions by either probability measure or conditional distribution methods and some of them have used them in the application of some diseases of kidney or cancer. Further Ibrahim *et al*. (2001) [9], Kheiri *et al*. (2007) [10], Santos *et al*. (2010) [18] and Parekh *et al*. (2016) [15] have obtained Bayesian frailty estimators.

In this paper we defined bivariate frailty models with the use of correlated variates. In section 2 we describe some literature on bi-variate survival function, $S(t_1, t_2)$ of two timings T_1 and T_2 such as T_1 is the time interval when catheter is inserted to a kidney patient and the removal time due to infection and T_2 is the time interval when catheter is again inserted second time and his returning time due to second time infection. Section 3 is devoted for correlated gamma frailty model (bi-variate gamma frailty distribution) for which its parameters are estimated. Section 4 discusses the correlated compound Poisson frailty model whereas correlated log normal frailty model is discussed for estimation of its parameters by using Marcov Chain Monte Carlo (MCMC) method and 3-level hierarchical prior distribution in section 5. For multivariate normal distribution frailty Bayesian estimators of parameters have been obtained by using prior distribution as normal and quadratic loss function in section 6. Section 7 is devoted for the estimation of Bayesian frailty parameters of correlated compound Poisson distribution.

Throughout this article Y represents logT where T is lifetime variable. In the special analysis we consider some of the baseline distributions and frailty prior distributions

2. Survival Function of Bi-variate frailty Model
Let T_1 and T_2 be time variables and Z as frailty and let us assume that T_1 and T_2 are independently distributed. Then hazard functions of Cox- model will be.

$$h(t_1, t_2) = Z.h_0(t_1, t_2)e^{X'\beta}$$

Where h_0 is base line hazard function, $\beta' = (\beta_1, \beta_2, \dots \beta_p)$ is a vector of fixed effect parameters, $X' = (X_1, X_2, \dots X_p)$ is vector of fixed observations and Z has frailty distribution with probability density function $f(z, \theta)$, where θ is frailty parameter. This is known as shared frailty model.

Let $H_0(t_1)$ and $H_0(t_2)$ be cumulative hazard functions of T_1 and T_2 respectively and let T_1 and T_2 be independent time variables given the frailty, Z and L_z be Laplace transformation of Z. Then the bivariate survival function of T_1 and T_2 in cox-model is

$$S(t_1, t_2) = L_z\left[\{H_0(t_1) + H_0(t_2)\}e^{X'\beta}\right]$$

If $P(Z = 0) = 1$ then T_1 and T_2 are independent otherwise if $P(Z > 0) = 1$ then T_1 and T_2 are dependent.

When conditional survival function $S_j(t_j \mid Z)$ is integrated out for Z, we get marginal survival functions

$$S_j(t_j) = ES_j(t_j \mid Z)$$

$$= ES_j(t_j)^z$$

$$= Ee^{-z(H_{0j}(t_j)}$$

$$= L(H_{0j}(t_j))$$

so that

$$S(t_1, t_2 \mid Z) = S_1(t_1)^z S_2(t_2)^z$$

$$= e^{-zH_{01}(t_1)} e^{-zH_{02}(t_2)}$$

$$= e^{-z(H_{01}(t_1)+H_{02}(t_2))}$$

and $\quad (t_1, t_2) = ES_1(t_1)^z S_2(t_2)^z$

$$= Ee^{-zH_{01}(t_1)} e^{-zH_{02}(t_2)}$$

$$= L(H_{01}(t_1) + H_{02}(t_2))$$

Generally gamma distribution with mean 1 and variance σ^2 is taken as standard assumption as frailty distribution. Using this property

$$\mathrm{S}(\mathrm{t}_1, \mathrm{t}_2) = \mathrm{L}\left(\mathrm{H}_{01}(\mathrm{t}_1) + \mathrm{H}_{02}(\mathrm{t}_2)\right)$$

$$= \left(1 + \sigma^2\left(\mathrm{H}_{01}(\mathrm{t}_1) + \mathrm{H}_{02}(\mathrm{t}_2)\right)\right)^{-\frac{1}{\sigma^2}}$$

$$= \left(\mathrm{S}_1(\mathrm{t}_1)^{-\sigma^2} + \mathrm{S}_2(\mathrm{t}_2)^{-\sigma^2} - 1\right)^{-\frac{1}{\sigma^2}}$$

Generalizing this result for T_1, T_2,…, T_p the unconditional survival function will be

$$\mathrm{S}(\mathrm{t}_1, \mathrm{t}_2, \dots, t_p) = \left(\sum_{t=1}^{p} \mathrm{S}_i(\mathrm{t}_\mathrm{i})^{-\sigma^2} - p + 1\right)^{-\frac{1}{\sigma^2}}$$

3. Bi-variate gamma frailty model

Correlated gamma frailty model was presented by Yashin *et al*. (1993, 1995) [25] and applied to related lifetimes in many different situations, e.g., Yashin *et al*. (1996), Yashin and Iachine (1997, 1999a,b), Iachine (2002), Zdravkovic *et al*. (2004), Wienke *et al*. (2005a).

Let $\alpha_0, \alpha_1, \alpha_2$ be some real positive numbers. Set $p_1 = \alpha_0 + \alpha_1$ and $p_2 = \alpha_0 + \alpha_2$. Let X_0, X_1, X_2 be independently gamma distributed random variables with $X_0 \sim G(\alpha_0, p_0), X_1 \sim G(\alpha_1, p_1), X_2 \sim G(\alpha_2, p_2)$. Consequently,

$$Z_1 = \frac{p_0}{p_1} X_0 + X_1 \sim G(\alpha_0 + \alpha_1, p_1) \quad (3.1)$$

$$Z_2 = \frac{p_0}{p_2} X_0 + X_2 \sim G(\alpha_0 + \alpha_2, p_2) \quad (3.2)$$

and

$$\mathrm{E}Z_1 = EZ_2 = 1, V(Z_1) = \frac{1}{p_1} := {\sigma_1}^2, V(Z_2) = \frac{1}{p_2} := {\sigma_2}^2$$

The following relations hold:

$$E{X_0}^2 = V(X_0) + (EX_0)^2 = \frac{\alpha_0}{{p_0}^2} + \left(\frac{\alpha_0}{p_0}\right)^2 = \frac{{\alpha_0}^2 + \alpha_0}{{p_0}^2}$$

$$EZ_1Z_2 = E\left(\frac{p_0}{p_1}X_0 + X_1\right)\left(\frac{p_0}{p_2}X_0 + X_2\right)$$

$$= E\left(\frac{{p_0}^2}{p_1p_2}{X_0}^2 + \frac{p_0}{p_1}X_0X_2 + \frac{p_0}{p_2}X_0X_1 + X_1X_2\right)$$

$$= \frac{{p_0}^2}{p_1p_2}\frac{{\alpha_0}^2+\alpha_0}{{p_0}^2} + \frac{p_0\alpha_0\alpha_2}{p_1p_0p_2} + \frac{p_0\alpha_0\alpha_1}{p_2p_0p_1} + \frac{\alpha_1\alpha_2}{p_1p_2}$$

$$= \frac{\alpha_0 + (\alpha_0+\alpha_1)(\alpha_0+\alpha_2)}{p_1p_2}$$

$$= \frac{\alpha_0}{(\alpha_0+\alpha_1)(\alpha_0+\alpha_2)} + 1$$

$$Cov(Z_1, Z_2) = EZ_1Z_2 - EZ_1EZ_2 = \frac{\alpha_0}{(\alpha_0+\alpha_1)(\alpha_0+\alpha_2)}$$

This leads to the correlation

$$\rho = \frac{Cov(Z_1,Z_2)}{\sqrt{V(Z_1)V(Z_2)}} = \frac{\alpha_0}{\sqrt{(\alpha_0+\alpha_1)(\alpha_0+\alpha_2)}} \tag{3.3}$$

Consequently, because of relation $\alpha_0 + \alpha_i = p_i = \frac{1}{{\sigma_i}^2}, (i = 1,2)$

it holds that $\alpha_0 = \frac{\rho}{\sigma_1\sigma_2}$

$$\alpha_i = \frac{1}{{\sigma_i}^2} - \alpha_0 = \frac{1 - \frac{\sigma_i}{\sigma_j}\rho}{{\sigma_i}^2}(i, j = 1,2; i \neq j)$$

We note that given σ_1, σ_2 and ρ, the values of α_0, α_i, α_2 can be obtained as estimates and vice varsa.
Now we can derive the unconditional model, applying the Laplace transform of gamma distributed random variables. Hence,

$$S(t_1, t_2) = ES(t_1, t_2 \mid Z_1, Z_2) \tag{3.4}$$

$$= ES_1(t_1 \mid Z_1)S_2(t_2 \mid Z_2)$$

$$= Ee^{-Z_1H_1(t_1)}e^{-Z_2H_2(t_2)}$$

$$= Ee^{-(\frac{p_0}{p_1}X_0+X_1)H_1(t_1)}e^{-(\frac{p_0}{p_2}X_0+X_2)H_2(t_2)}$$

$$= Ee^{-X_0\left(\frac{p_0}{p_1}H_1(t_1)+\frac{p_0}{p_2}H_2(t_2)\right)-X_1H_1(t_1)-X_2H_2(t_2)}$$

$$= \left(1 + \frac{1}{p_0}\left(\frac{p_0}{p_1}H_1(t_1) + \frac{p_0}{p_2}H_2(t_2)\right)\right)^{-\alpha_0}\left(1 + \frac{1}{p_1}H_1(t_1)\right)^{-\alpha_1}\left(1 + \frac{1}{p_2}H_2(t_2)\right)^{-\alpha_2}$$

$$= \left(1 + {\sigma_1}^2H_1(t_1) + {\sigma_2}^2H_2(t_2)\right)^{\frac{-\rho}{\sigma_1\sigma_2}}\left(1 + {\sigma_1}^2H_1(t_1)\right)^{\frac{-1+\frac{\sigma_1}{\sigma_2}\rho}{{\sigma_1}^2}} * \left(1 + {\sigma_2}^2H_2(t_2)\right)^{\frac{-1+\frac{\sigma_2}{\sigma_1}\rho}{{\sigma_2}^2}}$$

which results in the following representation of the correlated gamma frailty model:

$$S(t_1, t_2) = \frac{{S_1(t_1)}^{1-\frac{\sigma_1}{\sigma_2}\rho}{S_2(t_2)}^{1-\frac{\sigma_2}{\sigma_1}\rho}}{\left(S_1(t_1)^{-{\sigma_1}^2} + S_2(t_2)^{-{\sigma_2}^2} - 1\right)^{\frac{\rho}{\sigma_1\sigma_2}}}, \tag{3.5}$$

using the independence of the gamma distributed random variables X_0, X_1, X_2. The range of the correlation between frailties depends on the values of σ_1 and σ_2:

$$0 \leq \rho \leq min\left\{\frac{\sigma_1}{\sigma_2}, \frac{\sigma_2}{\sigma_1}\right\}.$$

Hence, if $\sigma_1 \neq \sigma_2$, it is always less than one. This property can be a serious limitation when the values of σ_1 and σ_2 differ strongly.

4. Bi-variate Compound Poisson Frailty Model

Sometimes some individuals like cancer patients may survive their cancer and in such cases the shared frailty model will fail to explain the frailty. If one of the individuals of the married couple has some problem in fertility and thereby they do not conceive a child which is known as zero susceptibility, so that they may take some time to divorce, which means couples have zero susceptibility. In such situations correlated compound Poisson frailty model is useful.

Yashin *et al*. (1999a) have introduced this frailty model as an extension of the correlated PVF frailty model. It is based on a bivariate extension of the compound Poisson frailty model, Aalen (1988, 1992). It is also related to the correlated gamma frailty cure model by Wienke *et al*. (2003a), which allows for a non-susceptible fraction in the population.

Let K_0, K_1 be some real positive variables and let X_0, X_1, X_2 be independently compound Poisson distributed random variables with $X_0 \sim CP(\alpha, k_0, \lambda)$, $X_1 \sim CP(\alpha, k_1, \lambda)$ and $X_2 \sim CP(\alpha, k_1, \lambda)$. Consequently, using a similar additive structure for the frailties as in (3.1) it holds that

$$Z_1 = X_0 + X_1 \sim CP(\alpha, k_0 + k_1, \lambda)$$
$$Z_2 = X_0 + X_2 \sim CP(\alpha, k_0 + k_1, \lambda)$$

Here we consider only the symmetric case, where the two life times are interchangeable. An extension to the non-symmetric case is straightforward. Also, the following relations are assumed:

$EZ_1 = EZ_2 = 1, V(Z_1) = V(Z_2) = \sigma^2.$
$\Rightarrow (k_0 + k_1)\lambda^{\alpha-1} = 1$ and $(k_0 + k_1)(1-\alpha)\,\lambda^{\alpha-2} = \sigma^2$.
also, $(k_0 + k_1)\lambda^{\alpha-2} = \frac{1}{\lambda}$ and $(k_0 + k_1)\lambda^{\alpha-2} = \frac{\sigma^2}{1-\alpha} = \frac{1}{\lambda}$.

Hence, $\lambda = \frac{1-\alpha}{\sigma^2}$, which results in

$$(k_0 + k_1)\lambda^{\alpha} = \lambda = \frac{1-\alpha}{\sigma^2} \tag{4.1}$$

It holds that

$$EX_0{}^2 = V(X_0) + (EX_0)^2 = k_0(1-\alpha)\lambda^{\alpha-2} + (k_0\lambda^{\alpha-1})^2$$
$$EZ_1Z_2 = E(X_0 + X_1)(X_0 + X_2)$$
$$= E(X_0{}^2 + X_0X_1 + X_0X_2 + X_1X_2)$$
$$= k_0(1-\alpha)\lambda^{\alpha-2} + k_0{}^2\lambda^{2\alpha-2} + k_0k_1\lambda^{2\alpha-2} + k_0k_1\lambda^{2\alpha-2} + k_1{}^2\lambda^{2\alpha-2}$$
$$= k_0(1-\alpha)\lambda^{\alpha-2} + (k_0 + k_1)^2\lambda^{2\alpha-2}$$
$$= k_0(1-\alpha)\lambda^{\alpha-2} + 1$$
$$cov(Z_1, Z_2) = EZ_1Z_2 - EZ_1EZ_2 = k_0(1-\alpha)\lambda^{\alpha-2}$$

This leads to the correlation

$$\rho = \frac{cov(Z_1, Z_2)}{\sqrt{V(Z_1)V(Z_2)}}$$
$$= \frac{k_0(1-\alpha)\lambda^{\alpha-2}}{(k_0+k_1)(1-\alpha)\lambda^{\alpha-2}}$$
$$= \frac{k_0}{k_0+k_1} \tag{4.2}$$

Consequently, because of (4.1) and (4.2)

$$k_0\lambda^{\alpha} = \frac{k_0}{k_0 + k_1}(k_0 + k_1)\lambda^{\alpha}$$
$$= \rho\frac{1-\alpha}{\sigma^2}. \tag{4.3}$$

Now we can derive the unconditional model, applying the Laplace transform of compound Poisson distributed random variables as,

$$S(t_1, t_2) = ES(t_1, t_2 \mid Z_1, Z_2)$$
$$= ES(t_1 \mid Z_1)S(t_2 \mid Z_2)$$
$$= Ee^{-Z_1H_0(t_1)}e^{-Z_2H_0(t_2)}$$
$$= Ee^{-(X_0+X_1)H_0(t_1)}e^{-(X_0+X_2)H_0(t_2)}$$
$$= Ee^{-X_0(H_0(t_1)+H_0(t_2))-X_1H_0(t_1)-X_2H_0(t_2)}$$
$$= e^{-\frac{k_0}{\alpha}\left(\left(\lambda+H_0(t_1)+H_0(t_2)\right)^{\alpha}-\lambda^{\alpha}\right)}e^{-\frac{k_1}{\alpha}\left(\left(\lambda+H_0(t_1)\right)^{\alpha}-\lambda^{\alpha}\right)}.e^{-\frac{k_1}{\alpha}\left(\left(\lambda+H_0(t_2)\right)^{\alpha}-\lambda^{\alpha}\right)} \tag{4.4}$$

The marginal survival function gives that

$$S(t) = e^{-\frac{k_0+k_1}{\alpha}((\lambda+H_0(t))^{\alpha}-\lambda^{\alpha})}, \tag{4.5}$$
$$\Rightarrow \lambda + H_0(t) = \left(\lambda^{\alpha} - \frac{\alpha}{k_0+k_1}lnS(t)\right)^{\frac{1}{\alpha}} \tag{4.6}$$

Hence, using (4.2) and (4.5)

$$e^{-\frac{k_1}{\alpha}\left(\left(\lambda+H_0(t)\right)^{\alpha}-\lambda^{\alpha}\right)} = e^{-\frac{k_1}{k_0+k_1}\frac{k_0+k_1}{\alpha}((\lambda+H_0(t))^{\alpha}-\lambda^{\alpha})}$$

$= e^{-(1-\rho)\frac{k_0+k_1}{\alpha}((\lambda+H_0(t))^{\alpha}-\lambda^{\alpha})}$

$= \left(e^{-\frac{k_0+k_1}{\alpha}((\lambda+H_0(t))^{\alpha}-\lambda^{\alpha})}\right)^{1-\rho}$

$= S(t)^{1-\rho}$

For the first term in (4.4) holds because of (4.6)

$$e^{-\frac{k_0}{\alpha}((\lambda+H_0(t_1)+H_0(t_2))^{\alpha}-\lambda^{\alpha})} = e^{-\frac{k_0}{\alpha}((\lambda+H_0(t_1)+\lambda+H_0(t_2)-\lambda)^{\alpha}-\lambda^{\alpha})}$$

$$= e^{-\frac{k_0}{\alpha}\left(\left(\left(\lambda^{\alpha}-\frac{\alpha}{k_0+k_1}lnS(t)\right)^{\frac{1}{\alpha}}+\left(\lambda^{\alpha}-\frac{\alpha}{k_0+k_1}lnS(t)\right)^{\frac{1}{\alpha}}-\lambda\right)^{\alpha}-\lambda^{\alpha}\right)}$$

$$= e^{\frac{k_0\lambda^{\alpha}}{\alpha}\left(1-\left(\left(1-\frac{\alpha}{(k_0+k_1)\lambda^{\alpha}}lnS(t_1)\right)^{\frac{1}{\alpha}}+\left(1-\frac{\alpha}{(k_0+k_1)\lambda^{\alpha}}lnS(t_2)\right)^{\frac{1}{\alpha}}-1\right)^{\alpha}\right)}$$

which results because of (4.1) and (4.3) in the following representation of the correlated compound Poisson frailty model:

$$S(t_1,t_2) = S(t_1)^{1-\rho}S(t_2)^{1-\rho} *exp\left\{\frac{\rho(1-\alpha)}{\alpha\sigma^2}\left[1-\left(\left(1-\frac{\alpha\sigma^2}{1-\alpha}lnS(t_1)\right)^{\frac{1}{\alpha}}+\left(1-\frac{\alpha\sigma^2}{1-\alpha}lnS(t_2)\right)^{\frac{1}{\alpha}}-1\right)^{\alpha}\right]\right\}$$

As special cases this model includes both gamma and inverse Gaussian correlated frailty models. For $\alpha = 0$ the gamma model obtained. For $\alpha \geq 0$ the correlated PVF frailty model is obtained, for $\alpha < 0$ the correlated compound Poisson frailty model is obtained.

5. Bi-variate Log-Normal Frailty Model

Let $(X_1, X_2)'$ have bi-variate normal distribution with mean vector $(m, m)'$and variance co-variance matrix as $\begin{pmatrix} s^2 & rs^2 \\ rs^2 & s^2 \end{pmatrix}$ and let $lnZ_j = X_j (j = 1, 2)$. Then $(Z_1, Z_2)'$ is said to have bi-variate log-normal distribution with frailty parameters σ^2 and ρ which can be described as under

$\mu = \mathrm{E}(\mathrm{Z_j}) = \mathrm{e}^{\mathrm{m}+\frac{s^2}{2}}$; $j = 1, 2$

$\sigma^2 = V(Z_j) = e^{2m+s^2}(e^{s^2} - 1)$; $j = 1, 2$

$\rho = corr(Z_1, Z_2) = \frac{e^{rs^2-1}}{e^{s^2}-1}$

Xue and Brookmeyer (1996) have introduced first time the correlated log-normal frailty model and applied it to mental health data to evaluate the health policy effects for inpatient psychiatric care. Cook *et al*. (1999) have used this frailty model in two-state mixed renewal processes for chronic disease.

5.1 MCMC method for the log-normal frailty model

Amongst gamma frailty model and lognormal frailty model, the lognormal approach is much more flexible then gamma model but gamma frailty model is mathematically easy and the likelihood function can be easily written down, whereas in lognormal frailty model maximum likelihood equations are not easily solvable and so Markov Chain Monte Carlo (MCMC) methods are used. In the gamma model Yashin *et al*. (1995), Wienke *et al*. (2003a, b) have applied procedures based on maximum likelihood methods. McGilchrist and Aisbett (1991) [13], McGilchrist (1993) [12], Lillard *et al*. (1995) [11], Sastry (1997) [19] and Ripatti *et al*. (2002) [16] have used maximum likelihood method in lognormal structure.

Here we apply this method for estimating parameters of correlated lognormal frailty model in following three Bayesian hierarchical (3 - levels) model in the following way

1. Likelihood function:

$$L(t,\delta \mid X,c,d) = \prod_{i=1}^{n}\prod_{j=1}^{2}\left(\exp(X_{ij})\, c\, \exp(dt_{ij})\right)^{\delta_{ij}} exp\left(-\exp(X_{ij})\frac{c}{d}(exp(dt_{ij})-1)\right)$$

2. Priors:

(i) $\begin{pmatrix} X_{i1} \\ X_{i2} \end{pmatrix} \sim N\left(\begin{pmatrix} -\frac{1}{2}\ln(\sigma^2+1) \\ -\frac{1}{2}\ln(\sigma^2+1) \end{pmatrix}, \begin{pmatrix} \ln(\sigma^2+1) & \ln(\rho\sigma^2+1) \\ \ln(\rho\sigma^2+1) & \ln(\sigma^2+1) \end{pmatrix}\right)$

(ii) $c \sim G(0.01, 0.1)$

(iii) $d \sim G(0.01, 0.1)$

3. Hyper priors:

(i) $\sigma^2 \sim \Gamma(0.01, 0.01)$

(ii) $\rho \sim U(-1, 1)$

where $X = (X_1, X_2, \ldots, X_n)$, $X_i = (X_{i1}, X_{i2})$, $t = (t_1, t_2, \ldots, t_n)$, $t_i = (t_{i1}, t_{i2})$, G and U denote the gamma and uniform distribution, respectively, and c and d are parameters of the Gompertz baseline hazard. The prior (i) assigned to the vector (X_{i1}, X_{i2}), is chosen in order to have, according to the traditional definition of frailty, a vector of log-normal distributed frailties $(Z_{i1}, Z_{i2}) = exp(X_{i1}, X_{i2})$, whose mean is equal to one. Finally, non-informative priors are assigned to the parameters of the Gompertz curve and to the frailty parameters. The full conditional distributions can be obtained because they are proportional to the joint distribution of all the random quantities of the model.

6. Bayesian Frailty estimation of Multivariate Normal distribution with Multivariate Normal distribution.

We extended the result obtained by Parekh *et al*. (2016) [15] for baseline distribution as univariate normal and prior frailty distribution as univariate normal. By taking baseline distribution as multivariate normal and prior frailty distribution as multivariate normal we obtained Bayesian frailty estimator of the parameters of prior distribution for quadratic loss function.

Theorem 6.1

Let $\underline{y} \mid \theta, \Sigma$ have $N_p(\theta, \Sigma)$ baseline distribution and θ have prior frailty distribution, $\pi(\theta)$ as $N_p(\underline{\mu}, A)$ where $\underline{\mu}$ is known. Σ and A are $(p \text{ x } p)$ known positive definite matrices. Then the Bayesian frailty estimate of θ is

$\underline{y} - \Sigma(A + \Sigma)^{-1}\left(\underline{y} - \underline{\mu}\right)$.

Proof: The joint density, $h(y, \theta)$ of $\underline{y}$ and $\underline{\theta}$ is

$$h(y,\theta) = \frac{1}{(2\pi)^p |\Sigma|^{\frac{1}{2}} |A|^{\frac{1}{2}}} exp\left\{-\frac{1}{2}\left[\left(\underline{y} - \underline{\theta}\right)' \Sigma^{-1}\left(\underline{y} - \underline{\theta}\right) + \left(\underline{\theta} - \underline{\mu}\right)' A^{-1}\left(\underline{\theta} - \underline{\mu}\right)\right]\right\} \quad (6.1)$$

Now

$$\left(\underline{y} - \underline{\theta}\right)' \Sigma^{-1}\left(\underline{y} - \underline{\theta}\right) + \left(\underline{\theta} - \underline{\mu}\right)' A^{-1}\left(\underline{\theta} - \underline{\mu}\right)$$
$$= \underline{y}'\Sigma^{-1}\underline{y} - 2\underline{\theta}'\Sigma^{-1}\underline{y} + \underline{\theta}'\Sigma^{-1}\underline{\theta} + \underline{\theta}'A^{-1}\underline{\theta} - 2\underline{\theta}'A^{-1}\underline{\mu} + \underline{\mu}'A^{-1}\underline{\mu}$$
$$= \underline{\theta}'(\Sigma^{-1} + A^{-1})\underline{\theta} - 2\underline{\theta}'\left(\Sigma^{-1}\underline{y} + A^{-1}\underline{\mu}\right) + \underline{y}'\Sigma^{-1}\underline{y} + \underline{\mu}'A^{-1}\underline{\mu}$$
$$= \underline{\theta}'C\underline{\theta} - \underline{\theta}'CC^{-1}\left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right) - \left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)' CC^{-1}\underline{\theta} + \underline{y}'\Sigma^{-1}\underline{y} + \underline{\mu}'A^{-1}\underline{\mu} + \left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)' C\left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right) - \left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)' C\left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right) \quad where\ C = \Sigma^{-1} + A^{-1}$$
$$= \left[\underline{\theta} - C^{-1}\left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)\right]' C\left[\underline{\theta} - C^{-1}\left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)\right] - \left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)' . C^{-1}\left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right) + \underline{y}'\Sigma^{-1}\underline{y} + \underline{\mu}'A^{-1}\underline{\mu} \quad (6.2)$$

As

$$C^{-1} = (\Sigma^{-1} + A^{-1})^{-1} = \Sigma - \Sigma(\Sigma + A)^{-1}\Sigma = A - A(A + \Sigma)^{-1}A$$

So that

$$\left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)' C^{-1}\left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)$$
$$= \left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)' [\Sigma - \Sigma(\Sigma + A)^{-1}\Sigma]\left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)$$
$$= \left[\underline{\mu}'A^{-1}\Sigma + \underline{y}'\Sigma^{-1}\Sigma - \underline{\mu}'A^{-1}\Sigma(\Sigma + A)^{-1}\Sigma - \underline{y}'\Sigma^{-1}\Sigma(\Sigma + A)^{-1}\Sigma\right]\left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)$$
$$= \left[\underline{\mu}'A^{-1}\Sigma + \underline{y}' - \underline{\mu}'A^{-1}\Sigma(\Sigma + A)^{-1}\Sigma - \underline{y}'(\Sigma + A)^{-1}\Sigma\right]\left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)$$
$$= \underline{\mu}'A^{-1}\Sigma A^{-1}\underline{\mu} + \underline{x}'A^{-1}\underline{\mu} - \underline{\mu}'A^{-1}\Sigma(\Sigma + A)^{-1}\Sigma A^{-1}\underline{\mu} - \underline{y}'(\Sigma + A)^{-1}\Sigma A^{-1}\underline{\mu} + \underline{\mu}'A^{-1}\underline{y} + \underline{y}'\Sigma^{-1}\underline{y} - \underline{\mu}'A^{-1}\Sigma(\Sigma + A)^{-1}\underline{y} - \underline{y}'(\Sigma + A)^{-1}\underline{y}$$

and hence

$$\underline{y}'\Sigma^{-1}\underline{y} + \underline{\mu}'A^{-1}\underline{y} - \left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)' C^{-1}\left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)$$
$$= \underline{y}'\Sigma^{-1}\underline{y} + \underline{\mu}'A^{-1}\underline{y} - \underline{\mu}'A^{-1}\Sigma A^{-1}\underline{\mu} - \underline{\mu}'A^{-1}\underline{y} - \underline{y}'\Sigma^{-1}\underline{y} + \underline{\mu}'A^{-1}\Sigma(\Sigma + A)^{-1}\Sigma A^{-1}\underline{\mu} + 2\underline{\mu}'A^{-1}\Sigma(\Sigma + A)^{-1}\underline{y} + \underline{y}'(\Sigma + A)^{-1}\underline{y}.$$
$$= \underline{y}'(\Sigma + A)^{-1}\underline{y} - \underline{\mu}'A^{-1}[\Sigma - \Sigma(\Sigma + A)^{-1}\Sigma]A^{-1}\underline{\mu} + \underline{\mu}'A^{-1}\underline{\mu} - 2\underline{\mu}'A^{-1}\underline{y} + 2\underline{\mu}'A^{-1}\Sigma(\Sigma + A)^{-1}\underline{y}$$
$$= \underline{y}'(\Sigma + A)^{-1}\underline{y} + \underline{\mu}'A^{-1}\underline{\mu} - \underline{\mu}'A^{-1}(A^{-1} + \Sigma^{-1})^{-1}A^{-1}\underline{\mu} - 2\underline{\mu}'A^{-1}\underline{y} + 2\underline{\mu}'A^{-1}\Sigma(\Sigma + A)^{-1}\underline{y}$$
$$= \underline{y}'(\Sigma + A)^{-1}\underline{y} + \underline{\mu}'[A^{-1} - A^{-1}(A^{-1} + \Sigma^{-1})^{-1}A^{-1}]\underline{\mu} - 2\underline{\mu}'[A^{-1} - A^{-1}\Sigma(\Sigma + A)^{-1}]\underline{y}$$
$$= \underline{y}'(\Sigma + A)^{-1}\underline{y} + \underline{\mu}'(\Sigma + A)^{-1}\underline{\mu} - 2\,\underline{\mu}'[A^{-1} - A^{-1}(A + \Sigma) - A(A + \Sigma)^{-1}]\underline{y}$$
$$= \underline{y}'(\Sigma + A)^{-1}\underline{y} + \underline{\mu}'(\Sigma + A)^{-1}\underline{\mu} - 2\underline{\mu}'(\Sigma + A)^{-1}\underline{y}$$
$$= \left(\underline{\mu} - \underline{y}\right)'(A + \Sigma)^{-1}(\underline{\mu} - \underline{y}) \quad (6.3)$$

Substituting (6.3) in (6.2), we get

$$= \left[\underline{\theta} - C^{-1}\left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)\right]' C\left[\underline{\theta} - C^{-1}\left(A^{-1}\underline{\mu} + \Sigma^{-1}\underline{y}\right)\right] - \left(\underline{\mu} - \underline{y}\right)'(A + \Sigma)^{-1}(\underline{\mu} - \underline{y})$$

and hence (6.1) will reduce to

$$h\left(\underline{y},\underline{\theta}\right)=\frac{1}{(2\pi)^{\frac{p}{2}}|\Sigma|^{\frac{1}{2}}|A|^{\frac{1}{2}}}exp\left\{-\frac{1}{2}\left[\underline{\theta}-C^{-1}\left(A^{-1}\underline{\mu}+\Sigma^{-1}\underline{y}\right)\right]'C\left[\underline{\theta}-C^{-1}\left(A^{-1}\underline{\mu}+\Sigma^{-1}\underline{y}\right)\right]\right\}.exp\left\{-\frac{1}{2}(\underline{y}-\underline{\mu})'(A+\Sigma)^{-1}(\underline{\mu}-\underline{y})\right\}$$

$$=\frac{1}{(2\pi)^{\frac{p}{2}}|C^{-1}|^{\frac{1}{2}}}exp\left\{-\frac{1}{2}\left(\underline{\theta}-\mu_y^{\pi}\right)'C\left(\underline{\theta}-\mu_y^{\pi}\right)\right\}.\frac{1}{(2\pi)^{\frac{p}{2}}|A+\Sigma|^{\frac{1}{2}}}exp\left\{-\frac{1}{2}(\underline{y}-\underline{\mu})'(A+\Sigma)^{-1}(\underline{y}-\underline{\mu})\right\}, \quad (6.4)$$

using $|\Sigma+A||C^{-1}| = |\Sigma||A|$

where

$$\mu_y^{\pi}=C^{-1}\left(A^{-1}\underline{\mu}+\Sigma^{-1}\underline{y}\right)$$
$$=(\Sigma^{-1}+A^{-1})^{-1}\left(A^{-1}\underline{\mu}+\Sigma^{-1}\underline{y}\right)$$
$$=(\Sigma^{-1}+A^{-1})^{-1}A^{-1}\underline{\mu}+(\Sigma^{-1}+A^{-1})^{-1}\Sigma^{-1}\underline{y}$$
$$=[A-A(A+\Sigma)^{-1}A]A^{-1}\underline{\mu}+[\Sigma-\Sigma(A+\Sigma)^{-1}\Sigma]\Sigma^{-1}\underline{y}$$
$$=\underline{\mu}-A(A+\Sigma)^{-1}\underline{\mu}+\underline{y}-\Sigma(A+\Sigma)^{-1}\underline{y}$$
$$=\underline{\mu}-(A+\Sigma-\Sigma)(A+\Sigma)^{-1}\underline{\mu}+\underline{y}-\Sigma(A+\Sigma)^{-1}\underline{y}$$
$$=\underline{\mu}-\underline{\mu}+\Sigma(A+\Sigma)^{-1}\underline{\mu}+\underline{y}-\Sigma(A+\Sigma)^{-1}\underline{y}$$
$$=\underline{y}-\Sigma(A+\Sigma)^{-1}(\underline{y}-\underline{\mu}) \quad (6.5)$$

Thus (6.4) shows that $h\left(\underline{y},\underline{\theta}\right)$ is the product of the conditional distribution of $\underline{\theta}$ given $\underline{y}$ which is $N_p(\mu_y^{\pi},C^{-1})$, where μ_y^{π} is given by (6.5) and the marginal distribution of $\underline{y}$ which is $N_p(\underline{\mu}\,,A+\Sigma)$ and hence for quadratic loss function the frailty Bayesian estimate of $\underline{\theta}$ is

$$\underline{y}-\Sigma(A+\Sigma)^{-1}(\underline{y}-\underline{\mu})$$

and covariance matrix is

$$A-A(A+\Sigma)^{-1}A.$$

Remark 6.1
If $\underline{y}_1,\underline{y}_2,\ldots,\underline{y}_n$ is a sample from $N_p(\underline{\theta}\,,\Sigma)$ and if $\underline{\theta}$ has frailty distribution $N_p(\underline{\mu}\,,A)$ where $\underline{\mu}\,,A,\Sigma$ are known then $\underline{\overline{y}}=(\overline{y}_1,\overline{y}_2,\ldots,\overline{y}_n)$ being sufficient for $\underline{\theta}$ and $\underline{\overline{y}}\mid\underline{\theta}$ has $N_p(\underline{\theta}\,,\frac{1}{n}\Sigma)$ distribution instead of taking $\pi(\underline{\theta}\mid\underline{y})$ we take $(\underline{\theta}\mid\underline{\overline{y}})$ as frailty prior distribution. Then the frailty Bayesian estimate of $\underline{\theta}$ will be obtained from (6.5) as
$\underline{\overline{y}}-\Sigma(nA+\Sigma)^{-1}(\underline{\overline{y}}-\underline{\mu})$
with Bayesian frailty covariance matrix as
$A-nA(nA+\Sigma)^{-1}A.$

7. Bayesian Frailty estimation of Compound Poisson distribution with Poisson distribution.
We consider the following theorem for the baseline distribution as Compound Poisson distribution with prior frailty distribution as Poisson distribution.

Theorem 7.1 (Bayesian estimate of compound Poisson frailty):
Let $X_1,X_2,\ldots,X_N$ be identically independently distributed as Gamma, $G(\alpha,\beta)$ variates with known scale parameter β and known shape parameter α and let N be prior frailty distribution $\pi(N)$, as Poisson with known mean ρ. Then for compound Poisson variate Z defined as

$$Z=\begin{cases}X_1+X_2+\cdots+X_N\,;\,if\ N>0\\ 0\,;\,if\ N=0\end{cases}$$

has the Bayesian frailty estimate, $\delta^{\pi}(N)$ of N will be

$$\delta^{\pi}(N)=\rho(\beta z)^{\alpha}.$$

Proof: Since $X_i\sim G(\alpha,\beta), i=1,2,\ldots,N$ with p.d.f.
$f(x_i;\alpha,\beta)=\frac{\beta^{\alpha}}{A\alpha}{x_i}^{\alpha-1}e^{-\beta x_i},\ i\ =\ 1,2,\ldots,n$
then $Z=\sum_{i=1}^{N}X_i\sim G(N\alpha,\beta)$ has p.d.f. as
$f(z\mid N)\ =\ \frac{\beta^{N\alpha}}{\Gamma(\alpha N)}z^{N\alpha-1}e^{-\beta z}\,,\alpha>0,\beta>0$
and as $N\sim\mathcal{P}(\rho)$, the p.d.f. of N is
$\pi(N)=\frac{e^{-\rho}\rho^{N}}{N!}\,,\,N=0,1,2,\ldots,\infty$

so that the joint p.d.f. of Z and N, $h(z,N)$ will be

$$h(z,N) = \frac{\beta^{N\alpha}\rho^N}{\Gamma(\alpha N)N!} z^{N\alpha-1} e^{-(\rho+\beta z)} . \tag{7.1}$$

Then the marginal distribution of $Z, m(z)$ will have p.d.f. as

$$m(z) = e^{-(\rho+\beta z)} \sum_{N=o}^{\infty} \frac{\beta^{N\alpha}\rho^N}{\Gamma(\alpha N)N!} z^{N\alpha-1}$$
$$= \frac{1}{z} e^{-(\rho+\beta z)} . \sum_{N=o}^{\infty} \frac{(\beta z)^{N\alpha}}{\Gamma(\alpha N)} . \frac{\rho^N}{N!} \tag{7.2}$$

and hence using (7.1) and (7.2) we get the posterior frailty distribution of N, $\pi(N, z)$ with p.d.f. as

$$\pi(N \mid z) = \frac{\frac{(\beta z)^{N\alpha}\rho^N}{\Gamma(\alpha N)N!}}{\sum_{N=0}^{\infty} \frac{(\beta z)^{N\alpha}\rho^N}{\Gamma(\alpha N)\ \ N!}}$$

Taking squared error loss function, the Bayesian frailty estimate, $\delta^{\pi}(z) = E(N \mid z)$ will be

$$\delta^{\pi}(N) = \rho(\beta z)^{\alpha}.$$

Conclusion

We have defined some of the shared frailty models such as Bivariate gamma frailty model, Compond Poisson shared frailty model as bivariate one, correlated log-normal frailty model, multivariate Normal frailty model and suggested that they may be fitted and estimated. Using MCMC method, bivariate log-normal frailty model is fitted. For multivariate Normal frailty model and Compound Poisson frailty model frailty Bayesian estimators have been derived for their parameters by using quadratic loss function.

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