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Modeling and analysis of a resource population due to increase in industrialization

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Abstract
In this paper, a non-linear mathematical model is proposed and analyzed to study the effect of rapid increase in industrialization produced by the human species affecting its own living resources and depleting the environment. It is shown that if the rate of industrialization increases, the density of the biological species settles down to a lower equilibrium than its original carrying capacity and its magnitude decreases as the equilibrium level of concentration of it increases. It may be noted that for very large increase in this field of development, the survival of the biological species is threatened.

Keywords: Modeling, and analysis, population due, industrialization

Introduction
There exists a situation where resource dependent population is affected when its resource is depleted by industrialization. This type of industrialization may be augmented by human actions (precursors). The effects of human activities and patterns of resource use on the structure and composition of resource have been studiedby many investigators, Bormann and Likens (1979), Padoch and Vayda (1983) [11], Hammond (1990), Brown (1992), Garcia – Montiel and Scatena (1994) [4], Woodwell (1970) [18], McLaughlin (1985) [10], Hari *et al.* (1986), Woodman and Cowling (1987) [19], Schulze (1989) [13]. Shukla *et al.* proposed and analyzed a mathematical model to study the effect of industrialization / population on resource depletion showing that the resource may be doomed to extinction if the rate of industrialization / population increase without control. The effect of changing habitats on survival of species has also been investigated, Shukla *et al.* (1996) [14]. Shukla and Dubey (1996) [14] have also studied the detrimental effects of industrialization and population on forestry resources. two toxicants on a biological species by using the same assumptions (see also Shukla and Dubey (1997)) [15]. Shukla *et al.* (2001) have further studied effect of a toxicant on the existence and survival of two competing species in a polluted environment.
In this paper a model has been proposed and analyzed for the survival of a resource dependent population (such as human beings) whose resource biomass is depleted by industrialization, the density of which is augmented by a precursor of the population.
Shukla *et al.* (2004) have also the model on survival of resource dependent species when resource is depleted by precursor augmented industrialization.
Generally the biological species is affected by toxicants through different pathways including uptake of toxicants from the environment or through food chains. In the case of less toxic substances, when the biological species is exposed to them for longer durations, these are uptaken by the species, the toxicant in the uptake phase interacts with tissues of body organ through various physiological processes, which in turn makes the species diseased. Such situations arise when the dusts of various substances enter into the human lung, Holma (1985), Folinsbee (1989).
In the modeling process, the following assumptions are made:
(1) The density of biological species and resource biomass is assumed to be governed by a generalized logistic equation with variable intrinsic growth rate and carrying capacity.

(2) The industrialization decreases the carrying capacity of the resource density in the habitat.

2. Mathematical Model

We consider a biological population growing logistically in its habitat, which is affected by the growth of industrialization, by the human population. It is assumed that the rate of increase of the industrial factor is proportional to the density of the population producing it.

Keeping in view the considerations and assumptions as mentioned above the problem is assumed to be governed by the following differential equations.

$$\frac{dN}{dt} = rN - \frac{r_0 N^2}{K} + \beta_1 NB$$

$$\frac{dB}{dt} = sB - \frac{s_{10} B^2}{L} - \beta_2 NB - s_1 IB - s_2 IB^2 \tag{1}$$

$$\frac{dI}{dt} = \lambda N + \mu IB - \theta I$$

$$N(0) \geq 0, \quad B(0) \geq 0, \quad I(0) \geq 0,$$

Here, $N(t)$ is the density of biological species with growth rate r and carrying capacity K, $B(t)$ is the density of resource biomass density with growth rate s and carrying capacity L, $I(t)$ is the density of industrialization produced by the population. β_1 is the growth rate of N and β_2 is the depletion rate of B, the constant λ is the growth rate coefficient of industrialization formed by population and θ is the natural depletion rate coefficient of industrialization. The coefficient $\mu \geq 0$ is the growth rate of I, s_1 is the depletion rate coefficient of the biomass by industrialization and s_2 is the coefficient causing biomass depletion due to crowding of industrial activity.

In our model (1) the coefficient r and s represent the intrinsic growth rate constants of the consumer species and the resource species respectively. The constants K and L denote the carrying capacities of population and the resource respectively. Also, it can be seen that if industrialization is caused only by the population species then also the model is meaningful.

2. Equilibrium analysis

The given model (1) has two non-negative equilibria in $N - B - I$ space, namely $E_0 = (0, B, I)$, $E_1 = (N, 0, I)$ and $E^*(N^*, B^*, I^*)$. We show the existence of E^* as follows. Here N^*, B^* and I^* are the positive solutions of the algebraic equations.

$$N = K\left[\frac{r + \beta_1 B}{r_0}\right]$$

$$B = \frac{(s - \beta_2 N - s_1 I)}{\left(\frac{s_{10}}{L} + s_2 I\right)}$$

$$I = \frac{\lambda N}{(\theta - \mu B)}$$

where, $\theta > \mu B^*$

Rewriting 2nd equation of the model using above notations we have

$$\frac{dB}{dt} = (s - \beta_2 N - s_1 I)B - \left(\frac{s_{10}}{L} + s_2 I\right)B^2$$

In this equation $(s - \beta_2 N - s_1 I)$ denotes the intrinsic growth rate of $\frac{dB}{dt}$, which would increase only if $(s - \beta_2 N - s_1 I) > 0$ for $B \geq 0$ and $I \geq 0$.

For $B = 0$ we then have

$$s - \frac{\beta_2 rK}{r_0} - s_1 I(B = 0) > 0$$

To prove the existence of E^*, let us take

$$F(B) = s_{10} B + s_2 ILB - sL + \beta_2 NL + s_1 IL$$

Then $F(0) = -L\left[s - \frac{\beta_2 rK}{r_0} - s_1 I(B = 0)\right] < 0$,

Also,

$$F\left(\frac{sL}{s_{10}}\right) = L\left[\frac{ss_2}{s_{10}} I\left(\frac{sL}{s_{10}}\right) + \beta_2 N\left(\frac{sL}{s_{10}}\right) + s_1 I\left(\frac{sL}{s_{10}}\right)\right] > 0$$

Therefore, their exists a root B^* in the interval $0 < B^* < \left(\frac{sL}{s_{10}}\right)$ which is obtained by solving, $F(B^*) = 0$.

For B^* to be unique, we must have $F'(B) > 0$, where $0 < B^* < \left(\frac{sL}{s_{10}}\right)$

$$F'(B) = s_{10} + s_2 IL + s_2 BL \frac{dI}{dB} + \beta_2 L \frac{dN}{dB} + s_1 L \frac{dI}{dB}$$

Or

$$F'(B) = s_{10} + L\left[s_2 I + s_2 B \frac{dI}{dB} + \beta_2 \frac{dN}{dB} + s_1 \frac{dI}{dB}\right]$$

It is noted that at $B = B^*$, $F'(B^*) > 0$.

Thus, the condition for unique and positive B^* is $F'(B) > 0$. Once B^* is determined N^* and I^* can be found from equations.

3. Stability analysis

(i) Local stability

By calculating Jacobian matrix it can be checked that E_0 is a saddle point with unstable manifold locally in N direction and stable manifold locally in $N - B - I$ space, E_1 is a saddle point with unstable manifold locally in B direction and stable manifold locally in $N - B - I$ space.

The stability behavior of E^* is not obvious from the corresponding Jacobian matrix, therefore, by using Liapunov's method, in the following theorem we have found sufficient conditions for E^* to be locally asymptotically stable.

Theorem: - Let the following inequalities hold-

$$\frac{\beta_1\left(s_1 + s_2 B^*\right)}{\beta_2 \mu I^*} < 2\frac{r_0\left(\theta - \mu B^*\right)}{K\lambda^2}$$

Where, $\theta > \mu B^*$

Then E^* is locally asymptotically stable.

Proof: - Using the following Liapunov's function for the linearlized system (1).

$N = N^* + n$, $B = B^* + b$ and $I = I^* + i$

Where, n , b and i are small perturbations around E^* ,
We get

$$\dot{n} = \frac{-r_0 N^*}{K} n + \beta_1 N^* b$$

$$\dot{b} = -\beta_2 B^* n - \left(\frac{s_{10} B^*}{L} + s_2 I^* B^*\right) b - \left(s_1 B^* + s_2 B^{*2}\right) i$$

$$\dot{i} = \lambda n + \mu I^* b + -\left(\theta - \mu B^*\right) i \quad \text{where, } \theta > \mu B^*$$

To study the local stability of E^*, we consider the following positive definite function,

$$V = \frac{1}{2}\frac{C_1 n^2}{N^*} + \frac{1}{2}\frac{C_2 b^2}{B^*} + \frac{1}{2} C_3 i^2$$

Which on differentiation gives

$$\dot{V} = \frac{C_1 n \dot{n}}{N^*} + \frac{C_2 b \dot{b}}{B^*} + C_3 i \dot{i}$$

On substituting the values of $\dot{n}$, $\dot{b}$ and $\dot{i}$ and we get.

$$\dot{V} = \frac{-C_1 rn^2}{K} - C_2\left(\frac{s_{10}}{L} + s_2 I^*\right) b^2 - C_3\left(\theta - \mu B^*\right) i^2 + \left(C_1\beta_1 - C_2\beta_2\right) nb + \left(C_3 \mu I^* - C_2\left(s_1 + s_2 B^*\right)\right) bi + C_3 \lambda ni$$

As $\theta > \mu B^*$

Choosing the values of the as

$C_1 = 1$, $C_2 = \dfrac{\beta_1}{\beta_2}$, $C_3 = \dfrac{\beta_1\left(s_1 + s_2 B^*\right)}{\beta_2 \mu I^*}$

The equation reduces to,

$$\dot{V} = \frac{-rn^2}{K} - \frac{\beta_1}{\beta_2}\left(\frac{s_{10}}{L} + s_2 I^*\right) b^2 - C_3\left(\theta - \mu B^*\right) i^2 + C_3 \lambda ni$$

For to be $\dot{V}$ negative definite the following conditions must be satisfied,

$$C_3\lambda^2 < 2\,\frac{r_0\left(\theta - \mu B^*\right)}{K}$$

$$C_3 < 2\,\frac{r_0\left(\theta - \mu B^*\right)}{K\lambda^2}$$

$$\Rightarrow \quad \frac{\beta_1\left(s_1 + s_2 B^*\right)}{\beta_2 \mu I^*} < 2\,\frac{r_0\left(\theta - \mu B^*\right)}{K\lambda^2}$$

Non – Linear Stability

Lemma: The set

$$A = \left\{ (N, B, I) : 0 \le N \le N_m, 0 \le B \le B_m, 0 \le I \le \frac{\lambda N_m}{\theta_1} \right\}$$

Where,
Proof: From the model we have,

$$\frac{dB}{dt} = sB - \frac{s_{10}B^2}{L} - \beta_2 NB - s_1 IB - s_2 IB^2$$

$$\le sB - \frac{s_{10}B^2}{L}$$

$$0 \le B \le \frac{sL}{s_{10}} = B_m$$

Also,

$$\frac{dN}{dt} \le rN - \frac{r_0 N^2}{K} + \beta_1 N\,\frac{sL}{s_{10}}$$

$$\le \left(r + \beta_1 \frac{sL}{s_{10}} \right) N - \frac{r_0 N^2}{K}$$

$$0 \le N \le \frac{K}{r}\left(r + \beta_1 \frac{sL}{s_{10}} \right) = N_m$$

Further, $\frac{dI}{dt} \le \lambda N_m + \left(\mu B - \theta\right) I$

$$= \lambda N_m - \left(\theta - \mu B\right) I$$

$$\le \lambda N_m - \theta_1 I$$

$$0 \le I \le \frac{\lambda N_m}{\theta_1}$$

Theorem : In addition to the above assumptions that growth rate r and s and the carrying capacity are assumed to be constant. Let the functions s and L satisfy in A. Then if the following conditions hold

$$\frac{\beta_1(s_1 + s_2 B^*)}{\beta_2 \mu I^*} < 2\frac{r_0(\theta - \mu B^*)}{K\lambda^2}$$

Then E^* is non linearly asymptotically stable in A.

Proof: Consider the following positive definite function around E^*.

$$W(N,B,I) = C_1\left(N - N^* - N^* \ln\frac{N}{N^*}\right) + C_2\left(B - B^* - B^* \ln\frac{B}{B^*}\right) + \frac{1}{2}C_3(I - I^*)^2$$

Differentiating with respect to t

$$\dot{W}(N,B,I) = C_1\left(\frac{N - N^*}{N}\right)\frac{dN}{dt} + C_2\left(\frac{B - B^*}{B}\right)\frac{dB}{dt} + C_3(I - I^*)\frac{dI}{dt}$$

Substituting from the above equations and doing manipulations,

$$\dot{W} = \frac{-C_1 r}{K}(N - N^*)^2 - C_2\left(\frac{s_{10}}{L} + s_2 I\right)(B - B^*)^2 - C_3(\theta - \mu B)(I - I^*)^2 + (C_1\beta_1 - C_2\beta_2)(N - N^*)(B - B^*)$$
$$+ [C_3\mu I^* - C_2(s_1 + s_2 B^*)](B - B^*)(I - I^*) + C_3\lambda(N - N^*)(I - I^*)$$

Choosing the values of the as

$$C_1 = 1, \quad C_2 = \frac{\beta_1}{\beta_2}, \quad C_3 = \frac{\beta_1(s_1 + s_2 B^*)}{\beta_2 \mu I^*}$$

The equation reduces to

$$\dot{W} = \frac{-C_1 r}{K}(N - N^*)^2 - C_2\left(\frac{s_{10}}{L} + s_2 I\right)(B - B^*)^2 - C_3(\theta - \mu B)(I - I^*)^2 + C_3\lambda(N - N^*)(I - I^*)$$

For $\dot{W}$ to be negative definite the following conditions must be satisfied,

$$C_3\lambda^2 < 2\frac{r_0(\theta - \mu B^*)}{K}$$

$$C_3 < 2\frac{r_0(\theta - \mu B^*)}{K\lambda^2}$$

$$\Rightarrow \frac{\beta_1(s_1 + s_2 B^*)}{\beta_2 \mu I^*} < 2\frac{r_0(\theta - \mu B^*)}{K\lambda^2}$$

Which is same as done in above theorem.

Conclusion

In this paper, a non – linear mathematical model has been proposed and analyzed to study the effect of growing industrialization by the human population and its impact on resource species. Although it affects both the species directly and indirectly yet in this paper on effect on resource biomass is being considered. The existence of non – trivial equilibrium has been proved and its stability behavior has analyzed. It is noted that the population settles down to a equilibrium level, which is lower than its carrying capacity. Also the equilibrium level of resource biomass reduces as the industrialization increases.

The results discussed here are more general than Shukla *et al.* (2005). It is noted that for large increase in industrialization, the toxicant and other wastage are induced in the environment, the possibility of extinction of the species does exist in the long run.

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