International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2018; 3(2): 507-510

www.mathsjournal.com
Received: 24-01-2018
Accepted: 25-02-2018

PL Powar
Department of Mathematics and Computer Science, Rani Durgawati University, Jabalpur, Madhya Pradesh, India

GRK Sahu
Department of Mathematics, Rani Durgawati University Govt. Model Science College, Jabalpur, Madhya Pradesh, India

Akhilesh Pathak
Department of Mathematics, St. Aloysius College, Rani Durgawati University, Jabalpur, Madhya Pradesh, India

Correspondence
PL Powar
Department of Mathematics and Computer Science, Rani Durgawati University, Jabalpur, Madhya Pradesh, India

A fixed point theorem for F-PGA: Contractions on complete metric spaces

PL Powar, GRK Sahu and Akhilesh Pathak

Abstract
Piri *et al.* had (H. Piri, S. Rahrovi, P. Kumam, Generalization of Khan fixed point theorem, J. Math. Computer Sci., 17 (2017), 76–83.) established generalized Khan contraction in 2017. By considering the concept of F-contraction on generalized Khan contraction, the authors have initiated the idea of F-Khan contraction and established the existence and uniqueness of fixed point.
In the present paper, the authors have introduced the following more general versions of contraction map:
- The extension of generalized Khan- contraction viz. PGA-contraction has been initiated at the first step.
- In the next step, referring the concept of F- contraction, the more general form of PGA- contraction, in particular F-PGA-contraction has been defined.

The existence and uniqueness of a fixed point has been established in view of F-PGA contraction.

Keywords: Khan contraction, PGA-contraction, F-PGA contraction

1. Introduction
Wardowski [3] in 2012 had initiated the idea of F-contraction. By considering this concept of F-contraction on Ciric contraction, Piri *et al.* [4] have introduced the more general form of contraction in 2017. Moreover, in continuation to this work, they have also introduced the existence of Khan- contraction by using the idea of F-contraction and tackled the existence and uniqueness problems of fixed point.
In the present paper, initially we have defined a PGA- contraction map which is based on the concept of generalized Khan-contraction.
Then by applying the idea of F-contraction on PGA – contraction, the more general form of contraction map has been introduced and under this newly defined contraction map, the existence and uniqueness of fixed point has been established.

2. Preliminaries
For our generalized analysis, we require the following basic definitions:
Definition 2.1 [5]. Let F_r be the family of all functions $F: (0,\infty) \rightarrow R$ such that
(F1) F is strictly increasing, i.e. for all $x, y \in (0,\infty)$ and $x < y \Rightarrow F(x) < F(y)$;
(F2) for each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F3) there exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.
Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is said to be an

F-contraction on (X, d), if there exist $F \in F_r$ and $\tau \in (0,\infty)$ such that
$\forall\, x, y \in X, [d(Tx, Ty) > 0 \Rightarrow \tau + F\big(d(Tx,Ty)\big) \leq F(d(x,y))]$.

Example 2.2 Let $F: (0,\infty) \rightarrow R$ defined by $F(x) = logx + x$ and a mapping defined on complete metric space R with respect to usual metric d is $T: R \rightarrow R, T(x) = \frac{x}{2}$, then it may be verified easily that T is F-contraction for $\{\alpha_n\} = \{\frac{1}{n}\}$.

Definition 2.3 [2] Let (X, d) be the metric space. A mapping $T: X \to X$ is said to be
Generalized Khan contraction if there exists $k \in [0,1)$ and for all $x, y \in X$ the following condition holds.

$$d(Tx,Ty) \leq \begin{cases} \dfrac{k\big(d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)\big)}{\max\{d(x,Ty),d(Tx,y)\}} & if \max\{d(x,Ty),d(Tx,y)\} \neq 0 \\ 0 \; if \; \max\{d(x,Ty),d(Tx,y)\} = 0 \end{cases}$$

Example 2.4 Let (R, d) be a usual metric space. A mapping
$T: R \to R$ is defined as $T(x) = \frac{x}{3}$, then T satisfies the condition of generalized Khan-contraction.
In view of Definition 2.2, we now introduce the more general form of contraction viz. PGA-contraction by using the idea of generalized Khan- contraction which was initiated by Piri et. al.(c.f.[2]).

Definition 2.5 Let (X, d) be the metric space. A mapping $T: X \to X$ is said to be **PGA-contraction** if there exists $k \in [0,1)$ and for all $x, y \in X$, the following condition holds;

$$d(Tx,Ty) \leq k.max\{G(x,y), d(x,y)\}.$$

Where $G(x,y) = \begin{cases} \frac{(d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx))}{\max\{d(x,Ty),d(Tx,y)\}} & if \max\{d(x,Ty),d(Tx,y)\} \neq 0 \\ 0 \; if \max\{d(x,Ty),d(Tx,y)\} = 0 \end{cases}$ (2.1)

Example 2.6 Let (X, d) be a metric space with respect to usual metric d, where $X = [0,\infty)$. A mapping: $X \to X, T(x) = \frac{2x}{3} + 1$, T satisfies the condition of PGA- contraction but it is not generalized Khan Contraction.
Referring the definition 2.1 and definition 2.3, we now introduce the F -PGA contraction.

Definition 2.7 Let (X, d) be the metric space. A mapping $T : X \to X$ is said to be an
F-PGA contraction if there exists $\tau \in (0,\infty)$ and $F \in F_r$ such that for all $x, y \in X$

$$d(Tx,Ty) > 0 \Rightarrow \tau + F\big(d(Tx,Ty)\big) \leq F(\max\{G(x,y), d(x,y)\}) .$$

Where $G(x,y)$ is as defined in (2.1).

Example 2.8 Let $F: (0,\infty) \to R$ defined by $F(x) = logx$**.** A mapping $T: X \to X$ which satisfy the condition of F-PGA contraction then

$$\tau + log\big(d(Tx,Ty)\big) \leq log\left(\max\left\{\left[\begin{cases} \dfrac{d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)}{\max\{d(x,Ty),d(Tx,y)\}} & if \max\{d(x,Ty),d(Tx,y)\} \neq 0 \\ 0 \; if \max\{d(x,Ty),d(Tx,y)\} = 0 \end{cases}\right], d(x,y)\right\}\right)$$

$$loge^{\tau}.\big(d(Tx,Ty)\big) \leq log\left(\max\left\{\left[\begin{cases} \dfrac{d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx)}{\max\{d(x,Ty),d(Tx,y)\}} & if \max\{d(x,Ty),d(Tx,y)\} \neq 0 \\ 0 \; if \max\{d(x,Ty),d(Tx,y)\} = 0 \end{cases}\right], d(x,y)\right\}\right)$$

If maximum is $\frac{(d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx))}{\max\{d(x,Ty),d(Tx,y)\}}$ and $\max\{d(x,Ty),d(Tx,y)\} \neq 0$ then

$$d(Tx,Ty) \leq e^{-\tau} \frac{(d(x,Tx)d(x,Ty) + d(y,Ty)d(y,Tx))}{\max\{d(x,Ty),d(Tx,y)\}}$$

i.e Khan contraction is a special case of F-PGA contraction.
If maximum is $d(x,y)$ then

$$d(Tx,Ty) \leq e^{-\tau} d(x,y)$$

It is clear that a classical Banach contraction is a special case of F-PGA contraction in this case..

3. Main Result

In view of the definitions and preliminaries described in section 2, we are now set to state the main result of this paper.

Theorem 3.1 Let (X, d) be a complete metric space and $T : X \to X$ be an F-PGA contraction. Then, T has a unique fixed point $x^* \in X$.

Proof. Let $x \in X$ and $\{x_n\}$ be a sequence of point of X such that $x_{n+1} = Tx_n$
for all $n = 0, 1, 2, \dots, \dots$

Consider

$F\big(d(x_n,x_{n+1})\big) = F\big(d(Tx_{n-1},Tx_n)\big)$

$\leq F(max\{G(x_{n-1},x_n), d(x_{n-1},x_n)\}) - \tau$ (by the F-PGA contraction)

$$\leq F\left(max\left\{\frac{d(x_{n-1},Tx_{n-1})d(x_{n-1},Tx_n) + d(x_n,Tx_n)d(x_n,Tx_{n-1})}{\max\{d(x_{n-1},Tx_n), d(Tx_{n-1},x_n)\}}, d(x_{n-1},x_n)\right\}\right) - \tau$$

$\leq F(max\{d(x_{n-1},x_n), d(x_{n-1},x_n)\}) - \tau$

$\leq F\big(d(x_{n-1},x_n)\big) - \tau$

This inequality holds for all $= 1,2,\dots,\dots$, we apply the same inequality on the right hand side for $n = n-1$ and continue the process till n= 0 and get the following:

$F\big(d(x_n,x_{n+1})\big) \leq F\big(d(x_{n-2},x_{n-1})\big) - 2\tau$

$\leq \dots\dots\dots$

$\leq F\big(d(x_0,x_1)\big) - n\tau$ (3.1)

Letting $n \to \infty$ in eq. (3.1), we get

$\lim_{n\to\infty} F\big(d(x_n,x_{n+1})\big) \leq \lim_{n\to\infty}[F\big(d(x_0,x_1)\big) - n\tau]$

$\leq -\infty$

$\lim_{n\to\infty} F\big(d(x_n,x_{n+1})\big) = -\infty$

Since $F \in F_K$, by using $(F2)$ of Definition 2.1, we arrive at
$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0.$ (3.2)
In view of (F3) (c.f. Definition 2.1) $\exists\ k \in (0,1)$ such that

$\lim_{d(x_n,x_{n+1})\to 0^+}[d(x_n,x_{n+1})]^k F\big(d(x_n,x_{n+1})\big) = 0$

$\lim_{n\to\infty}[d(x_n,x_{n+1})]^k F\big(d(x_n,x_{n+1})\big) = 0$ (3.3)

Consider (3.1) again

$F\big(d(x_n,x_{n+1})\big) \leq F\big(d(x_0,x_1)\big) - n\tau$

$[d(x_n,x_{n+1})]^k\big[F\big(d(x_n,x_{n+1})\big) - F\big(d(x_0,x_1)\big)\big] \leq -[d(x_n,x_{n+1})]^k n\tau \leq 0$

Let $\beta_n = d(x_n,x_{n+1})$
$[\beta_n]^k[F(\beta_n) - F(\beta_0)] \leq -[\beta_n]^k n\tau \leq 0$ (3.4)
Letting $n \to \infty$ in (3.4) and using (3.2) and (3.3), we appeal to
$\lim_{n\to\infty}[\beta_n]^k F(\beta_n) - \lim_{n\to\infty}[\beta_n]^k F(\beta_0) \leq \lim_{n\to\infty} -[\beta_n]^k n\tau \leq 0$
i.e. $\lim_{n\to\infty} n[\beta_n]^k = 0$ (3.5)
Now, let us observe from (3.5) that
Given $\epsilon > 0, \exists\ n_1 \in N$ such that

$|n[\beta_n]^k - 0| < \epsilon\ \forall\ n \geq n_1$

$n[\beta_n]^k < \epsilon$

$\beta_n < \frac{\epsilon}{n^{\frac{1}{k}}}\ \forall\ n \geq n_1\ \left(\epsilon = \epsilon^{\frac{1}{k}}\right)$

Claim: $\{x_n\}$ is a Cauchy sequence.

Consider $m, n \in N$ such that $m > n > n_1$.

$$d(x_m, x_n) \le d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2})$$

$$+ \ \dots\dots\dots + d(x_{n+1}, x_n)$$

$$\le \beta_{m-1} + \beta_{m-2} + \ \dots\dots\dots + \beta_n$$

$$< \sum_{i=n}^{\infty} \beta_i \ \le \sum_{i=n}^{\infty} \frac{\epsilon}{i^{\frac{1}{k}}}$$

Since $k \in (0,1)$ then $\frac{1}{k} > 1$.
By P-series test, the series $\epsilon \sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ is convergent for $\frac{1}{k} > 1$.
Hence, $\{x_n\}$ is Cauchy sequence.
Since X is complete, $\exists\ x^* \in X$ such that $x_n \to x^*$.
We now show that this x^* is a fixed point of T.
Consider

$$d(Tx^*, x^*) = \lim_{n\to\infty} d(Tx_n, x_n) = \lim_{n\to\infty} d(x_{n+1}, x_n)$$

$$d(x^*, Tx^*) = 0 \ \Rightarrow x^* = Tx^*$$

Claim: x^* is unique.
Let if possible $\exists$ two fixed points x_1 and x_2 such that $x_1 \neq x_2$.
By the definition of fixed point $Tx_1 = x_1$ and $Tx_2 = x_2$.
Now by the F-PGA contraction, we have

$$\tau \le F\big(d(x_1, x_2)\big) - F\big(d(Tx_1, Tx_2)\big)$$

$$\tau \le \ F\big(d(x_1, x_2)\big) - F\big(d(x_1, x_2)\big)$$

$\tau \le 0$ which is a contradiction.
Hence, T has unique fixed point.

4. Conclusion: The existence and uniqueness of a fixed point in a complete metric space has been established with the more generalized form of contraction map viz. PGA - contraction and F-PGA contraction which in its turn is a significant contribution.

5. References

1. Khan MS. A fixed point theorem for metric spaces, Rend. Inst. Math. Univ. Trieste. 1976; 8**:**69-72.
2. Piri H, Marasi H, Kumam P. A fixed point theorem for F-Khan-contractions on complete metric spaces and application to integral equations , J. Nonlinear Sci. Appl. 2017; 10:4564-4573.
3. Wardowski P. Fixed point theory of a new type of contractive mappings in complete metric spaces, Fixed Poin Theory Appl, 2012, 6.
4. Piri H, Rahrovi S, Kumam P. Generalization of Khan fixed point theorem, J. Math. Computer Sci. 2017; 17:76-83.
5. Powar PL, Sahu GRK, Pathak Akhilesh. A Serendipity Fixed Point using dual F-Contraction, IOSR Journal of Mathematics, 2015, 63-66.