International Journal of Statistics and Applied Mathematics 2018; 3(2): 652-663

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2018; 3(2): 652-663

www.mathsjournal.com
Received: 20-01-2018
Accepted: 21-02-2018

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The structure of the lattice of subgroups of the symmetric group S_5

A Vethamanickam and C Krishna Kumar

Abstract
In this paper we determine all the subgroups of the symmetric group S_5 explicitly by applying Lagrange's theorem, Sylow's theorem. We give the diagram of the lattice of subgroups of S_5. We also study some lattice theoretic properties of the lattice of subgroups of S_n, when $n \leq 5$.

Keywords: Symmetric group, Lagrange's theorem, Sylow's theorem, p-Sylow subgroup, lattice

1. Introduction
In mathematics, the notion of permutation relates to act of arranging all the members of a set into some sequence or order or the set is already ordered or rearranging its elements a process called permuting.

For any non-empty set S, define A(S) be the set of all bijections (permutations) on S. A(S) together with the binary operation of composition of functions is a group. If the set S contain n elements, then the set A(S) is denoted by S_n. The group S_n has n! Elements and is called the symmetric group of degree n. Lattice of subgroups of S_2, S_3 and S_4 have been studied already eg. Refer to [2], [3], [5] and [12]. In this paper we determine all the subgroups of S_5, and then draw the diagram of the lattice subgroups of S_5, also study some lattice theoretic properties. For the preliminaries and the basic definitions we refer to [2], [3], [4], [5] and [8].

2. Preliminary
We recall some definitions and results that will be used later.

Definition 2.1
The symmetric group S_5 is defined to be the group of all permutations on a set of five elements, ie, the symmetric group of degree five. In particular, it is a symmetric group of prime degree and it is denoted by S_5.

Definition 2.2
A group generated by a single element is called *cyclic* and we know that cyclic groups are abelian.

Definition 2.3
A subgroup of G is said to be a *normal subgroup* of G if for every $g \in G$ and $n \in N$, $gng^{-1} \in N$.

Definition 2.4
A partial order on a non-empty set P is a binary relation denoted by $\leq$ on P that is reflexive, anti- symmetric and transitive. If $\leq$ is a partial order on P, the pair $(P, \leq)$ is called a *partially ordered set or poset*. A Poset $(P, \leq)$ is called *totally ordered or a chain.* If any two elements x, $y \in P$ are *comparable*, that is either $x \leq y$ or $y \leq x$. A non-empty subset S of P is a chain in P, if S is totally ordered by $\leq$.

Definition 2.5

Let $(P, \leq)$ be a poset and let $S \subseteq P$. An upper bound for S is an element $x \in P$ for which $s \leq x$ for all $s \in S$. The least upper bound of S is called the supremum or join of S. A lower bound for S is an element $x \in P$ for which $x \leq s$ for all $s \in S$. The greatest lower bound of S is called the infimum or meet of S. Poset $(P, \leq)$ is called the lattice if every pair x, y elements of P has a Supremum and an Infimum, and denoted by $x \vee y$ and $x \wedge y$ respectively.

Definition 2.6

The set of all subgroups of a given group G is a partially ordered set under '$\subseteq$' inclusion relation. It is also a lattice denoted by L (G).

Definition 2.7

In the poset $(P, \leq)$, a covers b or b is covered by a (in notation, $a \succ b$ or $b \prec a$) if and only if $b < a$ and, for no $x \in P$, $b < x < a$.

Definition 2.8

Let G be a finite group and let $|G| = p^n m$ where $n \geq 1$, and p is a prime number such that (p, m) = 1. The subgroup of G of order p^n is called a p-sylow subgroup of G.

Definition 2.9

An element 'a' is an atom, if $a \succ 0$ and a dual atom, if $a \prec 1$.

Definition 2.10

A lattice L is said to be modular lattice, if $a \vee (b \wedge c) = (a \vee b) \wedge c$ for every a, b, c in L and $a \leq c$.

Definition 2.11

A Lattice L is said to be semi-modular if whenever a covers $a \wedge b$, then $a \vee b$ covers b, for all $a, b \in L$.

Definition 2.12

An element of a lattice L is called join irreducible if $x \vee y = a \Rightarrow x = a$ or $y = a$.

Definition 2.13

A Lattice L is said to be consistent if whenever j is a join-irreducible element in L, then for every x in L, $x \vee j$ is join irreducible in the upper interval [x, 1].

Theorem 3.1

If G is a finite group and H is a subgroup of G, then the order of H is a divisor of the order of G.

Theorem 3.2

If G is a finite group and $a \in G$, then order of a is a divisor of order of G.

Theorem 3.3

Let G be a finite group and let p be any prime number that divides the order of G. Then G contains an element of order p.

Theorem 3.4

If P is a prime number and $p^{\alpha} \setminus o$ (G), $p^{\alpha+1} \nmid o$ (G), then G has a subgroup of orderp^{α}.

Theorem 3.5

If G is a finite group, p is a prime number and $p^n \setminus o$ (G), but $p^{n+1} \nmid o$ (G), then any two subgroups of G of order p^n are conjugate.

Theorem 3.6

The number of p-sylow subgroups in G, for a given prime p, is of the form 1+kp.

Theorem 3.7

If a group G has a unique sylow p-subgroup, then it is normal.

Theorem 3.8

Let G be a group of order pq, where p and q are distinct primes and $p < q$. Then G has only one subgroup of order q and this subgroup of order q is normal in G.

Theorem 3.9

If N is a normal subgroup of G and H is any subgroup of G, then NH is a subgroup of G.

Theorem 3.10

If M and N are two normal subgroups of G, then NM is also normal subgroup of G.

Theorem 3.11

S_n has a normal subgroup of index 2 the alternating group A_n, consisting of all even permutation.

Notation

If G is a group, then we denote the lattice of all subgroups of G by the notation L (G).

Result

A non-empty subset H of the group G is a subgroup of G iff a, b $\in$ H$\Rightarrow$ab$\in$H and a $\in$H $\Rightarrow$ $a^{-1}\in$H.

We produce below the structure of the lattice of subgroups of the symmetric groups S_2, S_3 and S_4.

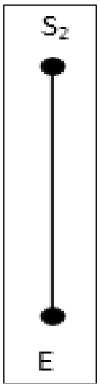

Fig 1: Lattice Structure of L (S_2)

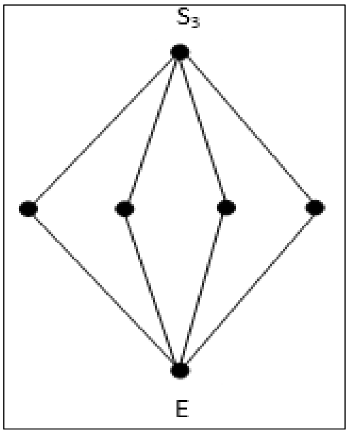

Fig 2: Lattice Structure of L (S_3)

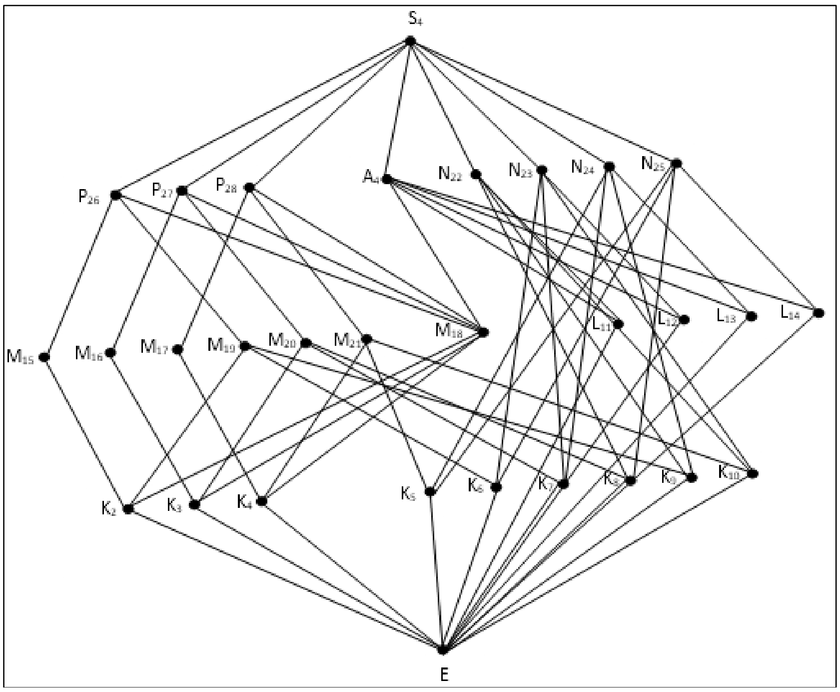

Fig 3: Lattice Structure of L (S_4)

4. Main Result

The symmetric group S_5 is the group of all permutations of the set S = {1, 2, 3, 4, 5}, we know that the order of S_5 is 120. We name the elements according to their orders.

Let $S_5 = \{i, \alpha_1, \alpha_2, \ldots\ldots, \alpha_{10}, \sigma_1, \sigma_2, \ldots\ldots, \alpha_{20}, \tau_1, \tau_2, \ldots\ldots, \tau_{30}, \gamma_1, \gamma_2 \ldots\ldots, \gamma_{15}, \beta_1, \beta_2, \ldots\ldots, \beta_{24}, \delta_1, \delta_2, \ldots\ldots, \delta_{20}\}$, where 'i' is

The identity permutation.

The cycles of length 2 are,

α_1=(4 5), α_2=(3 5), α_3=(3 4), α_4=(2 5), α_5=(2 3), α_6=(2 4), α_7=(1 5), α_8=(1 4), α_9= (1 3), α_{10}= (1 2).

The cycles of length 3 are,

σ_1=(1 2 3), σ_2=(1 3 2), σ_3=(1 2 4), σ_4=(1 4 2), σ_5=(1 2 5), σ_6=(1 5 2), σ_7=(1 3 4), σ_8=(1 4 3), σ_9=(1 4 5), σ_{10}=(1 5 4),

σ_{11}=(1 3 5), σ_{12}=(1 5 3), σ_{13}=(2 3 4), σ_{14}=(2 4 3), σ_{15}= (2 3 5), σ_{16}=(2 5 3), σ_{17}= (2 4 5), σ_{18}= (2 5 4), σ_{19}= (3 4 5), σ_{20}= (3 5 4).

The cycles of length 4 are,

τ_1=(2 3 4 5), τ_2=(2 5 4 3), τ_3=(2 3 5 4), τ_4=(2 4 5 3), τ_5=(2 4 3 5), τ_6=(2 5 3 4), τ_7=(1 2 3 4), τ_8=(1 4 3 2), τ_9=(1 2 3 5),

τ_{10}=(1 5 3 2), τ_{11}=(1 2 4 3), τ_{12}=(1 3 4 2), τ_{13}=(1 2 4 5), τ_{14}=(1 5 4 2), τ_{15}=(1 2 5 3), τ_{16}=(1 3 5 2), τ_{17}=(1 2 5 4),

τ_{18}=(1 4 5 2), τ_{19}=(1 3 4 5), τ_{20}=(1 5 4 3), τ_{21}=(1 3 5 4), τ_{22}=(1 4 5 3), τ_{23}=(1 3 2 4), τ_{24}=(1 4 2 3), τ_{25}=(1 3 2 5),

τ_{26}=(1 5 2 3), τ_{27}=(1 4 3 5), τ_{28}=(1 5 3 4), τ_{29}=(1 4 2 5), τ_{30}=(1 5 2 4).

The elements which are products of two transpositions are,

γ_1=(2 4)(3 5), γ_2=(2 5)(3 4), γ_3=(2 3)(4 5), γ_4=(1 3)(2 4), γ_5=(1 3)(2 5), γ_6=(1 4)(2 3), γ_7=(1 4)(2 5), γ_8=(1 5)(2 3),

γ_9=(1 5)(2 4), γ_{10}=(1 4)(3 5), γ_{11}= (1 5)(3 4), γ_{12}=(1 2)(3 4), γ_{13}=(1 2)(3 5), γ_{14}=(1 3)(4 5), γ_{15}=(1 2)(4 5).

The cycles of length 5 are,

β_1=(1 2 3 4 5), β_2=(1 3 5 2 4), β_3=(1 4 2 5 3), β_4=(1 5 4 3 2), β_5=(1 2 3 5 4), β_6=(1 3 4 2 5), β_7=(1 5 2 4 3), β_8=(1 4 5 3 2),

β_9=(1 2 4 5 3), β_{10}=(1 4 3 2 5), β_{11}=(1 5 2 3 4), β_{12}=(1 3 5 4 2), β_{13}=(1 2 4 3 5), β_{14}=(1 4 5 2 3), β_{15}=(1 3 2 5 4),

β_{16}=(1 5 3 4 2), β_{17}=(1 2 5 4 3), β_{18}=(1 5 3 2 4), β_{19}=(1 4 2 3 5), β_{20}=(1 3 4 5 2), β_{21}=(1 2 5 3 4), β_{22}=(1 5 4 2 3),

β_{23}=(1 3 2 4 5), β_{24}=(1 4 3 5 2).

The elements which are the products of a 3 cycle and a transposition are,

δ_1=(1 2 3)(4 5), δ_2=(1 3 2)(4 5), δ_3=(1 2 4)(3 5), δ_4=(1 4 2)(3 5), δ_5=(1 2 5)(3 4), δ_6=(1 5 2)(3 4), δ_7=(1 3 4)(2 5),

δ_8=(1 4 3)(2 5), δ_9=(1 4 5)(2 3), δ_{10}=(1 5 4)(2 3), δ_{11}=(1 3 5)(2 4), δ_{12}=(1 5 3)(2 4), δ_{13}=(2 3 4)(1 5), δ_{14}=(2 4 3)(1 5),

δ_{15}=(2 3 5)(1 4), δ_{16}=(2 5 3)(1 4), δ_{17}=(2 4 5)(1 3), δ_{18}=(2 5 4)(1 3), δ_{19}=(3 4 5)(1 2), δ_{20}=(3 5 4)(1 2).

In the following table we list the elements according to their order,

Table 1

Order	Elements
1	i
2	$\alpha_1, \alpha_2, \ldots\ldots, \alpha_{10}, \gamma_1, \gamma_2 \ldots\ldots, \gamma_{15}$
3	$\sigma_1, \sigma_2, \ldots\ldots, \alpha_{20}$
4	$\tau_1, \tau_2, \ldots\ldots, \tau_{30}$
5	$\beta_1, \beta_2, \ldots\ldots, \beta_{24}$
6	$\delta_1, \delta_2, \ldots\ldots, \delta_{20}$

5. Subgroups of S_5

According to Lagrange's theorem, the order of any non-trivial subgroups of S_5 divides the order of S_5. So we must have the subgroups of S_5 of orders 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60 and 120.

Subgroup of order 1

Obviously, the only subgroup of S_5 of order 1 is the trivial subgroup G_1 = {i}.

Subgroups of order 2

Let H be an arbitrary subgroup of S_5 of order 2, since 2 is a prime number, H is cyclic, Hence H is generated by an element of order 2. Thus all the subgroups of S_5 of order 2 are,

$H_2 = \{i, \alpha_1\} = < \alpha_1 >$, $H_3 = \{i, \alpha_2\} =< \alpha_2 >$, $H_4 = \{i, \alpha_3\} =< \alpha_3 >$, $H_5 = \{i, \alpha_4\} = < \alpha_4 >$, $H_6 = \{i, \alpha_5\} = < \alpha_5 >$, $H_7 = \{i, \alpha_6\} = < \alpha_6 >$, $H_8 = \{i, \alpha_7\} = < \alpha_7 >$, $H_9 = \{i, \alpha_8\} = < \alpha_8 >$, $H_{10} = \{i, \alpha_9\} =< \alpha_9 >$, $H_{11} = \{i, \alpha_{10}\} = < \alpha_{10} >$, $H_{12} = \{i, \gamma_1\} =< \gamma_1 >$, $H_{13} = \{i, \gamma_2\} =< \gamma_2 >$, $H_{14} = \{i, \gamma_3\} =< \gamma_3 >$, $H_{15} = \{i, \gamma_4\} =< \gamma_4 >$, $H_{16} = \{i, \gamma_5\} =< \gamma_5 >$, $H_{17} = \{i, \gamma_6\} =< \gamma_6 >$, $H_{18} = \{i, \gamma_7\} =< \gamma_7 >$, $H_{19} = \{i, \gamma_8\} =< \gamma_8 >$, $H_{20} = \{i, \gamma_9\} =< \gamma_9 >$, $H_{21} = \{i, \gamma_{10}\} =< \gamma_{10} >$, $H_{22} = \{i, \gamma_{11}\} =< \gamma_{11} >$, $H_{23} = \{i, \gamma_{12}\} =< \gamma_{12} >$, $H_{24} = \{i, \gamma_{13}\} =< \gamma_{13} >$, $H_{25} = \{i, \gamma_{14}\} =< \gamma_{14} >$, $H_{26} = \{i, \gamma_{15}\} =< \gamma_{15} >$.

Subgroups of order 3

Let K be an arbitrary subgroup of order 3, since 3 is a prime number, K is cyclic and generated by an element of order 3.
Since $3 \backslash o(G)$, $3^2 \nmid o(G)$, G has a 3- sylow subgroup of order 3. The number of 3-sylow subgroups is of the form 1+3k and $1+3k \backslash o(G)$, that is, $1+3k \backslash 2^3 \times 3 \times 5$. Therefore $1+3k \backslash 2^3 \times 5$, the possible values for k = 0, 1, 3. Therefore the maximum number of 3-sylow subgroups of order 3 is 10 when k = 3. Thus all the subgroups of S_5 of order 3 are,

$K_{27} = \{i, \sigma_1, \sigma_2\} =< \sigma_1 >=< \sigma_2 >$, $K_{28} = \{i, \sigma_3, \sigma_4\} =< \sigma_3 >=< \sigma_4 >$,

$K_{29} = \{i, \sigma_5, \sigma_6\} =< \sigma_5 >=< \sigma_6 >$, $K_{30} = \{i, \sigma_7, \sigma_8\} =< \sigma_7 >=< \sigma_8 >$,

$K_{31} = \{i, \sigma_9, \sigma_{10}\} =< \sigma_9 >=< \sigma_{10} >$, $K_{32} = \{i, \sigma_{11}, \sigma_{12}\} =< \sigma_{11} >=< \sigma_{12} >$,

$K_{33} = \{i, \sigma_{13}, \sigma_{14}\} =< \sigma_{13} >=< \sigma_{14} >$, $K_{34} = \{i, \sigma_{15}, \sigma_{16}\} =< \sigma_{15} >=< \sigma_{16} >$,

$K_{35} = \{i, \sigma_{17}, \sigma_{18}\} =< \sigma_{17} >=< \sigma_{18} >$, $K_{36} = \{i, \sigma_{19}, \sigma_{20}\} =< \sigma_{19} >=< \sigma_{20} >$.

Subgroups of order 4

Let L be the arbitrary subgroup of S_5 of order 4, then the elements of L must have order 1, 2 or 4. If L contains an element of order 4, then L is generated by an element of order 4. Therefore the subgroups generated by an element of order 4 are,

$L_{37} = \{i, \tau_1, \tau_2, \gamma_1\} =< \tau_1 >=< \tau_2 >$, $L_{38} = \{i, \tau_3, \tau_4, \gamma_2\} =< \tau_3 >=< \tau_4 >$,

$L_{39} = \{i, \tau_5, \tau_6, \gamma_3\} =< \tau_5 >=< \tau_6 >$, $L_{40} = \{i, \tau_7, \tau_8, \gamma_4\} =< \tau_7 >=< \tau_8 >$,

$L_{41} = \{i, \tau_9, \tau_{10}, \gamma_5\} =< \tau_9 >=< \tau_{10} >$, $L_{42} = \{i, \tau_{11}, \tau_{12}, \gamma_6\} =< \tau_{11} >=< \tau_{12} >$,

$L_{43} = \{i, \tau_{13}, \tau_{14}, \gamma_7\} =< \tau_{13} >=< \tau_{14} >$, $L_{44} = \{i, \tau_{15}, \tau_{16}, \gamma_8\} =< \tau_{15} >=< \tau_{16} >$,

$L_{45} = \{i, \tau_{17}, \tau_{18}, \gamma_9\} =< \tau_{17} >=< \tau_{18} >$, $L_{46} = \{i, \tau_{19}, \tau_{20}, \gamma_{10}\} =< \tau_{19} >=< \tau_{20} >$,

$L_{47} = \{i, \tau_{21}, \tau_{22}, \gamma_{11}\} =< \tau_{21} >=< \tau_{22} >$, $L_{48} = \{i, \tau_{23}, \tau_{24}, \gamma_{12}\} =< \tau_{23} >=< \tau_{24} >$,

$L_{49} = \{i, \tau_{25}, \tau_{26}, \gamma_{13}\} =< \tau_{25} >=< \tau_{26} >$, $L_{50} = \{i, \tau_{27}, \tau_{28}, \gamma_{14}\} =< \tau_{27} >=< \tau_{28} >$,

$L_{51} = \{i, \tau_{29}, \tau_{30}, \gamma_{15}\} =< \tau_{29} >=< \tau_{30} >$.

If L contains an element of order 2, then L does not contain an element of order 4. Therefore, the order of all the elements of L is 2, except the identity element.
The table below shows the multiplications of all the combinations of elements of order 2.

Table 2

·	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	γ_8	γ_9	γ_{10}	γ_{11}	γ_{12}	γ_{13}	γ_{14}	γ_{15}
α_1	i	σ_{20}	σ_{19}	σ_{18}	γ_3	σ_{17}	σ_{10}	σ_9	γ_{14}	γ_{15}	τ_5	τ_6	α_5	δ_{17}	δ_{18}	δ_9	τ_{29}	δ_{10}	τ_{30}	τ_{27}	τ_{28}	δ_{19}	δ_{20}	α_9	α_{10}
α_2	σ_{19}	i	σ_{20}	σ_{16}	σ_{15}	γ_1	σ_{12}	γ_{10}	σ_{11}	γ_{13}	α_6	τ_2	τ_1	δ_{11}	τ_{25}	δ_{15}	δ_{16}	τ_{26}	δ_{12}	α_8	τ_{20}	δ_{20}	α_{10}	τ_{19}	τ_{19}
α_3	σ_{20}	σ_{19}	i	γ_2	σ_{13}	σ_{14}	γ_{11}	σ_8	σ_7	γ_{12}	τ_4	α_4	τ_3	τ_{23}	δ_7	τ_{24}	δ_8	δ_{13}	δ_{14}	τ_{22}	α_7	α_{10}	δ_{19}	τ_{21}	δ_{20}
α_4	σ_{17}	σ_{15}	γ_2	i	σ_{16}	σ_{18}	σ_6	γ_7	γ_5	σ_5	τ_3	α_3	τ_4	δ_{18}	α_9	δ_{16}	α_8	τ_{10}	τ_{14}	δ_{15}	δ_6	δ_5	τ_9	δ_{17}	τ_{13}
α_5	γ_3	σ_{16}	σ_{14}	σ_{15}	i	σ_{13}	γ_8	γ_6	σ_2	σ_1	τ_6	τ_5	α_1	τ_{12}	τ_{16}	α_8	δ_{15}	α_7	δ_{13}	δ_{16}	δ_{14}	τ_{11}	τ_{15}	δ_2	δ_1
α_6	σ_{18}	γ_1	σ_{13}	σ_{17}	σ_{14}	i	γ_9	σ_4	γ_4	σ_3	α_2	τ_1	τ_2	α_9	δ_{17}	τ_8	τ_{18}	δ_{14}	α_7	δ_4	δ_{13}	τ_7	δ_3	δ_{18}	τ_{17}
α_7	σ_9	σ_{11}	γ_{11}	σ_5	γ_8	γ_9	i	σ_{10}	σ_{12}	σ_6	δ_{11}	δ_5	δ_9	δ_{12}	τ_{15}	δ_{10}	τ_{17}	α_5	α_6	τ_{21}	α_3	δ_6	τ_{16}	τ_{22}	τ_{18}
α_8	σ_{10}	γ_{10}	σ_7	γ_7	γ_6	σ_3	σ_9	i	σ_8	σ_4	δ_3	δ_7	δ_{10}	τ_{11}	δ_8	α_5	α_4	δ_9	τ_{13}	α_2	τ_{19}	τ_{12}	δ_4	τ_{20}	τ_{14}
α_9	γ_{14}	σ_{12}	σ_8	γ_5	σ_1	γ_4	σ_{11}	σ_7	i	σ_2	δ_{12}	δ_8	δ_1	α_6	α_4	τ_7	δ_7	τ_9	δ_{11}	τ_{28}	τ_{27}	τ_8	τ_{10}	α_1	δ_2
α_{10}	γ_{15}	γ_{13}	γ_{12}	σ_6	σ_2	σ_4	σ_5	σ_3	σ_1	i	δ_4	δ_6	δ_2	τ_{24}	τ_{26}	τ_{23}	τ_{30}	τ_{25}	τ_{29}	δ_3	δ_5	α_3	α_2	γ_{14}	α_1
γ_1	τ_6	α_6	τ_3	τ_4	τ_5	α_2	δ_{12}	δ_4	δ_{11}	δ_3	i	γ_3	γ_2	σ_{11}	β_{23}	β_{24}	β_8	β_7	σ_{12}	σ_4	β_{22}	β_5	σ_3	β_6	β_{21}
γ_2	τ_5	τ_1	α_4	α_3	τ_6	τ_2	δ_6	δ_8	δ_7	δ_5	γ_3	i	γ_1	β_{15}	σ_7	β_3	σ_8	β_{16}	β_4	β_{14}	σ_6	σ_5	β_1	β_2	β_{13}
γ_3	α_5	τ_2	τ_4	τ_3	α_1	τ_1	δ_{10}	δ_9	δ_2	δ_1	γ_2	γ_1	i	β_{20}	σ_{12}	σ_9	β_{19}	σ_{10}	β_{11}	β_{10}	β_{18}	β_9	β_{17}	σ_2	σ_1
γ_4	δ_{18}	δ_{12}	τ_{24}	δ_{17}	τ_{11}	α_9	δ_{11}	τ_{12}	α_6	τ_{23}	σ_{12}	β_{14}	β_{17}	i	σ_{17}	γ_{12}	β_{20}	β_{13}	σ_{11}	β_{16}	β_{19}	γ_6	β_{18}	σ_{18}	β_{15}
γ_5	δ_{17}	τ_{26}	δ_8	α_9	τ_{15}	δ_{18}	τ_{16}	δ_7	α_4	τ_{25}	β_{22}	σ_8	β_9	σ_{18}	i	β_{21}	σ_7	γ_{13}	β_{12}	β_{11}	β_{24}	β_{10}	γ_8	σ_{17}	β_{23}
γ_6	δ_{10}	δ_{16}	τ_{23}	δ_{15}	α_8	τ_7	δ_9	α_5	τ_8	τ_{24}	β_{21}	β_2	σ_{10}	γ_{12}	β_{24}	i	σ_{15}	σ_9	β_1	σ_{16}	β_{23}	γ_4	β_3	β_4	β_{22}
γ_7	τ_{30}	δ_{15}	δ_7	α_8	δ_{16}	τ_{17}	τ_{18}	α_4	δ_8	τ_{29}	β_5	σ_7	β_{18}	β_{17}	σ_8	σ_{16}	i	β_8	γ_{15}	σ_{15}	β_{20}	β_6	β_{19}	β_7	γ_9
γ_8	δ_9	τ_{25}	δ_{14}	τ_9	α_7	δ_{13}	α_5	δ_{10}	τ_{10}	τ_{26}	β_6	β_{13}	σ_9	β_{16}	γ_{13}	σ_{10}	β_5	i	σ_{13}	β_{15}	σ_{14}	β_7	γ_5	β_8	β_{14}
γ_9	τ_{29}	δ_{11}	δ_{13}	τ_{13}	δ_{14}	α_7	α_6	τ_{14}	δ_{12}	τ_{30}	σ_{11}	β_1	β_{10}	σ_{12}	β_9	β_4	γ_{15}	σ_{14}	i	β_{12}	σ_{13}	β_{11}	β_2	β_3	γ_7
γ_{10}	τ_{28}	α_8	τ_{21}	δ_{16}	δ_{15}	δ_3	τ_{22}	α_2	τ_{27}	δ_4	σ_3	β_{15}	β_{11}	β_{13}	β_{10}	σ_{15}	σ_{16}	β_{14}	β_9	i	γ_{14}	β_{12}	σ_4	γ_{11}	β_{11}
γ_{11}	τ_{27}	τ_{19}	α_7	δ_5	δ_{13}	δ_{14}	α_3	τ_{20}	τ_{28}	δ_6	β_{23}	σ_5	β_{19}	β_{18}	β_{21}	β_{22}	β_{17}	σ_{13}	σ_{14}	γ_{14}	i	σ_6	β_{20}	γ_{10}	β_{24}
γ_{12}	δ_{20}	δ_{19}	α_{10}	δ_6	τ_{12}	τ_8	δ_5	τ_{11}	τ_7	α_3	β_8	σ_6	β_{12}	γ_6	β_{11}	γ_4	β_7	β_6	β_{10}	β_9	σ_5	i	σ_{19}	β_5	σ_{20}
γ_{13}	δ_{19}	α_{10}	δ_{20}	τ_{10}	τ_{16}	δ_4	τ_{15}	δ_3	τ_9	α_2	σ_4	β_4	β_{20}	β_{19}	γ_8	β_2	β_{18}	γ_5	β_3	σ_3	β_{17}	σ_{20}	i	β_1	σ_{19}
γ_{14}	α_9	τ_{20}	τ_{22}	δ_{18}	δ_1	δ_{17}	τ_{21}	τ_{19}	α_1	δ_2	β_7	β_3	σ_1	σ_{17}	σ_{18}	β_1	β_6	β_5	β_2	γ_{11}	γ_{10}	β_8	β_4	i	σ_2
γ_{15}	α_{10}	δ_{20}	δ_{19}	τ_{14}	δ_2	τ_{18}	τ_{17}	τ_{13}	δ_1	α_1	β_{24}	β_{16}	σ_2	β_{14}	β_{22}	β_{23}	γ_9	β_{15}	γ_7	β_{13}	β_{21}	σ_{19}	σ_{20}	σ_1	i

We observe row by row of table-2. From the second row we conclude that

$\{\alpha_1, \alpha_2\}$, $\{\alpha_1, \alpha_3\}$, $\{\alpha_1, \alpha_4\}$, $\{\alpha_1, \alpha_6\}$, $\{\alpha_1, \alpha_7\}$, $\{\alpha_1, \alpha_8\}$, $\{\alpha_1, \gamma_1\}$, $\{\alpha_1, \gamma_2\}$, $\{\alpha_1, \gamma_4\}$, $\{\alpha_1, \gamma_5\}$, $\{\alpha_1, \gamma_6\}$, $\{\alpha_1, \gamma_7\}$, $\{\alpha_1, \gamma_8\}$,

$\{\alpha_1, \gamma_9\}$, $\{\alpha_1, \gamma_{10}\}$, $\{\alpha_1, \gamma_{11}\}$, $\{\alpha_1, \gamma_{12}\}$, $\{\alpha_1, \gamma_{13}\}$

Are not subsets of L. Using similar argument to the other rows, we obtain the subgroups of order 4 as,

L_{52} = {i, $\alpha_2, \alpha_6, \gamma_1$}, L_{53} = {i, $\alpha_3, \alpha_4, \gamma_2$}, L_{54} = {i, $\alpha_1, \alpha_5, \gamma_3$}, L_{55} = {i, $\alpha_6, \alpha_9, \gamma_4$}, L_{56} = {i, $\alpha_4, \alpha_9, \gamma_5$}, L_{57} = {i, $\alpha_5, \alpha_8, \gamma_6$},

L_{58} = {i, $\alpha_4, \alpha_8, \gamma_7$}, L_{59} = {i, $\alpha_5, \alpha_7, \gamma_8$}, L_{60} = {i, $\alpha_6, \alpha_7, \gamma_9$}, L_{61} = {i, $\alpha_2, \alpha_8, \gamma_{10}$}, L_{62} = {i, $\alpha_3, \alpha_7, \gamma_{11}$}, L_{63} = {i, α_3, α_{10},

γ_{12}}, L_{64} = {i, $\alpha_2, \alpha_{10}, \gamma_{13}$}, L_{65} = {i, $\alpha_1, \alpha_9, \gamma_{14}$}, L_{66} = {i, $\alpha_1, \alpha_{10}, \gamma_{15}$}, L_{67} = {i, $\gamma_4, \gamma_6, \gamma_{12}$}, L_{68} = {i, $\gamma_1, \gamma_2, \gamma_3$}, L_{69} = {i,

$\gamma_5, \gamma_8, \gamma_{13}$}, L_{70} = {i, $\gamma_7, \gamma_9, \gamma_{15}$}, L_{71} = {i, $\gamma_{10}, \gamma_{11}, \gamma_{14}$}.

Subgroups of order 5

Let M be an arbitrary subgroup of S_5, since 5 is a prime number, the subgroup M is cyclic and is generated by an element of order 5.

Since $|G|$ = 2^3×3×5 and 5\o (G), 5^2 ∤o (G), therefore G has 5- sylow subgroup of order 5. The number of 5-sylow subgroups of G of order 5 is of the form 1+5k and 1+5k\o(G), that is, 1+5k\$2^3$×3× 5. Therefore 1+5k\$2^3$×3, the possible values for k = 0 and 1. Therefore, the maximum number of 5-sylow subgroup of order 5 is 6 when k = 1.

Thus all the subgroups of S_5 of order 5 are,

M_{72} = {i, $\beta_1, \beta_2, \beta_3, \beta_4$}= $<\beta_1> = <\beta_2> = <\beta_3> = <\beta_4>$

M_{73} = {i, $\beta_5, \beta_6, \beta_7, \beta_8$}= $<\beta_5> = <\beta_6> = <\beta_7> = <\beta_8>$

M_{74} = {i, $\beta_9, \beta_{10}, \beta_{11}, \beta_{12}$}= $<\beta_9> = <\beta_{10}> = <\beta_{11}> = <\beta_{12}>$

M_{75} = {i, $\beta_{13}, \beta_{14}, \beta_{15}, \beta_{16}$}= $<\beta_{13}> = <\beta_{14}> = <\beta_{15}> = <\beta_{16}>$

M_{76} = {i, $\beta_{17}, \beta_{18}, \beta_{19}, \beta_{20}$}= $<\beta_{17}> = <\beta_{18}> = <\beta_{19}> = <\beta_{20}>$

M_{77} = {i, $\beta_{21}, \beta_{22}, \beta_{23}, \beta_{24}$}= $<\beta_{21}> = <\beta_{22}> = <\beta_{23}> = <\beta_{24}>$.

The subgroups of order 6

Let N be any arbitrary subgroup of S_5 of order 6, since $|N|$ = 2×3, according to the third sylow's theorem, N has exactly one subgroup of order 3 and this group is normal in N.

Now we generate the elements of S_5 of order 6, If N contains an element of order 6 then it generates N, there are 10 subgroups of order 6 contain the elements of order 2, order 3 and order 6 and is isomorphic to the cyclic group Z_6, they are,

N_{78} = {i, $\sigma_1, \sigma_2, \delta_1, \delta_2, \alpha_1$}, N_{79} = {i, $\sigma_3, \sigma_4, \delta_3, \delta_4, \alpha_2$}, N_{80} = {i, $\sigma_5, \sigma_6, \delta_5, \delta_6, \alpha_3$}, N_{81} = {i, $\sigma_7, \sigma_8, \delta_7, \delta_8, \alpha_4$}, N_{82} = {i, $\sigma_9, \sigma_{10}, \delta_9, \delta_{10}, \alpha_5$}, N_{83} = {i, $\sigma_{11}, \sigma_{12}, \delta_{11}, \delta_{12}, \alpha_6$}, N_{84} = {i, $\sigma_{13}, \sigma_{14}, \delta_{13}, \delta_{14}, \alpha_7$}, N_{85} = {i, $\sigma_{15}, \sigma_{16}, \delta_{15}, \delta_{16}, \alpha_8$}, N_{86} = {i, $\sigma_{17}, \sigma_{18}, \delta_{17}, \delta_{18}, \alpha_9$}, N_{87} = {i, $\sigma_{19}, \sigma_{20}, \delta_{19}, \delta_{20}, \alpha_{10}$}.

If N does not contain an element of order 6, we get the other 10 subgroups of S_5 of the elements order 2 and the elements of order 3. Therefore the subgroups are as follows,

N_{88} = {i, $\sigma_1, \sigma_2, \alpha_5, \alpha_9, \alpha_{10}$}, N_{89} = {i, $\sigma_3, \sigma_4, \alpha_6, \alpha_8, \alpha_{10}$}, N_{90} = {i, $\sigma_5, \sigma_6, \alpha_4, \alpha_7, \alpha_{10}$}, N_{91} = {i, $\sigma_7, \sigma_8, \alpha_3, \alpha_8, \alpha_9$}, N_{92} = {i, $\sigma_9, \sigma_{10}, \alpha_1, \alpha_7, \alpha_8$}, N_{93} = {i, $\sigma_{11}, \sigma_{12}, \alpha_2, \alpha_7, \alpha_9$}, N_{94} = {i, $\sigma_{13}, \sigma_{14}, \alpha_3, \alpha_5, \alpha_6$}, N_{95} = {i, $\sigma_{15}, \sigma_{16}, \alpha_2, \alpha_4, \alpha_5$}, N_{96} = {i, $\sigma_{17}, \sigma_{18}, \alpha_1, \alpha_4, \alpha_6$}, N_{97} = {i, $\sigma_{19}, \sigma_{20}, \alpha_1, \alpha_2, \alpha_3$}.

Also the other subgroups of order 6 of S_5 by taking the product of the elements of S_5 of order 2 and the elements of order 3. Therefore such subgroups are,

N_{98} = {i, $\sigma_1, \sigma_2, \gamma_3, \gamma_{14}, \gamma_{15}$}, N_{99} = {i, $\sigma_3, \sigma_4, \gamma_1, \gamma_{10}, \gamma_{13}$}, N_{100} = {i, $\sigma_5, \sigma_6, \gamma_2, \gamma_{11}, \gamma_{12}$}, N_{101} = {i, $\sigma_7, \sigma_8, \gamma_2, \gamma_5, \gamma_7$}, N_{102} = {i, $\sigma_9, \sigma_{10}, \gamma_3, \gamma_6, \gamma_8$}, N_{103} = {i, $\sigma_{11}, \sigma_{12}, \gamma_1, \gamma_4, \gamma_9$}, N_{104} = {i, $\sigma_{13}, \sigma_{14}, \gamma_8, \gamma_9, \gamma_{11}$}, N_{105} = {i, $\sigma_{15}, \sigma_{16}, \gamma_6, \gamma_7, \gamma_{10}$}, N_{106} = {i, $\sigma_{17}, \sigma_{18}, \gamma_4, \gamma_5, \gamma_{14}$}, N_{107} = {i, $\sigma_{19}, \sigma_{20}, \gamma_{12}, \gamma_{13}, \gamma_{15}$}.

Subgroups of order 7

Let Q be an arbitrary subgroup of S_5, Since $|G|$ = 2^3×3×5 and 2^3\o(G), 2^4 ∤o(G), therefore the number of 2-sylow subgroups of G of order 8 is of the form 1+2k and 1+2k\o(G), that is, 1+2k\$2^3$×3× 5. Therefore 1+2k\5×3, the possible values for k = 0, 1, 2 and 7. Therefore the maximum number of 2-sylow subgroup of order 8 is 15 when k = 7.

Since there is no element of order 8, so every element of Q must have order 1, 2 and 4, such subgroups are got by the product of the elements of S_5 of order 2 and the elements of order 4 and they are,

Q_{108} = { i, $\tau_1, \tau_2, \gamma_1, \gamma_2, \gamma_3, \alpha_2, \alpha_6$}, Q_{109} = { i, $\tau_3, \tau_4, \gamma_1, \gamma_2, \gamma_3, \alpha_3, \alpha_4$},

Q_{110} = { i, $\tau_5, \tau_6, \gamma_1, \gamma_2, \gamma_3, \alpha_1, \alpha_5$}, Q_{111} = { i, $\tau_7, \tau_8, \gamma_4, \gamma_6, \gamma_{12}, \alpha_6, \alpha_9$},

Q_{112} = { i, $\tau_9, \tau_{10}, \gamma_5, \gamma_8, \gamma_{13}, \alpha_4, \alpha_9$}, Q_{113} = { i, $\tau_{11}, \tau_{12}, \gamma_4, \gamma_6, \gamma_{12}, \alpha_5, \alpha_8$},

Q_{114} = { i, $\tau_{13}, \tau_{14}, \gamma_7, \gamma_9, \gamma_{15}, \alpha_4, \alpha_8$}, Q_{115} = { i, $\tau_{15}, \tau_{16}, \gamma_5, \gamma_8, \gamma_{13}, \alpha_5, \alpha_7$},

Q_{116} = { i, $\tau_{17}, \tau_{18}, \gamma_7, \gamma_9, \gamma_{15}, \alpha_6, \alpha_7$}, Q_{117} = { i, $\tau_{19}, \tau_{20}, \gamma_{10}, \gamma_{11}, \gamma_{14}, \alpha_2, \alpha_8$},

Q_{118} = { i, $\tau_{21}, \tau_{22}, \gamma_{10}, \gamma_{11}, \gamma_{14}, \alpha_3, \alpha_7$}, Q_{119} = { i, $\tau_{23}, \tau_{24}, \gamma_4, \gamma_6, \gamma_{12}, \alpha_3, \alpha_{10}$},

Q_{120} = { i, $\tau_{25}, \tau_{26}, \gamma_5, \gamma_8, \gamma_{13}, \alpha_2, \alpha_{10}$}, Q_{121} = { i, $\tau_{27}, \tau_{28}, \gamma_{10}, \gamma_{11}, \gamma_{14}, \alpha_1, \alpha_9$},

Q_{122} = { i, $\tau_{29}, \tau_{30}, \gamma_7, \gamma_9, \gamma_{15}, \alpha_1, \alpha_{10}$}.

Subgroups of order 8

Let P be any arbitrary subgroup of S_5 of order 10, since $|P|$ = 2×5, according sylow's theorem, P has exactly one subgroup of order 5, there is no element of order 10 in S_5. Therefore P is not cyclic and P contains only elements of order 2 and order 5, by taking the product of subgroups of order 2 and order 5, we get the following subgroups of order 10 and they are,

P_{123} = {i, $\beta_1, \beta_2, \beta_3, \beta_4, \gamma_2, \gamma_6, \gamma_9, \gamma_{13}, \gamma_{14}$},

P_{124} = {i, $\beta_5, \beta_6, \beta_7, \beta_8, \gamma_1, \gamma_7, \gamma_8, \gamma_{12}, \gamma_{14}$, },

P_{125} = {i, $\beta_9, \beta_{10}, \beta_{11}, \beta_{12}, \gamma_3, \gamma_5, \gamma_9, \gamma_{10}, \gamma_{12}$},

P_{126} = {i, $\beta_{13}, \beta_{14}, \beta_{15}, \beta_{16}, \gamma_2, \gamma_4, \gamma_8, \gamma_{10}, \gamma_{15}$},

P_{127} = {i, $\beta_{17}, \beta_{18}, \beta_{19}, \beta_{20}, \gamma_3, \gamma_4, \gamma_7, \gamma_{11}, \gamma_{13}$},

P_{128}= {i, $\beta_{21}, \beta_{22}, \beta_{23}, \beta_{24}, \gamma_1, \gamma_5, \gamma_6, \gamma_{11}, \gamma_{15}$}.

Subgroups of order 9

Let R be any arbitrary subgroup of S_5 of order 12, since $|R|$ = 2^2×3, the elements of R must be of order 2, 3 or 6 as there is no element of order 12 in S_5. We obtain 10 subgroups of S_5 of order 12 is the combination of the elements of order 2, order 3 and order 6 are,

R_{129} = {i, $\sigma_1, \sigma_2, \delta_1, \delta_2, \gamma_3, \gamma_{14}, \gamma_{15}, \alpha_1, \alpha_5, \alpha_9, \alpha_{10}$},

R_{130} = {i, $\sigma_3, \sigma_4, \delta_3, \delta_4, \gamma_1, \gamma_{10}, \gamma_{13}, \alpha_2, \alpha_6, \alpha_8, \alpha_{10}$},

R_{131}= {i, $\sigma_5, \sigma_6, \delta_5, \delta_6, \gamma_2, \gamma_{11}, \gamma_{12}, \alpha_3, \alpha_4, \alpha_7, \alpha_{10}$},

R_{132} = {i, $\sigma_7, \sigma_8, \delta_7, \delta_8, \gamma_2, \gamma_5, \gamma_7, \alpha_3, \alpha_4, \alpha_8, \alpha_9$},

R_{133} = {i, $\sigma_9, \sigma_{10}, \delta_9, \delta_{10}, \gamma_3, \gamma_6, \gamma_8, \alpha_1, \alpha_5, \alpha_7, \alpha_8$},

R_{134} = {i, $\sigma_{11}, \sigma_{12}, \delta_{11}, \delta_{12}, \gamma_1, \gamma_4, \gamma_9, \alpha_2, \alpha_6, \alpha_7, \alpha_9$},

R_{135}= {i, $\sigma_{13}, \sigma_{14}, \delta_{13}, \delta_{14}, \gamma_8, \gamma_9, \gamma_{11}, \alpha_3, \alpha_5, \alpha_6, \alpha_7$},

R_{136}= {i, $\sigma_{15}, \sigma_{16}, \delta_{15}, \delta_{16}, \gamma_6, \gamma_7, \gamma_{10}, \alpha_2, \alpha_4, \alpha_5, \alpha_8$},

R_{137} = {i, $\sigma_{17}, \sigma_{18}, \delta_{17}, \delta_{18}, \gamma_4, \gamma_5, \gamma_{14}, \alpha_1, \alpha_4, \alpha_6, \alpha_9$},

R_{138} = {i, $\sigma_{19}, \sigma_{20}, \delta_{19}, \delta_{20}, \gamma_{12}, \gamma_{13}, \gamma_{15}, \alpha_1, \alpha_2, \alpha_3, \alpha_{10}$},

Also we obtain the other 5 subgroups of S_5 of order 12 of the combination of the elements of order 2 and order 3. Therefore the 5 subgroups of order 12 are,

R_{139} = {i, $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_7, \sigma_8, \sigma_{13}, \sigma_{14}, \gamma_4, \gamma_6, \gamma_{12}$},

R_{140} = {i, $\sigma_1, \sigma_2, \sigma_5, \sigma_6, \sigma_{11}, \sigma_{12}, \sigma_{15}, \sigma_{16}, \gamma_6, \gamma_7, \gamma_{10}$},

R_{141} = {i, $\sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_9, \sigma_{10}, \sigma_{17}, \sigma_{18}, \gamma_4, \gamma_5, \gamma_{14}$},

R_{142} = {i, $\sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{19}, \sigma_{20}, \gamma_{12}, \gamma_{13}, \gamma_{15}$},

R_{143} = {i, $\sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}, \sigma_{20}, \gamma_8, \gamma_9, \gamma_{11}$}.

Subgroups of order 10

Subgroups of order 15 do not exist, since 15 = 3×5 and as $5 \not\equiv 1 \pmod 3$, a group of order 15 is cyclic, therefore it must be generated by an element of order 15, hence a subgroup of G of order 15 does not exist.

Subgroups of order 11

Let T be any arbitrary subgroup of S_5 of order 20, since $|T|$ = 2^2× 5, the number of 2-sylow subgroups of order 4 in T is 1+2k and 1+2k\ 5, the possible values of k are 0 and 2. Hence the number of subgroups of T of order 4 is either 1 or 5, thus the maximum number of subgroups of order 4 is 5, similarly the number of 5-sylow subgroups of T of order 5 is 1+5k and 1+5k\4, the possible values of k is 0 only. Hence the number of subgroups of order 5 is 1. Therefore the number of subgroups of T of order 4 is 5 and the number of subgroups of order 5 is 1. Therefore each subgroup of order 20 contains a unique subgroup of order 5 and each contains a subgroup of order 10. Hence there are 6 subgroups of S_5 of order 20 are,

T_{144} = {i, $\beta_1, \beta_2, \beta_3, \beta_4, \gamma_2, \gamma_6, \gamma_9, \gamma_{13}, \gamma_{14}, \tau_3, \tau_4, \tau_{11}, \tau_{12}, \tau_{17}, \tau_{18}, \tau_{25}, \tau_{26}, \tau_{27}, \tau_{28}$},

T_{145} = {i, $\beta_5, \beta_6, \beta_7, \beta_8, \gamma_1, \gamma_7, \gamma_8, \gamma_{12}, \gamma_{14}, \tau_1, \tau_2, \tau_{13}, \tau_{14}, \tau_{15}, \tau_{16}, \tau_{23}, \tau_{24}, \tau_{27}, \tau_{28}$},

T_{146} = {i, $\beta_9, \beta_{10}, \beta_{11}, \beta_{12}, \gamma_3, \gamma_5, \gamma_9, \gamma_{10}, \gamma_{12}, \tau_5, \tau_6, \tau_9, \tau_{10}, \tau_{17}, \tau_{18}, \tau_{19}, \tau_{20}, \tau_{23}, \tau_{24}$},

T_{147} = {i, $\beta_{13}, \beta_{14}, \beta_{15}, \beta_{16}, \gamma_2, \gamma_4, \gamma_8, \gamma_{10}, \gamma_{15}, \tau_3, \tau_4, \tau_7, \tau_8, \tau_{15}, \tau_{16}, \tau_{19}, \tau_{20}, \tau_{29}, \tau_{30}$},

T_{148} = {i, $\beta_{17}, \beta_{18}, \beta_{19}, \beta_{20}, \gamma_3, \gamma_4, \gamma_7, \gamma_{11}, \gamma_{13}, \tau_5, \tau_6, \tau_7, \tau_8, \tau_{13}, \tau_{14}, \tau_{21}, \tau_{22}, \tau_{25}, \tau_{26}$},

T149= {i, $\beta_{21}, \beta_{22}, \beta_{23}, \beta_{24}, \gamma_1, \gamma_5, \gamma_6, \gamma_{11}, \gamma_{15}, \tau_1, \tau_2, \tau_9, \tau_{10}, \tau_{11}, \tau_{12}, \tau_{21}, \tau_{22}, \tau_{29}, \tau_{30}$}.

Subgroups of order 12

Let V be any arbitrary subgroup of S_5 of order 24, since $|V| = 2^3\times 3$, the number of 2-sylow subgroups of order 8 in V is 1+2k and 1+2k\3, the possible values of k are 0 and 1. Hence the number of subgroups of V of order 8 is either 1 or 3, thus the maximum number of subgroups of order 8 is 3, similarly the number of 3-sylow subgroups of order 3 in V is 1+3k and 1+3k\8, the possible values of k are 0 and 1. Hence the number of subgroups of order 3 in V is either 1 or 4, thus the maximum number of subgroups of order 3 is 4, Therefore by multiplying a subgroups of order 8 with subgroups of order 3, we determine 5 subgroups of S_5 of order 24 are,

V_{150} = {i, $\alpha_3, \alpha_5, \alpha_6, \alpha_8, \alpha_9, \alpha_{10}, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_7, \sigma_8, \sigma_{13}, \sigma_{14}, \gamma_4, \gamma_6, \gamma_{12}, \tau_7, \tau_8, \tau_{11}, \tau_{12}, \tau_{23}, \tau_{24}$},

V_{151} = {i, $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \sigma_{13}, \sigma_{14}, \sigma_{15}, \sigma_{16}, \sigma_{17}, \sigma_{18}, \sigma_{19}, \sigma_{20}, \gamma_1, \gamma_2, \gamma_3, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6$},

V_{152} = {i, $\alpha_1, \alpha_2, \alpha_3, \alpha_7, \alpha_8, \alpha_9, \sigma_7, \sigma_8, \sigma_9, \sigma_{10}, \sigma_{11}, \sigma_{12}, \sigma_{19}, \sigma_{20}, \gamma_{10}, \gamma_{11}, \gamma_{14}, \tau_{19}, \tau_{20}, \tau_{21}, \tau_{22}, \tau_{27}, \tau_{28}$},

V_{153} = {i, $\alpha_1, \alpha_4, \alpha_6, \alpha_7, \alpha_8, \alpha_{10}, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_9, \sigma_{10}, \sigma_{17}, \sigma_{18}, \gamma_7, \gamma_9, \gamma_{15}, \tau_{13}, \tau_{14}, \tau_{17}, \tau_{18}, \tau_{29}, \tau_{30}$},

V_{154} = {i, $\alpha_2, \alpha_4, \alpha_5, \alpha_7, \alpha_9, \alpha_{10}, \sigma_1, \sigma_2, \sigma_5, \sigma_6, \sigma_{11}, \sigma_{12}, \sigma_{15}, \sigma_{16}, \gamma_5, \gamma_8, \gamma_{13}, \tau_9, \tau_{10}, \tau_{15}, \tau_{16}, \tau_{25}, \tau_{26}$}.

Subgroup of order 13

The subgroup of order 30 cannot exist since 30 = 2×3×5, by multiplying a subgroup of order 3 and subgroup of order 10 or multiplying a subgroup of order 5 and subgroup of order 6, ie) by finding KiPj or MiNj for all i and j we get in each case an element of order 4 which cannot exist in a subgroup of order 30.

Subgroup of order 14

The subgroup of order 40 also cannot exist, since $40 = 2^3\times 5 = 4 \times 10$ by multiplying a subgroup of order 5 and subgroup of order 8 that is by finding MiQj for all i and j we get elements of order 3 or elements of order 6 which cannot exist in a subgroup of order 40.

Also multiplying a subgroup of order 4 and a subgroup of order 10, that is by finding LiPj for all i and j, we get elements of order 3 or 6 which cannot exist in a subgroup of order 40.

Subgroup of order 15

The only subgroup of S_5 of order 60 is the alternating group A_5 consisting of all even permutations in S_5. Therefore the only one subgroup of order 60 is,

A_5 = {i, $\sigma_1, \sigma_2, \sigma_3, \dots\dots, \alpha_{20}, \gamma_1, \gamma_2, \gamma_3, \dots\dots, \gamma_{15}, \beta_1, \beta_2, \beta_3, \dots\dots, \beta_{24}$}.

Subgroup of order 16

Every group is a subgroup of itself; hence the whole group S_5 is a subgroup of S_5 of order 120.

According these results, we have the diagram of lattice subgroups of S_5 is shown below,

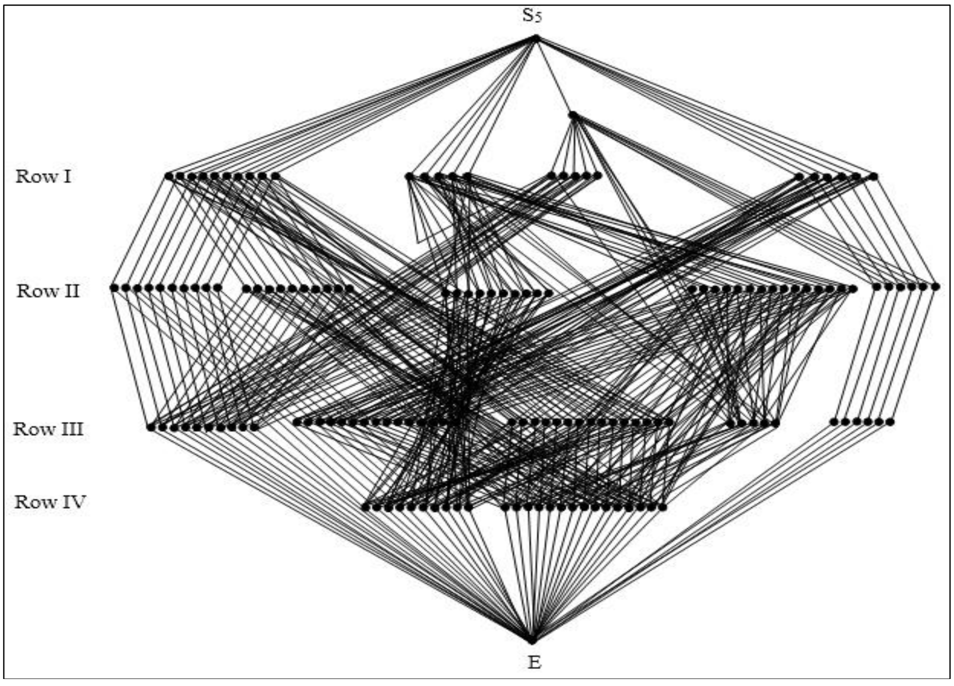

Fig 3

Row I: (Left to right): R_{129} to R_{138}, V_{150} to V_{154}, R_{139} to R_{143} and T_{144} to T_{149}
Row II: (Left to right): N_{78} to N_{87}, N_{98} to N_{107}, N_{88} to N_{97}, Q_{108} to Q_{122} and P_{123} to P_{128}
Row III: (Left to right): K_{27} to K_{36}, L_{37} to L_{51}, L_{52} to L_{66}, L_{67} to L_{71} and M_{72} to M_{77}
Row IV: (Left to right): H_2 to H_{11} and H_{12} to H_{26}.

6. Lattice Identities in the Subgroup Lattices of the Symmetric Group S_n

6.1 Consistency in Lattices of Subgroups of the Symmetric Group S_n

Lemma: 6.1

L (S_n) is consistent if n = 2 and n = 3 and L (S_n) is not consistent if n = 4 and n = 5.

Proof:

From the Fig.1 and Fig.2, it is easily seen that whenever j is a join-irreducible element in L(S_n), then $x \vee j$ is join-irreducible in the upper interval [x,1] for every $x \in L(S_n)$ when $n \leq 3$. So, L (S_n) is consistent, when $n \leq 3$.

When n = 4,

We choose the join irreducible element $K_2 \in L(S_4)$, and now $K_2 \vee K_5 = S_4 = M_{21} \vee N_{24}$ in the upper interval [K_5, S_4]. So L (S_4) is not consistent.

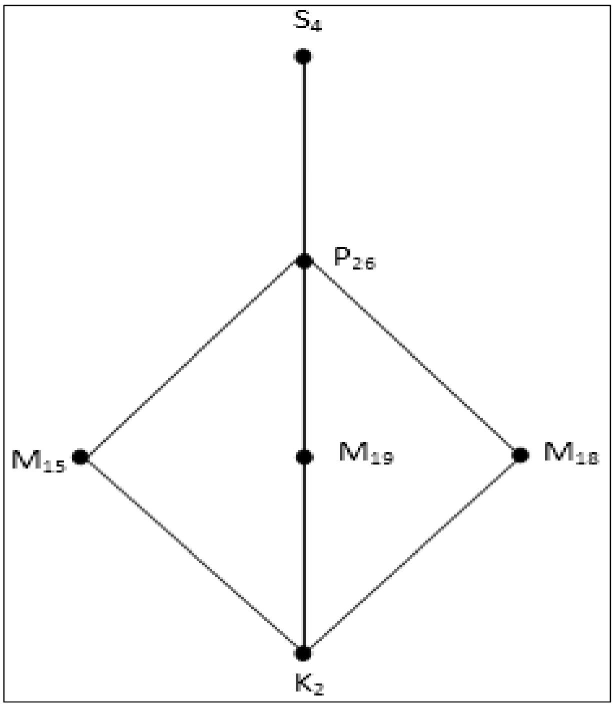

Fig 4

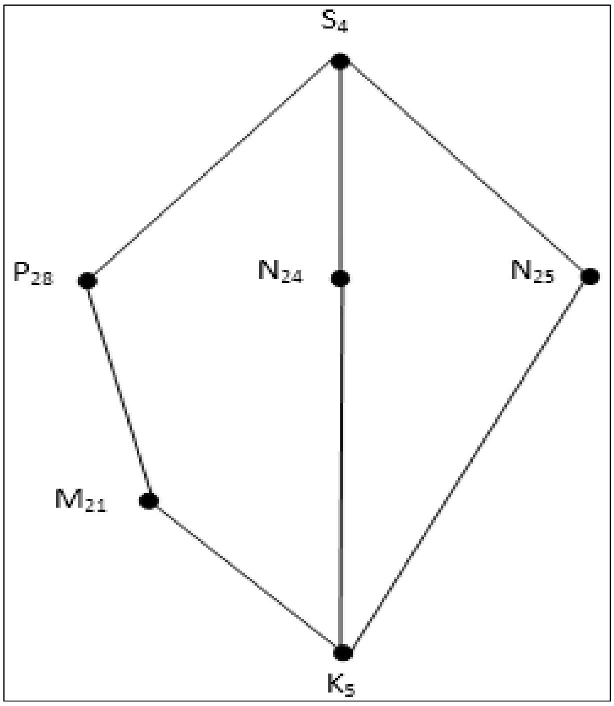

Fig 5

When n = 5,
Now we choose join irreducible element $L_{27} \in L(S_5)$, and $K_{27} \vee P_{97} = S_5 = T_{138} \vee V_{151}$ in the upper interval $[P_{97}, S_5]$. So $L(S_5)$ is not consistent.

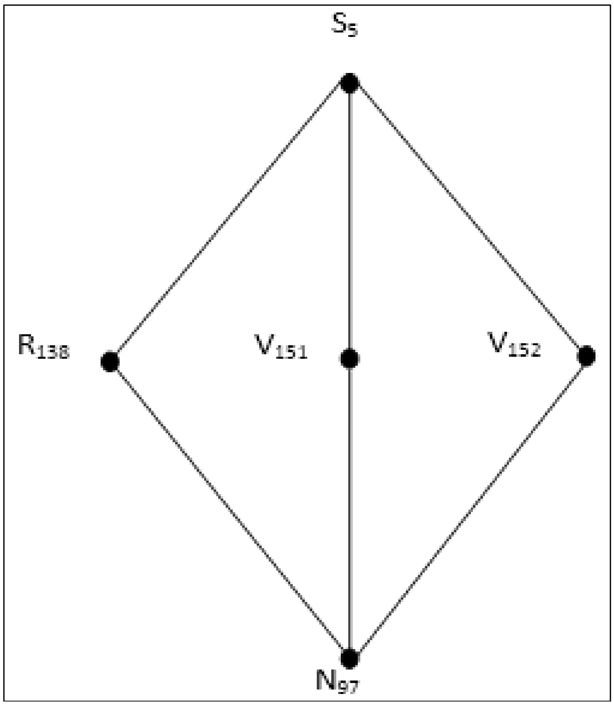

Fig 6

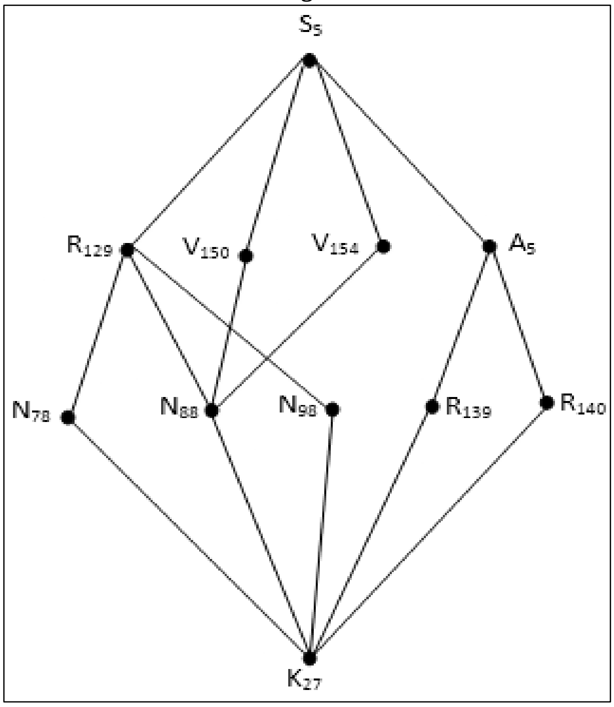

Fig 7

6.2 Modularity in Lattices of Subgroups of Symmetric Group S_n
Lemma: 6.2
$L(S_n)$ is modular if n = 2 and n = 3 and $L(S_n)$ is not modular if n = 4 and n = 5.
Proof:
Since $L(S_2)$ is isomorphic to B_1, the 2-element chain, it is modular. So that $L(S_2)$ is modular.
Now $L(S_3)$ is isomorphic to M_4, it is modular, so that $L(S_3)$ is modular.
But $L(S_4)$ is not modular, since we know that N_5 is not modular and it contains the sublattice $\{\{e\}, K_4, M_{18}, L_{11}, A_4\}$ which is isomorphic to the pentagon N_5. Hence $L(S_4)$ is not modular.
Also, $L(S_5)$ is not modular, since we know that N_5 is not modular and it contains the sublattice $\{\{e\}, K_{27}, H_{23}, L_{67}, R_{139}\}$ which is isomorphic to the pentagon N_5. Hence $L(S_5)$ is not modular.

6.3 Semi-Modularity in Lattices of Subgroups of Symmetric Group S_n

Lemma: 6.3

L (S_n) is semi-modular if n = 2 and n = 3 and L (S_n) is not semi-modular if n = 4 and n = 5

Proof:
We know that L (S_2) and L (S_3) are modular, so they are also semi- modular.
When n = 4,

In L (S_4), there are two elements K_5 and L_{11} such that $K_5 \wedge L_{11} = \{e\}$ which is covered by K_5 while $K_5 \vee L_{11} = S_4$ which does not cover L_{11}. Therefore, L (S_4) is not semi-modular.
When n = 5,

In L (S_5), there are two elements K_{27} and M_{77} such that $K_{27} \wedge M_{77} = \{e\}$ which is covered by K_{27} while $K_{27} \vee M_{77} = S_4$ which does not cover M_{77}. Therefore, L (S_5) is not semi-modular.

7. Conclusion

Consistency of L (S_n), Modularity of L (S_n), Semi-Modularity of L (S_n) for $n \leq 5$ were investigated in this section. But to decide the properties for $n > 5$ seems to be difficult.

8. References

1. Baumslag B. Theory and Problems of Group Theory: Schaum's outline Series, McGraw-Hill, New York, 1968.
2. Fraleigh JB. A First Course in Abstract Algebra, Addison-Wesley, London, 1992.
3. Gardiner CF. A First Course in Group Theory, Springer-Verlag, Berlin, 1997.
4. Gratzer G. Lattice Theory: Foundation. Birkhauser Veslag, Basel, 1998.
5. Herstien IN. Topics in Algebra, John Wiley and sons, New York, 1975.
6. Herstien IN. Abstract Algebra, Third Edition, Prentice Hall Inc. Upper Saddle River, 1996.
7. Jebaraj Thiraviam D. A Study on some special types of lattices, Ph.D thesis, Manonmaniam Sundaranar University, 2015.
8. Ralph Mckenzie N, George Mcnulty F, Walter Taylor F. Algebras Lattices, Varieties, Wadsworth and Brooks/Cole, Monterey California, 1987, I.
9. Roman S. Lattice and Ordered set, Springer, New York, 2008.
10. Samila D. Counting the Subgroups of the One Headed Group S_5 upto Automorphism, IOSR Journal of Mathematics. 2013; 8(3):87-93.
11. Scott WR. Group Theory, New York, Dover, 1987.
12. Sulaiman R. Subgroups Lattice of Symmetric Group S_4; International Journal of Algebra. 2012; 6(1):29-35.
13. Vethamonickam A, Jebaraj Thiraviam. On Lattices of Subgroups, Int. Journal of Mathematical Archiv. 2015; 6(9):1-11.