ISSN: 2456-1452
Maths 2018; 3(2): 492-498

www.mathsjournal.com
Received: 22-01-2018
Accepted: 23-02-2018

Mihir B Trivedi
Research Scholar at Rai University, Ahmedabad, Gujarat, India

Dr. Pradeep J Jha
Professor of Mathematics at Rai University, Ahmedabad, Gujarat, India

Correspondence
Dr. Pradeep J Jha
Professor of Mathematics at Rai University, Ahmedabad, Gujarat, India

Additional mathematical properties of Narayana numbers and Motzkin numbers

Mihir B Trivedi and Dr. Pradeep J Jha

Abstract
The work that follows in the lines below is, in gross, outcome of our work in observing and deriving inter-terms relation between Narayana numbers and their associated properties with Catalan numbers and Motzkin Numbers. Again Motzkin Numbers and its associated determinant and minors have hidden properties which is also explored giving proof in terms of theorems. To this, we have successively evolved inter connectivity among Motzkin numbers, Narayana Numbers and Catalan Numbers.

Keywords: Catalan numbers, Narayana numbers, Motzkin numbers, principle of mathematical induction

1. Introduction
The contents of this paper have triangularised division, though different but all assembled and are mutually interwoven.
The first vertex, to some extent, prescribes some basic introduction to Catalan Numbers and describes our reframed work on the same. The second vertex deals with beautifully break-up of Catalan Numbers. These are Narayana Numbers, denoted as $N(n,k)$ with $n,k \in N$ and $n \geq k$ with $1 \leq k \leq n.$ These numbers are devotional to Catalan numbers but within each break-up such that their row wise sum for each value of n and k gives Catalan number.
What we have designed is their row wise and column wise inter relation which helps harmonized composition of the table.
The third and last in order is vertex with the label of Motzkin number which are strictly designed to follow some rules. Motzkin number (0, 0) to (n, 0), with some condition in a way that length of each segment of a path can be either 1 or $\sqrt{2}$. In addition to this, we have attempted to interpret planner graph associated with certain Motzkin Number. Also it is followed by some important properties.

2. Catalan Number
2.1 A brief introduction to our present work
Catalan Numbers are, as some call them superstitious numbers, backed by glorious beginning from the time of Chinese scientist Ming'antu (c. 1692 – c. 1763) who designed its pattern arising from geometrical construction. Along with the flow of time in 1751, Leonhard Euler, working on deriving the ways of triangulation of an (n+2)-gon found the result which we call Catalan Numbers. In this important work he was assisted by Christian Goldbach and more substantially by Johann Segner. Euler then observed that successive ratios have a pattern which can be given as

$$C_n = \frac{2*6*10*...*(4n-2)}{(n+1)!} \; for \; n \geq 1 \qquad \text{………. (1)}$$

Segner, on further work on the same issue, derived the recurrence relation between Catalan numbers and derived that with,

$$C_0 = 1, \; C_{n+1} = \sum_{i=0}^{n} C_i \, C_{n-i} \; for \; n \geq 0 \qquad \text{………. (2)}$$

It was the only time in 1838 when Belgium mathematician Charles Catalan, a tutor at Ecole

Polytechnique, was inspired and devoted some years of life in the parallel type of work and proved the useful results. The first one of its own type is

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1} \; for\ n \in N \qquad \text{.......... (3)}$$

Following the same work he worked on same subject and perceived many applications in numerous real life situations.
In this section, with the above brief notes we like to first put before our valued readers of what we have already done in our previous work.
As a part of our previous paper [1], what we have establish put below which may be noted and be used for the future purpose. These are as follows,

(1) $\begin{vmatrix} \binom{2n-1}{n-1} & \binom{2n-1}{n-1} \\ \frac{2n}{n+1} & 2 \end{vmatrix} = C_n, n \in N$

(2) $\begin{vmatrix} \binom{2n-2}{n-1} & \binom{2n-2}{n-2} \\ \binom{2n}{n} & \binom{2n}{n-1} \end{vmatrix} = C_n * C_{n-1}$, for $n \in N - \{2\}$,

(3) $C_n = 2^{2n-2} \prod_{k=2}^{n} \left(1 - \frac{3}{2k}\right) < 2^{2n-2}$

(4) $\lim_{n\to\infty} \frac{C_{n+1}}{C_n} = 4$

3. Narayana Number

Narayana Numbers, closely related to Catalan Numbers, are important and claim growing space in literature in the field of Mathematics. They are named after Indian mathematician T. V. Narayana (1930–1987)
On the same line working on combinatorics, Narayana suggested break-up of Catalan Numbers. The break-up is perfectly symmetrical on either side on the distribution pattern of the Catalan number C_n.
He designed the division format in such a way that

$$\sum_{k=1}^{k=n} N(n,k) = C_n$$

where $n, k \in N, 1 \le k \le n$ and $N(n,k)$ is the k^th^ Narayana number for a given n. .... (4)

Each term for a fixed n and k under above restriction and confined to the following formula.

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}; \; 1 \le k \le n \qquad \text{.......... (5)}$$

It is obvious from the above formula that N(n, 1) = 1 = N(n, n)

3.1 Narayana Number _ Tabular form

In this section we give a tabular form of N(n, k) for some initial values of $n \in N$ and dependent value of parameter k,

Table 1

k =	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
n = 1	1							
2	1	1						
3	1	3	1					
4	1	6	6	1				
5	1	10	20	10	1			
6	1	15	50	50	15	1		
7	1	21	105	175	105	21	1	
8	1	28	196	490	490	196	28	1

3.2 Some Important Clues

A close look on the study of distribution of Narayana numbers we have been prompted to work in deriving some important clues.
As we know that for a given $n \ge 1$ and each $k \in [1, n], n \in N$

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}; \; 1 \le k \le n$$

We are interested in investigating $N(n, k+1) - N(n, k)$.
This is the difference between values of two successive Narayana numbers for a given n and k.

Theorem 1
For a given system of Narayana numbers with above mentioned condition that $n \in N, n \geq 1, k \in [1, n]$ we have $N(n, k+1) - N(n, k) = (n-1)! \binom{n}{k} \left[\frac{(n-2k)(n+1)}{(k+1)!(n+1-k)!}\right]$.
In order to prove above equality we start with the definition of Narayana Numbers

$$N(n, k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$

L.H.S = $N(n, k+1) - N(n, k)$

$$= \frac{1}{n}\binom{n}{k+1}\binom{n}{k} - \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$

$$= \frac{1}{n}\binom{n}{k}\left[\frac{n!}{(k+1)!\,(n-k-1)!} - \frac{n!}{(k-1)!\,(n-k+1)!}\right]$$

$$= \frac{n!}{n}\binom{n}{k}\frac{1}{(k-1)!}\frac{1}{(n-k-1)!}\left[\frac{1}{k(k+1)} - \frac{1}{(n-k)(n-k+1)}\right]$$

$$= \frac{n!}{n}\binom{n}{k}\frac{1}{(k+1)!}\frac{1}{(n-k+1)!}\left[n^2 - 2nk - 2k + n\right]$$

$$= (n-1)!\binom{n}{k}\left[\frac{(n-2k)(n+1)}{(k+1)!(n-k+1)!}\right]$$

=R.H.S

Therefore,

$$N(n, k+1) - N(n, k) = (n-1)!\binom{n}{k}\left[\frac{(n-2k)(n+1)}{(k+1)!(n-k+1)!}\right] \quad \text{………. (6)}$$

In the same sequence we continue to establish one more theorem. This theorem aims at finding the difference between N(n+1, k) and N(n, k).

Theorem 2
In the notational terminology of Narayana numbers we will prove

$$N(n+1, k) - N(n, k) = N(n, k)\left[\frac{(k-1)(2n+2-k)}{(n+2-k)(n+1-k)}\right]$$

L.H.S = $N(n+1, k) - N(n, k)$

$$= \frac{1}{n+1}\binom{n+1}{k}\binom{n+1}{k-1} - \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$

$$= \frac{1}{n+1}\left[\frac{n+1}{n+1-k}\binom{n}{k}\frac{n+1}{n+2-k}\binom{n}{k-1}\right] - \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$

$$= \binom{n}{k}\binom{n}{k-1}\left[\frac{1}{n+1}\left(\frac{n+1}{n+1-k}\right)\left(\frac{n+1}{n+2-k}\right) - \frac{1}{n}\right]$$

$$= \frac{1}{n}\binom{n}{k}\binom{n}{k-1}\left[\frac{(k-1)(2n-k+2)}{(n+2-k)(n+1-k)}\right]$$

= R.H.S

Therefore,

$$N(n+1, k) - N(n, k) = N(n, k)\left[\frac{(k-1)(2n+2-k)}{(n+2-k)(n+1-k)}\right] \quad \text{………. (7)}$$

Theorem 3
Any Narayana Number N(n, k), $1 \leq k \leq n$ can be represented as a determinant of 2×2 Matrix M_n where Matrix M_n is as follows,

$$M_n = \frac{1}{n-k-1}\begin{bmatrix}\binom{n+1}{k+1} & \binom{n-1}{k+1} \\ \binom{n}{k+1} & \binom{n-1}{k+1}\end{bmatrix}; n > k+1$$

To give the proof of above statement we start with the definition of Narayana Numbers which can also be define as,

$$N(n,k) = \frac{1}{k+1}\binom{n}{k}\binom{n-1}{k} \qquad \text{………. (8)}$$

Using the following two properties of Binomial coefficient

a. $\binom{n}{k} = \binom{n+1}{k+1} - \binom{n}{k+1}$

b. $\binom{n-1}{k} = \binom{n}{k+1} - \binom{n-1}{k+1}$

and by result (8)

$$N(n,k) = \frac{1}{k+1}\left[\left\{\binom{n+1}{k+1} - \binom{n}{k+1}\right\}\left\{\binom{n}{k+1} - \binom{n-1}{k+1}\right\}\right] \qquad \text{………. (9)}$$

Now we prove that $\binom{n}{k+1} - \binom{n-1}{k+1} = \frac{k+1}{n-k-1}\binom{n-1}{k+1}$

$$\binom{n}{k+1} - \binom{n-1}{k+1} = \frac{n!}{(k+1)!\,(n-k-1)!} - \frac{(n-1)!}{(k+1)!\,(n-k-2)!}$$

$$= \frac{(n-1)!}{(k+1)!\,(n-k-2)!}\left[\frac{n}{n-k-1} - 1\right]$$

$$= \frac{k+1}{n-k-1}\binom{n-1}{k+1}$$

Using result (9)

$$N(n,k) = \frac{1}{k+1}\left[\left\{\binom{n+1}{k+1} - \binom{n}{k+1}\right\}\left\{\frac{k+1}{n-k-1}\binom{n-1}{k+1}\right\}\right]$$

$$= \frac{1}{n-k-1}\left[\binom{n+1}{k+1}\binom{n-1}{k+1} - \binom{n}{k+1}\binom{n-1}{k+1}\right]$$

Therefore,

$$N(n,k) = \frac{1}{n-k-1}\begin{vmatrix} \binom{n+1}{k+1} & \binom{n-1}{k+1} \\ \binom{n}{k+1} & \binom{n-1}{k+1} \end{vmatrix}$$

Remarks

1. The Determinant form of Narayana Number can also be represented as

$$N(n,k) = \begin{vmatrix} \binom{n-1}{k} & \binom{n}{k+1} \\ \binom{n}{k} & \binom{n+1}{k+1} \end{vmatrix}$$

Note that the binomial coefficient appears in above determinant form of $N(n,k)$ can be read in the Pascal Triangle also.

2. We can extract the Narayana numbers from the Pascal Triangle as a determinant of 2×2 matrix

Pascal Triangle

Table 2

N																	
0									1								
1								1		1							
2							1		2		1						
3						1		3		3		1					
4					1		4		6		4		1				
5				1		5		10		10		5		1			
6			1		6		15		20		15		6		1		
7		1		7		21		35		35		21		7		1	
8	1		8		28		56		70		56		28		8		1

The first Narayana Number N(1,1) = 1 is obtained by computing the determinant of matrix $\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$.

Similarly the next two Narayana Number N(2,1) and N(2,2) can be obtained by solving the determinants $\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$ and $\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$

3.3 Targeting a Cell in Narayan Number Tabulation

The study of Narayana number table and its property lead us to two important properties aiming at particular cell and its neighborhood. These are as follows.

Using equations (6) and (7) one can also derive the formulae to find entry of each cell of the Narayana Triangle. In this section we also aimed to derive the same using two equations mentioned above.

Using equation (6), we can write,

$$N(n,k+1) = N(n,k) + (n-1)!\binom{n}{k}\left[\frac{(n-2k)(n+1)}{(k+1)!\,(n-k+1)!}\right]$$

$$= \frac{1}{n}\binom{n}{k}\binom{n}{k-1} + (n-1)!\binom{n}{k}\left[\frac{(n-2k)(n+1)}{(k+1)!\,(n-k+1)!}\right]$$

$$= \frac{1}{n}\binom{n}{k}\left(\frac{n!}{(k-1)!\,(n-k+1)!}\right) + (n-1)!\binom{n}{k}\left[\frac{(n-2k)(n+1)}{(k+1)!\,(n-k+1)!}\right]$$

$$= \binom{n}{k}\left(\frac{(n-1)!}{(k-1)!\,(n+1-k)!}\right)\left(1 + \frac{(n+1)(n-2k)}{k(k+1)}\right)$$

$$= \binom{n}{k}\left(\frac{(n-1)!}{(k-1)!\,(n-k)!}\right)\left(\frac{(n-k)^2 + (n-k)}{k(k+1)(n-k+1)}\right)$$

$$= \frac{1}{n}\binom{n}{k}\left(\frac{n!}{k!\,(n-k)!}\right)\left(\frac{n-k}{k+1}\right)$$

$$= \binom{n}{k}^2\left(\frac{n-k}{n(k+1)}\right)$$

Therefore,

$$N(n,k+1) = \binom{n}{k}^2\left(\frac{n-k}{n(k+1)}\right) \;;\; k = 0,1,2,3,\ldots,n-1 \qquad \ldots\ldots\ldots (10)$$

Again, using equation (7)

$$N(n+1,k) = N(n,k)\left[1 + \frac{(k-1)(2n+2-k)}{(n+2-k)(n+1-k)}\right]$$

$$= \frac{1}{n}\left(\frac{n!}{k!\,(n-k)!}\right)\left(\frac{n!}{(k-1)!\,(n-k+1)!}\right)\left(\frac{n(n+1)}{(n-k+1)(n-k+2)}\right)$$

$$= \frac{(n!)(n!)(n+1)}{k!\,(k-1)!\,(n-k+1)!\,(n-k+2)!}$$

$$= \frac{(n+1)!}{k!\,(n-k+1)!}\frac{n!}{(k-1)!\,(n-k+2)!}$$

$$= \binom{n+1}{k}\frac{(n+1)!}{k!\,(n-k+1)!}\frac{k}{(n+1)(n-k+2)}$$

$$= \frac{k}{(n+1)(n-k+2)}\binom{n+1}{k}^2$$

Therefore,

$$N(n+1,k) = \frac{k}{(n+1)(n-k+2)}\binom{n+1}{k}^2 \qquad \ldots\ldots\ldots (11)$$

Thus we have proved above results as stated; its importance lies in directly locating the cell value on a given value of ‘n’ and ‘k’. The difference results helps advancing from a given cell.

4. Motzkin Numbers and Determinant Properties

Pacing with Catalan number to some extent, geometrically we have another system known as Motzkin number system. They, in geometric nature, are the advancing path of 2n steps or n steps from (0, 0) to (n, n) of total length 2n or (0, 0) to (n, 0) of total length n as the corresponding case be Catalan number denoted as C_n or Motzkin number denoted as M_n.
Some Motzkin Numbers are 1, 1, 2, 4, 9, 21, 51, 127…
They satisfy the following recurrence relation,

$M_{n+1} = M_n + \sum_{k=o}^{n-1} M_k * M_{n-k-1}$; $n \in N$ and $M_0 = 1 = M_1$

4.1 Determinant value Properties

Array of Motzkin numbers of size $(n \times 1) \times (n \times 1)$ with $n \geq 0, n \in N$ where

$$M_n = \sum_{k=0}^{k=\left[\frac{n}{2}\right]} \binom{n}{2k} C_k \text{ where } C_k \text{is the } k^{th} \text{ Catalan number.} \qquad \text{………. (12)}$$

$$\text{And } C_k = \frac{1}{k+1}\binom{2k}{k} \; with\; C_0 = 1 \qquad \text{………. (13)}$$

On considering a sequence of Motzkin Number we have form an infinite set of rectangular arrays of order $(n \times 1) \times (n \times 1)$ where $n \geq 1$.
Let us denote the set by notation $M_n = \{M_1, M_2, M_3, \dots, M_n\}$, where each M_i for $i \in [1, n]$ is a square matrix.

e.g. $M_1 = \begin{bmatrix} M_0 & M_1 \\ M_1 & M_2 \end{bmatrix}$, $M_2 = \begin{bmatrix} M_0 & M_1 & M_2 \\ M_1 & M_2 & M_3 \\ M_2 & M_3 & M_4 \end{bmatrix}$

and in general,

$$M_n = \begin{bmatrix} M_0 & M_1 & M_2 & \cdots & M_n \\ M_1 & M_2 & M_3 & \cdots & M_{n+1} \\ M_2 & M_3 & M_4 & \cdots & M_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_n & M_{n+1} & M_{n+2} & \cdots & M_{2n} \end{bmatrix} \qquad \text{………. (14)}$$

In connection to this we have two following properties stated in the form of theorems.

Theorem 4
To show that in above connection $|M_n| = 1$ where M_n is given by (14)

Proof: The proof is given by Principle of Mathematical Induction.

Part-1 for $n = 1$ we show that $|M_1| = 1$

$$M_1 = \begin{vmatrix} M_0 & M_1 \\ M_1 & M_2 \end{vmatrix}$$

We plug in Motzkin numbers and get

$$M_1 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$$

Part-2 We accept the result to hold true for $n = k$ i.e. we accept the

$$M_k = 1, \text{ where } M_k = \begin{vmatrix} M_0 & M_1 & M_2 & \cdots & M_k \\ M_1 & M_2 & M_3 & \cdots & M_{k+1} \\ M_2 & M_3 & M_4 & \cdots & M_{k+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_k & M_{k+1} & M_{k+2} & \cdots & M_{2k} \end{vmatrix} \qquad \text{………. (15)}$$

Part-3 We establish the result for $n = k + 1$ i.e. we will show that $M_{k+1} = 1$

$$\text{where} \quad M_{k+1} = \begin{vmatrix} M_0 & M_1 & M_2 & \cdots & M_{k+1} \\ M_1 & M_2 & M_3 & \cdots & M_{k+2} \\ M_2 & M_3 & M_4 & \cdots & M_{k+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_k & M_{k+1} & M_{k+2} & \cdots & M_{2k+2} \end{vmatrix} \qquad \text{………. (16)}$$

$$M_{k+1} = \begin{bmatrix} 0 & 0 & & 0 & \cdots & 0 & M_{k+1} \\ 0 & 0 & & 0 & \cdots & 0 & M_{k+2} \\ 0 & 0 & 0 & \cdots & & 0 & M_{k+3} \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & & 0 & M_{2k+1} \\ M_{k+1} & M_{k+2} & M_{k+3} & \cdots & & M_{2k+1} & M_{2k+2}-1 \end{bmatrix}$$

$$+ \begin{bmatrix} M_0 & M_1 & M_2 & \cdots & M_k & 0 \\ M_1 & M_2 & M_3 & \cdots & M_{k+1} & 0 \\ M_2 & M_3 & M_4 & \cdots & M_{k+2} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ M_k & M_{k+1} & M_{k+2} & \cdots & M_{2k} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

The first component on the RHS has a value zero while using the accepted result (15) of part-2, second component equals 1.
Therefore, $M_{k+1} = 0 + 1 = 1$
Thus the result is true for $n = k + 1$ and hence by Principle of Mathematical Induction we have $M_n = 1, n \geq 1$ without loss of generality.

Theorem 5
In the set of Motzkin rectangular array $M_n, n \geq 2$ we have the determinant value of minor of M_o ($M_o = 1$) has always an order of the minor.

e.g. $M_2 = \begin{bmatrix} M_0 & M_1 & M_2 \\ M_1 & M_2 & M_3 \\ M_2 & M_3 & M_4 \end{bmatrix}$

Minor of M_0 is $\begin{vmatrix} M_2 & M_3 \\ M_3 & M_4 \end{vmatrix}$

Plug in the values of Motzkin number minor of M_0 is $\begin{vmatrix} M_2 & M_3 \\ M_3 & M_4 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 4 & 9 \end{vmatrix} = 2$
which is the order of the corresponding minor.
In general it can be written as,

$$\begin{vmatrix} M_2 & M_3 & M_4 & \cdots & M_{n+1} \\ M_3 & M_4 & M_5 & \cdots & M_{n+2} \\ M_4 & M_5 & M_6 & \cdots & M_{n+3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ M_{n+1} & M_{n+2} & M_{n+3} & \cdots & M_{2n} \end{vmatrix} = n, \text{ Order of minor; } M_n = n^{th} \text{Motzkin Number}$$

This property can be established using the Principle of Mathematical Induction and properties of determinant.

5. Conclusion
As it happens in slim cases, we were inspired with certain units to be included in this segments and working on it have diversified in different areas connecting vertices, edges, and faces in Motzkin paths. This has widened our scope of research work leaving some open problems and to some extent these, with proven version, shall follow in the next research paper that will follow very shortly.

6. References

1. Mihir B Trivedi, Dr. Pradeep J Jha. Approaches differ: Catalan numbers. International Journal of Advanced Science and Research. 2017; 2(6):82-89.
2. Roy Oste, Joris Van der Jeugt. Motzkin paths, Motzkin polynomials and recurrence Relations, 2015.
3. Paul Berry. A Note on Narayana Triangles and Related Polynomials, Riordan Arrays, and MIMO Capacity Calculations. Journal of Integer Sequences. 2011, 14, Article 11.3.8.
4. Tom Davis. Catalan Numbers, 2016.
5. Donaghey R, Shapiro LW. Motzkin numbers. J. Comb. Theory A. 1977; 23:291-301.