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Specification and stabiity of two equations vector autoregressive model for time series data analysis

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Abstract
The most fertile areas of contemporary time series research concerns multiequation models. The vector autoregressive (VAR) model is the multivariate counter part of the univariate autoregressive model. A VAR model can be used in examining the relationships among a set of variables. The estimates of the parameters of the VAR model can be used for forecasting purposes. Forecasting with a VAR is a multivariable extension of forecasting using a simple autoregression. The VAR analysis tools such as Granger Causality tests, Impulse Response Analysis and Variance Decomposition models can be used in establishing relationships among economic variables; and in the formulation of structured economic models.
In the present study a Two-Equations VAR model has been specified and stability conditions have been derived. Using Lag operators, the Two-Equations VAR model has been expressed in terms of Lag operators. This specification facilitates to study further inferential aspects of the VAR model for time series data analysis.

Keywords: Specification and stability, autoregressive model, time series data analysis

1. Introduction

The most fertile areas of contemporary time series research concerns multiequation models. We can get number of dynamic relationships using single equation time series method. The first chapter contains Transfer Function Analysis. These techniques are used to generalize the univariate methodology by allowing the time path of the dependent variable influenced by the exogenous variable. If it is not known, there is no feedback and transfer function analysis can be very effective tools for forecasting and hypothesis testing.

Transfer Function is an appropriate, if the sequence is stochastic. If $\{y_t\}$ is endogenous and $\{z_t\}$ is exogenous. The procedure is a straight forward modification of the standard Box-Jenkins methodology. The resulting impulse response function traces out the time path of $\{z_t\}$ realizations on the $\{y_t\}$ sequence.

Each variable is allowed to depend on its past realizations and on the past realizations of all other variables in the system. There is no special attention paid to parsimony since the imposition of the "incredible identification restrictions" may be inconsistent. Granger Causality Tests, block erogeneity, and lag length tests can help in a parsimonious model. Ordinary least squares gives efficient estimates of the VAR coefficients.

VAR Analysis is the underlying structural model cannot be recovered from estimated VAR. For each variable in the system, accounting techniques can be used.

(i) The percentage of the forecast error variance attributable to each of the other variables.
(ii) The impulse responses to the various innovations.

In VAR analysis, the system of equation is over parameterized. The Bayesian approach combines a set of prior with the VAR methods.

Litterman (1980) proposes a sensible set of Bayesian priors that have become the standard in Bayesian VAR models. The selection of such variables is intimately bound up with the time series concepts of cointegration and causality.

The modeling approach followed so far assumes that there is no feedback from the univariate or dependent variable to the independent or explanatory variables. The dynamic response of each variable to various economic shocks can be obtained and the restrictions of the model tested. Long run restrictions can aid in decomposing a series into its temporary and permanent components. As, opposed to the class of univariate decompositions. Decomposition in a VAR model can be exactly specified.
Here we develop a techniques, structural VARS that blend economic theory and multiple time series analysis. In a structural VAR, the restrictions of a particular economic model are imposed on the contemporaneous relationship among the variables.

2. Multivariate Linear Model for Time Series

A Multivariate time series model provides an adequate unrestricted approximation to be reduced form of an unknown structural specification of a simultaneous equations system. Under the assumption that an underlying structure exists, Zellner (1979) and Zellner and Palm (1974) have shown that any structural model can be written in the form of multivariate time series models. Among the different models, the multivariate time series models can take, additional or explanatory variables can be exogeneous. They can be event determined as in the case of intervention models, or they can be embody stochastic variation, as in the case of Transfer Function models.
Multivariate time series models have been developed, such as Vector Autoregressive models (VAR) that make no distinction between the dependent and explanatory variables.
Some Specifications of Multivariate Linear Time series models:

VAR (p) $\phi_P(L)X(t) = C + V(t)$

VMA (q) $X(t) = C + \theta_q(L)\,V(t)$

VARMA (p,q) $\varphi_p(L)x(t) = C + \theta_q(L)V(t)$

VARIMA (p, d, q) $\varphi_p(L)[1-L]^d \times(t) = C + \theta_q(L)V(t)$

BVAR (p) $x_i(t) = \sum_{j=1}^{n}\sum_{k=1}^{p}\alpha_{i,jk}x_j(t-k) + c_i + v_i(t)$

Where

$$\alpha_{i,ik} \sim N\{\delta_{ik}, \pi_5\pi_1/[k\exp(\pi_4)]\}$$

$$\alpha_{i,ik} \sim N\{0, \pi_5\pi_2\sigma_i^2/[k\exp(2\pi_4)\sigma_j^2]\}\ \forall i \neq 0$$

$$C_i \sim N\ (0,\ \pi_5\pi_3\pi_2\sigma_i^2)$$

$$\sum_{i=1}^{p}\alpha_{i,jk} \sim N(\delta_{ij}, \sigma_j^2/\pi_6 2\sigma_i^2)$$

ECM (p,q) $\phi_p(L)[1-L]x(t) = -Az(t-1) + \theta_q(L)V(t)$

3. Vector Autoression (VAR) Models

Sims (1980) suggested Vector Auto regression (VAR) models for forecasting macro time series. VAR assumes that all the variables are endogenous for instance, consider the following three macro series; money supply, interest rate and output. Vector of three independent variables as a (AR) function of its lagged values. If the number of lags (x) and number of equations (g) increase, then the degrees of freedom problem becomes more difficult. Generally, the number of parameters to be estimated becomes $g + xg^2$
For small samples, individual parameters may not be estimated. So, only simple VAR model can be considered for a small sample. The system of equations has the same set of variables in each equation SUR on the system is equivalent to OLS on each equation.
Under Normality of the disturbances, MLE as well as likelihood ratio test can be performed. One important application of LR tests in the context of VAR is in determining the choice of lags to be used. In this case, the Log-likelihood for restricted model with m lags and the unrestricted model with q> m lags.
LR test is asymptotically distributed as $\chi^2_{(q-m)g^2}$. In case of sample size T (large) estimate the large number of parameters (qg^2+g) for the unrestricted model. VAR model have been used to test the hypothesis that some variables do not Granger cause some other variables.

3.1 A Simple Vector Auto Regression (VAR)

Consider the simple form as

$$y_{1t} = m_1 + a_{11} y_{1,t-1} + a_{12} y_{2,t-1} + \in_{1t}$$

$$y_{2t} = m_2 + a_{21} y_{1,t-1} + a_{22} y_{2,t-1} + \in_{2t}$$

$$y_t = m + A y_{t-1} + \in_t \qquad \ldots (3.1)$$

Each variable is expressed as a linear combination of lagged values of itself. The VAR equations may be expanded to consider deterministic time trends and other exogeneous variable.
Where as in univariate case, the behaviour of y's will depend on the properties of the A matrix.
Let the eigen values and eigen vectors of a matrix A be

$$\wedge = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad c = \begin{bmatrix} \vdots & \vdots \\ c_1 & c_2 \\ \vdots & \vdots \end{bmatrix}$$

Provided that the eigen values are distinct. The eigen vectors are linearly independent and C will be non-singular.
Then

$$C^{-1} AC = \wedge \quad \text{And} \quad A = C \wedge C^{-1} \qquad \ldots (3.2)$$

Let us consider a new vector of variables z_r as

$$Z_t = c^{-1} y_t \quad \text{Or} \quad y_t = cZ_t \qquad \ldots (3.3)$$

By pre multiplying equation (3.1) by c^{-1} gives,

$$c^{-1} y_t = c^{-1} m + c^{-1} A y_{t-1} + c^{-1} \in_t$$

$$\Rightarrow Z_t = m^* + \wedge Z_{t-1} + \eta_t \qquad \ldots (3.4)$$

Where $c^{-1} y_t = Z_t$; $c^{-1} m = m^*$ and $c^{-1} \in_t = \eta_t$ which is a white noise vector. Thus.

$$Z_{1t} = m_1^* + \lambda_1 Z_{1,t-1} + \eta_{1t}$$

$$Z_{2t} = m_2^* + \lambda_2 Z_{2,t-1} + \eta_{2t}$$

Each Z variable follows a separate AR (1) process and is stationary.
I (0), if the eigen values has modulus < 1 is a random walk with drift
I (1), if the eigen value is 1 and is explosive if the eigen value exceeds 1 in numerical value.

3.2 A Three variable Vector Auto Regression (VAR)

By expanding the system of a first order VAR to three variables. Suppose the eigen values of the A matrix are $\lambda_1 = 1$, $|\lambda_2| < 1$ and $|\lambda_3| < 1$. Thus there exists a (3x3) Nonsingular matrix C of eigen vectors A. By defining a three element Z vector as in equation (3.3) follows that Z_{1t} is I (1); Z_{2t} and Z_{3t} are each I (0). If all 'Y' variables are I (1) then y vector may be expressed as

$$y_t = \begin{bmatrix} \vdots \\ c_1 \\ \vdots \end{bmatrix} Z_{1t} + \begin{bmatrix} \vdots \\ c_2 \\ \vdots \end{bmatrix} Z_{2t} + \begin{bmatrix} \vdots \\ c_3 \\ \vdots \end{bmatrix} Z_{3t}$$

Consider a linear combination of the y variables i.e. I (0), we need to eliminate Z_{1t} element. Let $c^{(2)}$ and $c^{(3)}$ denote the second and third rows of c^{-1}, two co-integrating relations are available in

$Z_{2t} = c^{(2)} y_t$ And $Z_{3t} = c^{(3)} y_t$ … (3.5)

A linear combination of I (0) variables is itself I (0) Thus, any linear combination of the variables in equation (3.5) is also a co-integrating relation with an co-integrating vector. When two or more cointegrating vectors are found there is an infinity of cointegrating vectors.

We consider π matrix then the eigen values are

$$\pi = C(1-\wedge)C^{-1}$$

$$= \begin{bmatrix} \vdots & \vdots & \vdots \\ c_1 & c_2 & c_3 \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix} \begin{bmatrix} \cdots & c^{(1)} & \cdots \\ \cdots & c^{(2)} & \cdots \\ \cdots & c^{(3)} & \cdots \end{bmatrix} \quad \ldots (3.6)$$

$$= \begin{bmatrix} \vdots & \vdots \\ \mu_2 c_2 & \mu_3 c_3 \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} \cdots & c^{(2)} & \cdots \\ \cdots & c^{(3)} & \cdots \end{bmatrix}$$

Thus π splits into the product of a (3x2) matrix of rank two. The matrix contains the two cointegrating vectors with which both cointegrating vectors enter into the Error Correction formulation for each Δy_i the

$$\Delta y_{it} = m_1 - (\mu_2 c_{12}) Z_{2,t-1} - (\mu_3 c_{13}) Z_{3,t-1} + \in_{1t}$$

$$\Delta y_{2t} = m_2 - (\mu_2 c_{22}) Z_{2,t-1} - (\mu_3 c_{23}) Z_{3,t-1} + \in_{2t}$$

$$\Delta y_{3t} = m_3 - (\mu_3 c_{32}) Z_{2,t-1} - (\mu_3 c_{33}) Z_{3,t-1} + \in_{3t}$$

The factorization of π is written

$$\pi = \alpha \beta^1$$

Where α and β are (3x2) matrices of rank two i.e., the rank of π is two and there are two cointegrating vectors. By substitution $\alpha \beta^1$ in.

$$\Delta y_t = m - \pi y_{t-1} + \in_t$$

i.e. $\Delta y_t = m - \alpha \beta^1 y_{t-1} + \in_t$

$$= \Delta y_t = m - \alpha Z_{t-1} + \in_t \quad \ldots (3.7)$$

Where $Z_{t-1} = \beta^1 y_{t-1}$ contains two cointegrating variables. Suppose that the eigen values are $\lambda_1 = \lambda_2 = 1$ and (λ_3)<1. Then it is possible to find a non-singular matrix p such that $P^{-1}AP = j$ where j is a Jordan matrix

$$J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

By considering a three element vector $Z_t = P^{-1} y_t$ it follows that Z_1 is I(2), Z_2 is I(1) and Z_3 is I (0) generally all three variables are I(2), then

$$y_t = \begin{bmatrix} \vdots \\ p_1 \\ \vdots \end{bmatrix} Z_{1t} + \begin{bmatrix} \vdots \\ p_2 \\ \vdots \end{bmatrix} Z_{2t} + \begin{bmatrix} \vdots \\ p_3 \\ \vdots \end{bmatrix} Z_{3t}$$

Premultiplying by the second row of p^{-1} namely $P^{(2)}$ gives $P^{(2)} y_t = Z_{2t}$

Similarly $P^{(3)}$ gives $P^{(3)} y_t = Z_{3t}$. Which is I (0)

Thus there are two cointegrating vectors, but only one produces a stationary linear combination of y's. The eigen values are $\lambda_1 = 1,\ \lambda_2 = 1$ and $\lambda_3 = a$. Where the last eigen value is assumed to have modulus less than one. The first two Y variables are random walks with drift, and thus I (1) and the last equation in the VAR connects all three variables. So, that $y^{(3)}$ is I (1).

Then π matrix is

$$\pi = I - A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 1-a \end{bmatrix}$$

The rank of π is one, and it may be factorized as

$$\pi = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & a-1 \end{bmatrix}$$

When the row vector is the cointegrating vector. This result may be seen from

$$Z_t = y_{1t} + y_{2t} + (a-1) y_{3t}$$

$$= y_{1t} + y_{2t} + (a-1)\left(y_{1,t-1} + y_{2,t-1} + a y_{3,t-1} + m_3 + \in_{3t} \right)$$

$$= \Delta y_{1t} + \Delta y_{2t} + a Z_{t-1} + (a-1) m_3 + (a-1) \in_{3t}$$

= constant $+ a Z_{t-1} + v_t$

Where $v_t = \in_{1t} + \in_{2t} + (a-1) \in_{3t}$

Thus Z_t follows a stable AR (1) process and I (0).

3.3 VAR Higher Order Systems

Consider Second Order System

$$y_t = m + A_1 y_{t-1} + A_2 y_{t-2} + \in_t \qquad \dots (3.8)$$

By subtracting y_{t-1} from both sides, we get

$$\Delta y_t = m + \left(A_1 - I \right) y_{t-1} + A_2 y_{t-2} + \in_t$$

The process of adding or subtracting $(A_1 - I)\, y_{t-2}$ on the right side, we get

$$\Delta y_t = m + \left(A_1 - I \right) \Delta y_{t-1} - \pi y_{t-2} + \in_t \qquad \dots (3.9)$$

Where $\pi = I - A_1 - A_2$

Similarly an alternative change in parameter is

$$\Delta y_t = m - A_2 \Delta y_{t-1} - \pi y_{t-1} + \in_t \quad \ldots (3.10)$$

In second order system, there will be one lagged first difference term on the right hand side.
By continuing the procedure upto Var (p) system defined in equation (3.5) may be change in parameters. i.e,

$$\Delta y_t = m + B_1 \Delta y_{t-1} + B_{p-1} \Delta y_{t-p+1} - \pi y_{t-1} + \in_t \quad \ldots (3.11)$$

Where the B's are the functions of the A's and $\pi = 1 - A_1 - \ldots\ldots\ldots - A_p$. The behaviour of the Y vector depends on the values of λ that solve

$$\left| \lambda^p I - \lambda^{p-1} A_1 - \ldots\ldots - \lambda A_{p-1} - A_p \right| = 0$$

For explosive roots, consider three possibilities

(i) Rank (π) = k If each root has modulus less than one, π will have full rank and be non-singular. All the 'y' variables in equation (3.5) will be I (0) and unrestricted OLS estimates of equation (3.5) and equation (3.11) will give same inferences about the parameters.

(ii) Rank (π) = r < k This situation will occur if there is a unit root with multiplicity (k-r) and the remaining 'r' roots are numerically less than one.

(iii) Rank (π) =0. This is a special case. It will only occur if $A_1 + \ldots\ldots + A_p = I$, in this case π =0 and equation (3.11) shows the VAR should be specified in terms of first differences of the variables.

4. Vector Autoregressive Model for Time Series

The two important time series analysis techniques namely, Intervention Analysis and Transfer function Analysis generalize the univariate techniques by allowing the time path of the study variable or dependent variable to be influenced by the time path of an exogenous or explanatory variable. When there is no feedback, these two analysis techniques can be considered as very effective techniques for forecasting and testing of hypotheses.

The main limitation of Intervention and Transfer Function models is that many economic systems do exhibit feedback. Generally, it is not always known if the time path of a series designated to be the Independent variable has been unaffected by the time path of the study variable. The basic form of a Vector Autoregression (VAR) treats all variables symmetrically without making reference to the issue of dependence verses independence.

The VAR analysis tools such as Granger Causality tests, Impulse Response Analysis and Variance Decomposition models can be used in establishing relationships among economic variables and in the formulation of a more structured economic model.

Economic theories contain behavioral, structural, reduced form relationships that can be incorporated into a VAR analysis. Generally in the structural VAR, the restrictions of a particular economic variable are imposed on the contemporaneous relationships among the variables. The dynamic effect of each variable to various economic shocks can be obtained and the restrictions of the model be tested.

The research work on vector Autoregression VAR models was started around 1980 after Sims (1980) article. The VAR model is very useful in the analysis of the interrelationships between the different time series. When one considers several time series, one may need to take into account the interdependence between them. One method of doing this is to estimate a dynamic simultaneous equations model, which is a simultaneous emotions model with lags in all this variables.

The specification of the dynamic simultaneous equations model involves two steps: First one has to classify the variables into two categories, Endogenous Exogenous variables and second one has to impose some constraints on the parameters to achieve identification. Sims (1980) argues that both these steps involves many arbitrary decisions and suggest as an alternative, the VAR approach. It is a multiple time series generalization of the Autoregressive model. The multiple time series analog of the ARMA model is the VARMA model. The VARMA model was initiated by Tsiao and Box (1981).

5. Specification of Two Equations Var Model

Consider two economic time series $\{Y_{1t}\}$and $\{Y_{2t}\}$and let the time path of $\{Y_{1t}\}$ be affected by current and past realizations of $\{Y_{2t}\}$ sequence and let the time path of $\{Y_{2t}\}$ be affected by current and past realizations of the $\{Y_{1t}\}$ sequence. A simple bivariate system of simultaneous equations model can be specified as

$$Y_{1t} = \beta_{10} - \beta_{12} Y_{2t} + \gamma_{11} y_{1t-1} + \gamma_{12} y_{2t-1} + u_{1t} \quad \ldots. (4.1)$$

$$Y_{2t} = \beta_{20} - \beta_{21} Y_{1t} + \gamma_{21} y_{1t-1} + \gamma_{22} y_{2t-1} + u_{2t} \quad \ldots (4.2)$$

It is assumed that,

(i) Both y_{1t} and y_{2t} are stationary;

(ii) u_{1t} and u_{2t} are white – noise disturbances with standard deviations of σ_1 and σ_2 respectively;
(iii) $\{u_{1t}\}$ and $\{u_{2t}\}$ are uncorrelated white noise disturbances.

The equations (4.1) and (4.2) constitute the "First order vector Auto regression (VAR) model, because the longest lag length is unity. Since, y_{1t} and y_{2t} are allowed to affect each other, $-\beta_{12}$ gives the contemporaneous effect of a unit change of y_{2t} on y_{1t} and γ_{12} is the effect of a unit change in y_{2t-1} on y_{1t}

The disturbances u_{1t} and u_{2t} are pure innovations (or shocks) in y_{1t} and y_{2t} respectively.
If $\beta_{21} \neq 0$ then u_{1t} has an indirect contemporaneous effect on y_{2t} and
If $\beta_{12} \neq 0$ then u_{2t} has an indirect contemporaneous effect on y_{1t}.
Using matrices equations (4.1) and 4.2) can be written in the compact form:

$$\begin{bmatrix} 1 & \beta_{12} \\ \beta_{21} & 1 \end{bmatrix} \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} \beta_{10} \\ \beta_{20} \end{bmatrix} + \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix} \begin{bmatrix} y_{1t-1} \\ y_{2t-1} \end{bmatrix} + \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix} \quad \text{…. (4.3)}$$

$$\Rightarrow \beta y_t = \delta_0 + \Gamma y_{t-1} + U_t \quad \text{…. (4.4)}$$

Where, $\beta = \begin{bmatrix} 1 & \beta_{12} \\ \beta_{21} & 1 \end{bmatrix}$, $y_t = \begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix}$, $\delta_0 = \begin{bmatrix} \beta_{10} \\ \beta_{20} \end{bmatrix}$

$$\Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}, y_{t-1} = \begin{bmatrix} y_{1t-1} \\ y_{2t-1} \end{bmatrix}, u_t = \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix}$$

$$\Rightarrow y_t = \beta^{-1}\delta_0 + \beta^{-1}\Gamma y_{t-1} + \beta^{-1}U_t \quad \text{… (4.5)}$$

(Or) $y_t = A_0 + A_1 y_{t-1} + \epsilon_t$ … (4.6)

Where $A_0 = \beta^{-1}\delta_0$, $A_1 = \beta^{-1}\Gamma$, $E_t = \beta^{-1}u_t$

The model (4.6) gives VAR model in standard form.
In general, two equations VAR model can be written as

$$y_{1t} = a_{10} + a_{11}y_{1t-1} + a_{12}y_{2t-1} + \in_{1t} \quad \text{… (4.7)}$$

$$y_{2t} = a_{20} + a_{21}y_{1t-1} + a_{22}y_{2t-1} + \in_{2t} \quad \text{… (4.8)}$$

The system containing equations (4.1) and (4.2) is known as 'Structural VAR model' or the 'Primitive System'. The equivalent system containing equations (4.7) and (4.8) is known as 'Standard VAR model'.
Since, $\in_t = \beta^{-1}u_t$, one may obtain,

$$\in_{1t} = (u_{1t} - \beta_{12}u_{2t}) / (1 - \beta_{12}\beta_{21}) \quad \text{… (4.9)}$$

$$\in_{2t} = (u_{2t} - \beta_{21}u_{1t}) / (1 - \beta_{12}\beta_{21}) \quad \text{… (4.10)}$$

$$\Rightarrow E[\in_{1t}] = E[(u_{1t} - \beta_{12}u_{2t}) / (1 - \beta_{12}\beta_{21})] = 0 \quad [\because E(u_{1t}) = 0 E(u_{2t})]$$

and $\text{var}[\in_{1t}] = E[\in^2_{1t}] = E[(u_{1t} - \beta_{12}u_{2t}) / (1 - \beta_{12}\beta_{21})]^2$

$$= [\sigma^2_{y_1} + \beta^2_{12}\sigma^2_{y_2}] / (1 - \beta_{12}\beta_{21})^2 \quad \text{… (4.11)}$$

[$\because u_{1t}$ And u_{2t} are white-noise processes with zero means and constant variances and are uncorrelated]
Thus, the Var (ϵ_{1t}) is time-independent,
The Auto Correlations of ϵ_{1t} and ϵ_{1t-i}

$$E[\epsilon_t \epsilon_{1t-i}] = \frac{E[u_{1t} - \beta_{12} u_{2t}][u_{1t-i} - \beta_{12} u_{2t-i}]}{(1-\beta_{12}\beta_{21})^2} = 0, \forall i \neq 0 \quad \text{… (4.12)}$$

Similarly, it can be easily shown that, E [ϵ_{2t}] = 0,

$$E[\epsilon_{2t}] = [\sigma^2_{y_2} + \beta^2{}_{21} \sigma^2_{y_1}] / (1-\beta_{12}\beta_{21})^2, \quad \text{... (4.13)}$$

And $E[\epsilon_{2t} \; \epsilon_{2t-i}] = 0, \forall i \neq 0$

Thus, ϵ_{1t} and ϵ_{2t} are the stationary processes with zero means, constant variances and all auto covariances equal to zero.
Consider, the Covariance of ϵ_{1t} and ϵ_{2t} as

$$E[\epsilon_{1t} \; \epsilon_{2t}] = \frac{E[(u_{1t} - \beta_{12} \; u_{2t})(u_{2t} - \beta_{21} u_{1t})]}{(1-\beta_{12} \; \beta_{21})^2}$$

$$= -[\beta_{21} \sigma^2_{y_1} + \beta_{12} \; \sigma^2_{y_2}] / (1-\beta_{12} \; \beta_{21})^2 \quad \text{… (4.14)}$$

Generally (4.14) will not be zero. i.e., ϵ_{1t} and ϵ_{2t} are correlated.
If $\beta_{12} = \beta_{21} = 0$ then ϵ_{1t} and ϵ_{2t} will be uncorrelated
The variable – covariance matrix of $\epsilon_{1t}$and $\epsilon_{2t}$is given by

$$\sum = \begin{bmatrix} var(\epsilon_{1t}) & cov(\epsilon_{1t}, \epsilon_{2t}) \\ cov(\epsilon_{1t}, \epsilon_{2t}) & Var(\epsilon_{2t}) \end{bmatrix} \quad \text{… (4.15)}$$

Since all the elements of $\sum$ are time independent, one may obtain $\sum$ as

$$\sum = \begin{bmatrix} \sigma^2_{y_1} & \sigma_{y_1 y_2} \\ \sigma_{y_2 y_1} & \sigma^2_{y_2} \end{bmatrix}$$

Where Var (ϵ_{it}) = $\sigma^2_{y_i}$ and Cov ($\epsilon_{1t}, \epsilon_{2t}$) = $\sigma_{y_1 y_2}$ = $\sigma_{y_2 y_1}$
Simply one may use the more compact form as,

$$\sum = \begin{bmatrix} \sigma^2_1 & \sigma_{12} \\ \sigma_{21} & \sigma^2_2 \end{bmatrix} \quad \text{… (4.16)}$$

Where Var (ϵ_{it}) = σ^2_i and cov ($\epsilon_{1t}, \epsilon_{2t}$) = σ_{12} = σ_{21}

6. Stability of Two Equations Var Model
Consider the first order standard VAR model in matrix notation as

$$Y_t = A_0 + A_1 Y_{t-1} + \epsilon_t \quad (5.1)$$

Where Y_t, Y_{t-1}, A_0, A_1 and ϵ_t are defined in the equations (4.4) and (4.5)
Here, $a_0 = ((a_{io}))_{nx1}$; $A_1 = ((a_{ij}))_{nxn}$; $\epsilon_t = ((\epsilon_{it}))_{nx1}$ and $Y_t = ((Y_{it}))_{nx1}$ matrices.
By iterating (5.1) backwards, one may obtain,
$Y_t = A_0 + A_1 [A_0 + A_1 Y_{t-2} + \epsilon_{t-1}] + \epsilon_t$
(Or)

$Y_t = [I + A_1] A_0 + A_1^2 Y_{t-2} + A_1 \epsilon_{t-1} + \epsilon_t$ ..... (5.2)

Consequently, for n iterations, one may obtain,

$$Y_t = [I + A_1 + A_1^2 + \dots A_1^n] A_0 + \sum_{i=0}^{n} A_1^i \in_{t-i} + A_1^{n+1} Y_{t-n-1} \quad \dots (5.3)$$

In practice, the stability condition for the first order autoregressive model

$x_t = a_0 + a_1 x_{t-1} + u_t$ Is that $|a_1| < 1$.

For the stability of the first order VAR model, one may examine the homogeneous equation.

$$Y_t = A_1 Y_{t-1} \quad \dots (5.4)$$

One can use the method of undetermined coefficients and for each Y_{it}, posit a solution of the form

$$Y_{it} = C_i \lambda^t \quad \dots (5.5)$$

Where C_i is an arbitrary constant.
By substituting $Y_{it} = C_i\lambda^t$ and $Y_{it-1} = C_i\lambda^{t-1}$ in the equation (5.4), one may obtain

$$C_1 \lambda^t = a_{11} c_1 \lambda^{t-1} + a_{12} c_2 \lambda^{t-1} + \dots + a_{1n} c_n \lambda^{t-1}$$

$$C_2 \lambda^t = a_{21} c_1 \lambda^{t-1} + a_{22} c_2 \lambda^{t-1} + \dots + a_{2n} c_n \lambda^{t-1}$$

$$C_n \lambda^t = a_{n1} c_1 \lambda^{t-1} + a_{n2} c_2 \lambda^{t-1} + \dots + a_{nn} c_n \lambda^{t-1}$$

Dividing each equation by λ^{t-1} gives

$$c_1 (a_{11} - \lambda) + c_2 a_{12} + c_3 a_{13} + \dots + c_n a_{1n} = 0$$

$$c_1 \; a_{21} + c_2 (a_{22} - \lambda) + c_3 a_{23} + \dots + c_n a_m = 0$$

. . . .
. . . .
. . . .

$$c_1 a_{n1} + c_2 a_{n2} + c_3 a_{23} + \dots + c_n (a_{nn} - \lambda) = 0$$

$$\Rightarrow \begin{bmatrix} (a_{11} - \lambda) a_{12}\; a_{13} \dots a_{1n} \\ a_{21}\; (a_{22} - \lambda) a_{23} \dots a_{23} \\ . \quad . \quad . \quad . \\ . \quad . \quad . \quad . \\ a_{n1}\; a_{n2}\; a_{n3} \dots (a_{nn} - \lambda) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix} \quad \dots (5.6)$$

For a nontrivial solution to this system of equations, one must have,

$$\begin{vmatrix} (a_{11} - \lambda)\; a_{12}\; a_{13} \dots a_{1n} \\ a_{21}\; (a_{22} - \lambda)\; a_{23} \dots a_{2n} \\ . \; . \; . \; . \; . \\ a_{n1}\; a_{n2}\; a_{n3} \dots (a_{nn} - \lambda) \end{vmatrix} = 0 \quad \dots (5.7)$$

This implies that the determinant gives an n^{th} order polynomial which is satisfied by n values of λ. Thus, one may obtain n values of λ as n characteristic roots $\lambda_1, \lambda_2 \ldots \lambda_n$.

Since each λ_i, i = 1, 2 …n is a solution to the homogenous equation, the linear combination of the homogeneous solutions

$Y_{it} = d_1 \lambda_1^t + d_2 \lambda_2^t + \ldots + d_n \lambda_n^t$ Is also a homogeneous solution.

It should be noted that each sequence $\{y_{it}\}$ will have the same roots.

Hence, the necessary and sufficient condition for the stability of a first order VAR is that all characteristics roots lie within the unit circle.

If one iterates backward continuously, then convergence requires that the expression A_1^n, approaches zero as n approaches infinity.

If the stability condition holds, then the sequences $\{y_{1t}\}$ and $\{y_{2t}\}$ will be jointly covariance stationary. Each sequence has a finite and time-invariant mean and a finite and time-invariant variance.

Remarks: LAG OPERATORS: The lag operator L is defined to be a linear operator such that for any value y_t, $L^i y_t \equiv y_{t-i}$

Properties of LAG Operators

1. LC = C, where c is a constant.
2. $\left[L^i + L^j\right] y_t = L^i y_t + L^j y_t = y_{t-i} + y_{t-j}$
3. $\left[L^i L^j\right] y_t = L^i \left[L^j y_t\right] = L^i y_{t-j} = y_{t-i-j}$ and $L^0 y_t = y_t$.
4. L raised to a negative power is actually a Lead operator. i.e. $L^{-i} y_t = y_{t+i}$
5. For $|a| < 1$, the infinite sum $(1+aL+a^2L^2+a^3L^3+\ldots\ldots)\, y_t = \dfrac{y_t}{1-aL}$

6. **Lag Operators in Higher Order Systems**: One can use lag operators to transform the n^{th} order equation

$y_t = a_o + a_1 y_{t-1} + a_2 y_{t-2} + \ldots.. + a_n y_{t-n} + \in_t$ Into $\left(1 - a_1 L - a_2 L^2 - \ldots\ldots. - a_n L^n\right) y_t = a_o + \in_t$

Or $y_t = \dfrac{[a_0 + \in_t]}{[1 - a_1 L - a_2 L^2 - \ldots. a_n L^n]}$

7. The stability condition is that for the characteristic roots of $[1-a_1L-\ldots\ldots a_nL^n]$ to lie outside of the unit circle.

7. Two Equations VAR model in terms of LAG Operators.

Consider the two equations VAR model.

$$y_{1t} = a_{10} + a_{11} y_{1t-1} + a_{12} y_{2t-1} + \in_{1t} \quad \ldots (6.1)$$

$$y_{2t} = a_{20} + a_{21} y_{1t-1} + a_{22} y_{2t-1} + \in_{2t} \quad \ldots (6.2)$$

Using Lag operators, the VAR model can be rewritten as

$$y_{1t} = a_{10} + a_{11} L y_{1t} + a_{12} L y_{2t} + \in_{1t} \quad \ldots (6.3)$$

$$y_{2t} = a_{20} + a_{21} L y_{1t} + a_{22} L y_{2t} + \in_{2t} \quad \ldots (6.4)$$

Or

$$(1 - a_{11} L) y_{1t} = a_{10} + a_{12} L\, y_{2t} + \in_{1t} \quad \ldots (6.5)$$

$$(1 - a_{22} L) y_{2t} = a_{20} + a_{21} L\, y_{1t} + \in_{2t} \quad \ldots (6.6)$$

If one uses equation (6.6) to solve for y_{2t}, it follows that Ly_{2t} is given by

$$Ly_{2t} = L\left[a_{20} + a_{21}Ly_{1t} + \in_{2t}\right]/\left(1 - a_{22}L\right) \quad \ldots (6.7)$$

From (6.5) and (6.7), one may obtain

$$(1-a_{11} L)\, y_{1t} = a_{10} + a_{12} \left\{L\left(a_{20} + a_{21}Ly_{1t} + \in_{2t}\right)/\left(1 - a_{22}L\right)\right\} + \in_{1t} \quad \ldots (6.8)$$

It should be noted that the first-order VAR model in the $\{y_{1t}\}$ and $\{y_{2t}\}$ sequences is transformed into second-order stochastic difference equation in the $\{y_{1t}\}$ sequence.
Solving for y_{1t} explicitly, one may obtain

$$y_{1t} = \frac{a_{10}\left(1 - a_{22}\right) + a_{12}a_{20} + \left(1 - a_{22}L\right)\in_{1t} + a_{12} \in_{2t-1}}{\left(1 - a_{11}L\right)\left(1 - a_{22}L\right) - a_{12}a_{21}L^2} \quad \ldots (6.9)$$

Similarly, one may obtain

$$y_{2t} = \frac{a_{20}\left(1 - a_{11}\right) + a_{21}a_{10} + \left(1 - a_{11}L\right)\in_{2t} + a_{21} \in_{1t} - 1}{\left(1 - a_{11}L\right)\left(1 - a_{22}L\right) - a_{12}a_{21}L^2} \quad \ldots (6.10)$$

Equations (6.9) and (6.10) both have the same characteristic equation. Convergence or stability of the VAR requires that the roots of the polynomial $\left[\left(1 - a_{11}^2\right)\left(1 - a_{22}^2\right) - a_{12}a_{21}L^2\right]$ must lie outside the unit circle.
As long as a_{12} and a_{21} do not both equal zero, the solutions for the two sequences have the same characteristic roots. Thus, both y_{1t} and y_{2t} will exhibit similar time paths.

8. Conclusions
The two important time series analysis techniques namely, Intervention Analysis and Transfer function Analysis generalize the univariate techniques by allowing the time path of the study variable or dependent variable to be influenced by the time path of an exogenous or explanatory variable. When there is no feedback, these two analysis techniques can be considered as very effective techniques for forecasting and testing of hypotheses.
The main limitation of Intervention and Transfer Function models is that many economic systems do exhibit feedback. Generally, it is not always known if the time path of a series designated to be the Independent variable has been unaffected by the time path of the study variable. The basic form of a Vector Autoregression (VAR) treats all variables symmetrically without making reference to the issue of dependence verses independence.
In the present study, a Two – Equations Vector Autoregressive (VAR) model has been specified and stability conditions have been developed. Using Lag operators, the two – equations VAR model has been expressed in terms of these operators.

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