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Rachana Khandelwal
Department of Mathematics, Maharishi Arvind University, Rajasthan, India

Padama Kumawat
Department of Mathematics, Maharishi Arvind University, Rajasthan, India

Yogesh Khandelwal
Department of Mathematics, Maharishi Arvind University, Rajasthan, India

Correspondence
Rachana Khandelwal
Department of Mathematics, Maharishi Arvind University, Rajasthan, India

A study of natural transform based on decomposition method for solving non-linear ordinary differential equation

Rachana Khandelwal, Padama Kumawat and Yogesh Khandelwal

Abstract
Integral transform method is most useful technique for solving differential equation of Mathematics. The natural transform is derived from the Fourier Integral. In this research paper, natural transform converges to Laplace and Sumudu transform. For inverse natural transform, we have Bromwich contour integral and Heaviside's expansion formula. Here, we are using natural decomposition method (NDM) to find the exact solution of different types of non-linear ordinary differential equation which is based on natural transform method (NTM) and Adomian Decomposition Method (ADM).

Keywords: Ordinary differential equation, natural transform method (NTM), adomian decomposition method (ADM), natural decomposition method (NDM)

1. Introduction

1.1 Natural transform method: Z.H. khan *et al*. [2008] gave some properties and applications of natural transform "in the name of N transform ".natural transform method is associated to Laplace and sumudu transform method. Some basic property such as first shift, change of scale transform of derivative and integral of N transform. Application of Laplace transform in particular to the fractional partial differential equation. Sumudu transform was given by GK Watugala in1993 and inverse formula given by S Weerakoon.

Let the real function $\xi(x) > 0$ and $\xi(x) = 0$ for $x < 0$ is piecewise continuous, exponential order and define by

$$A = \{\xi(x): \exists\, M, \tau_1, \tau_2 > 0, |\, \xi(x) < Me^{\frac{t}{\tau_i}}, \text{if } x\ Ɛ\ (-1)^j * [0, \infty)\}; \quad (1)$$

The N transform of the function $\xi(x) > 0$ and $\xi(x) = 0$ for $x < 0$ is given by

$$N^+[\xi(x)] = \Phi[s, u] = \int_0^\infty e^{-sx} \xi(ux) dx, s > 0, u > 0; \quad (2)$$

Where s and u are the transform variables and following calculated formula is used for inverse of natural transform. The inverse of natural transform is given by

$$N^{-1}[\Phi(s, u)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\frac{sx}{u}} = \Phi(s, u) ds \quad (3)$$

1.2 Adomian decomposition method: Adomian decomposition method was introduced by George Adomian in 1981. The main advantage of this method is that we can apply this to all types of differential and integral equation, linear or non-linear, homogeneous non-homogeneous with constant or variable co-efficients.it reduce the size of computation work and maintains the high accuracy of the numerical solution. In ADM, a solution can be decomposed into an infinite series that converged rapidly into the exact solution. The linear and non-linear portion of the equation can be separated by Adomian decomposition method. The inversion of linear operator can be represented by the linear operator any given condition

is taken into consideration. The decomposition of a series is obtained by non-linear portion which is called Adomian polynomials. By the using Adomian polynomials we can find a solution in the form of a series which can be determined by the recursive relationship.
Let the equation is

$$H = h, \tag{4}$$

Where H is non-linear operator and h can be any function and value. Now H can be represented by N operator as follow, which is invertible

$$Ny + Ry + N^{*}y = h, \tag{5}$$

Here the liner operator 'R' represents the reminder of the linear portion and N* is a non-linear operator representing the non-linear terms in H. Now, apply inverse operator N^{-1} on both side of the equation of (5) we have following equation

$$N^{-1}Ny = N^{-1}h – N^{-1} - RyN^{-1}N^{*}y, \tag{6}$$

Now equation (6) becomes

$$y(x) = g(x) - N^{-1}Ry - N^{-1}N^{*}y \tag{7}$$

Where $g(x)$ represent the function obtained by integrating h and putting within the given boundary condition.
Let the unknown function $y(x)$ be an infinite series which is given by

$$y(x) = \sum_{n=0}^{\infty} y_n(x), 0 < n < \infty \tag{8}$$

Let take $y_0 = g(x)$ (9)

We can get other terms by using recursive relationship. The non –linear terms can be decomposed into a series that is called adomian polynomial, A_n.
Now, the non-linear term can be written as

$$N^{*}y(x) = \sum_{n=0}^{\infty} A_n \,. \tag{10}$$

To find the value of A_n, we introduce a grouping parameter. Hence,A_n is given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}N^{*}\mathrm{y}(\lambda)|_{\lambda=0}$$

Thus the recursive relation obtained

$$y_0 = g(x)$$

$$y_{n+1} = N^{-1}Ry_n + N^{-1}A_n.$$

The ADM gives a convergent series solution which is absolute and uniformly convergent.

1.3 Natural decomposition method: Ordinary differential equations solve by natural decomposition method, which is comprised from natural transform and adomian decomposition method. To prove the efficiency and the accuracy of ADM, we several application in the field of engineering and physics. The Adomian decomposition method (ADM) was proposed by George Adomian. This method is applicable to a wide class of linear and non- linear ordinary differential equation. To supply the exact and analytical approximate solutions, the natural decomposition method gives reliable results for the non-linear models. The given solution converges rapidly to the exact solution. The main objective of this paper is to enhance an efficient algorithm by natural decomposition method for non-linear ordinary differential equation. A solution in the form of convergent series is provided by the natural decomposition method. Let the form of general non-linear ordinary differential equation is given by

$$L(v) + U(v) + V(v) = g(x); \tag{11}$$

The initial condition is given by $v(0) = h(x)$ (12)
Where L, U and V are highest derivative, reminder of the differential operator and non-linear term respectively.
Let the differential operator $'L'$ is of the first order, then, taking N- transform of equation (11), we get

$$\frac{s}{u}F(s,u) - \frac{F(0)}{u} + N^{+}[U(v)] + N^{+}[V(v)] = N^{+}[g(x)] \tag{13}$$

Now, apply the initial condition in equation (13) with the help of equation (12), we have

$$F(s,u) = \frac{h(x)}{s} + \frac{u}{s}N^{+}[g(x)] - \frac{u}{s}N^{+}[V(v) + U(v)] \quad (14)$$

Now, taking N^{-1} transform in equation (14), we obtain:

$$v(x) = G(x) - N^{-1}\left\{\frac{u}{sN^{+}[V(v)+U(v)]}\right\} ; \quad (15)$$

Where $G(x)$ is the non-homogenous source term.
Let suppose an infinite series solution of the unknown function $v(x)$of the form:

$$v(x) = \sum_{n=0}^{\infty} v_n(x), \quad (16)$$

Using equation (16) in equation (15), we get

$$\sum_{n=0}^{\infty} v_n(x) = G(x) - N^{-1}\left[\frac{u}{sN^{+}\{U\sum_{n=0}^{\infty} v_n(x) + \sum_{n=0}^{\infty} A_n(x)\}}\right], \quad (17)$$

where the non-linear term is represented by $A_n(x)$ which is an Adomian polynomial.
Comparing both side of equation (17), we can easily use the recursive relationship as given below:

$$v_0(x)v_0(x) = G(x)$$

$$v_1(x) = -N^{-1}[u/sN^{+}\{Uv_0(x) + A_0(x)\}]$$

$$v_2(x) = -N^{-1}\left[\frac{u}{s}N^{+}\{Uv_1(x) + A_1(x)\}\right]$$

$$v_3(x) = -N^{-1}[u/sN^{+}\{Uv_2(x) + A_2(x)\}]$$

Generally, we can write this recursive relation as follows:

$$v_{n+1}(x) = -N^{-1}[u/sN^{+}\{Uv_n(x) + A_n(x)\}], \text{n} \geq 0 ; \quad (18)$$

Hence, the exact solution is given as follow:

$$v(x) = \sum_{n=0}^{\infty} v_n(x) \quad (19)$$

2. Application of Natural decomposition method: Our objective of this research paper to find out the solution of non-linear ordinary differential equation by using natural decomposition method.
Consider the first order non-linear ordinary differential equation of the form:

$$\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2 + z^2(x) = 1 + sinx \quad (20)$$

Subject to the initial condition $\text{z}(0) = 0, \text{z}'(0) = 1$ (21)

Taking the N- transform to both side of equation (20) we obtain

$$\frac{s^2}{u^2}z(s,u) - \frac{z(0)}{u^2} - \frac{z'(0)}{u} + N^{+}\left[\left(\frac{dz}{dx}\right)^2 + z^2(x)\right] = \frac{1}{s} + \frac{u}{s^2+u^2}$$ Substituting equation (21) into equation (22) we obtain: (22)

$$\frac{s^2}{u^2}z(s,u) + \frac{1}{u} = \frac{1}{s} + \frac{u}{s^2+u^2} - N^{+}\left[\left(\frac{dz}{dx}\right)^2 + z^2(x)\right]$$

$$\frac{s^2}{u^2}z(s,u) = \frac{1}{s} + \frac{u}{s^2+u^2} - \frac{1}{u} - N^{+}\left[\left(\frac{dz}{dx}\right)^2 + z^2(x)\right],$$

$$\frac{s^2}{u^2}z(s,u) = \frac{1}{s} - \frac{s^2}{u(s^2+u^2)} - N^{+}\left[\left(\frac{dz}{dx}\right)^2 + z^2(x)\right],$$

$$Z(s,u) = \frac{u^2}{s^3} - \frac{u^2}{u(s^2+u^2)} - N^{+}\left[\left(\frac{dz}{dx}\right)^2 + z^2(x)\right], \quad (23)$$

Taking the inverse N-transform of equation (23), we have

$$z(x) = \frac{x^2}{2!} - sinx - N^{-1}\left[\frac{u^2}{s^2}N^+\left[\left(\frac{dz}{dx}\right)^2 + z(x)\right]\right]$$

$$z(x) = \frac{x^2}{2!} - sinx - N^{-1}\left[\frac{u^2}{s^2}N^+\left[\left(\frac{dz}{dx}\right)^2 + z^2(x)\right]\right] \quad (24)$$

Now let an infinite series solution of unknown function $z(x)$,

Let $z(x) = \sum_0^\infty z_n(x)$; (25)

$$\sum_{n=0}^\infty z_n(x) = \frac{x^2}{2!} - sinx - N^{-1}\left[\frac{u^2}{s^2}N^+\sum_{n=0}^\infty[A_n + B_n]\right],$$

Where A_n and B_n are adomainpolynomails of the nonlinear term $\left(\frac{dz}{dx}\right)^2$and $z^2(x)$ respectively.
Comparing both sides by equation (24), we can drive the general recursive as follow

$$z_0(x) = \frac{x^2}{2!} - sinx$$

$$z_1(x) = -N^{-1}\left[\frac{u^2}{s^2}N^+[A_0 + B_0]\right]$$

$$z_2(x) = -N^{-1}\left[\frac{u^2}{s^2}N^+[A_1 + B_1]\right]$$

$$z_3(x) = -N^{-1}\left[\frac{u^2}{s^2}N^+[A_2 + B_2]\right]$$

In general form,

$$z_{n+1}(x) = -N^{-1}\left[\frac{u^2}{s^2}N^+[A_n + B_n]\right]$$

$$z_1(x) = -N^{-1}\left[\frac{u^2}{s^2}N^+[A_0 + B_0]\right]$$

$$z(x) = -N^{-1}\left[\frac{u^2}{s^2}N^+[(z_0')^2 + (z_0)^2]\right]$$

$$z_0 = \frac{x^2}{2!} - sinx \quad (26)$$

$$z'_0 = \frac{2x}{2!} - cosx \quad (27)$$

Putting equation (26) & (27) in equation (25) we get,

$$z_1(x) = -N^{-1}\left[\frac{u^2}{s^2}N^+\left[(x - cosx)^2 + \left(\frac{x^2}{2} - sinx\right)^2\right]\right]$$

$$z_1(x) = -N^{-1}\left[\frac{u^2}{s^2}N^+\left[x^2 + \cos^2 x - 2xcosx + \frac{x^4}{4} + \sin^2 x - 2x^2 sinx\right]\right]$$

$$z_1(x) = -N^{-1}\left[\frac{u^2}{s^2}N^{+1}(1)\right] + \ldots\ldots$$

$$z_1(x) = -N^{-1}\left[\frac{u^2}{s^3}\right]$$

$z_1(x) = -\frac{x^2}{2!} + \ldots..$

$z_1(x)$Satisfies the given differential equation which gives the exact solution of the term

$z(x) = \sin(x)$

Another problem we have as follow
$\frac{dz}{dx} + z(x) = 1 + x^2$ Subject to the initial condition $z(0) = 0$ (28)
Taking the N-transform to both side of equation (28) we obtain:

$$\frac{s}{u} z(s, u) - \frac{z(0)}{u} + N^{+1}[z(x)] = \frac{1}{s} + \frac{2u^2}{s^3}$$

$$\frac{s}{u} z(s, u) = \frac{1}{s} + \frac{2u^3}{s^3} - N^{+}[z(x)]$$

$$z(s, u) = \frac{u}{s^2} + \frac{2u^3}{s^4} - \left[\frac{u}{s} N^{+}[z(x)]\right] \quad (29)$$

Taking the inverse N-transform of equation (29) we have

$$z(x) = x + \frac{2x^3}{3!} - N^{-1}\left[\frac{u}{s} N^{+}[z(x)]\right] \quad (30)$$

Now let an infinite series solution of unknown function $z(x)$,

let $z(x) = \sum_{n=0}^{\infty} z_n(x)$ (31)

Using equation (31) we can re-write equation (30),

$$\sum_{n=0}^{\infty} z_n(x) = x + \frac{2x^2}{3!} - N^{-1}\left[\frac{u}{s} N^{+}\left[\sum_{n=0}^{\infty} A_n\right]\right]$$

Where A_n are the adomain polynomials of the nonlinear term $z(x)$.
Comparing both sides equation (17) we can drive the general relation as follow

$$z_0(x) = x + \frac{x^3}{3!}$$

$$z_1(x) = -N^{-1}\left[\frac{u}{s} N^{+}(A_0)\right]$$

$$z_2(x) = -N^{-1}\left[\frac{u}{s} N^{+}(A_1)\right]$$

$$z_3(x) = -N^{-1}\left[\frac{u}{s} N^{+}(A_2)\right]$$

General form of the recessive relation can be written as,

$$z_{n+1}(x) = -N^{-1}\left[\frac{u}{s} N^{+1}(A_n)\right] \quad (32)$$

$$z_1(x) = -N^{-1}\left[\frac{u}{s} N^{+}(A_0)\right]$$

$$z_1(x) = -N^{-1}\left[\frac{u}{s} N^{+}\big(z_0(x)\big)\right]$$

$$z_1(x) = -N^{-1}\left[\frac{u}{s} N^{+}\left(x + \frac{x^3}{3!}\right)\right]$$

$$z_1(x) = -N^{-1}\left[\frac{u}{s}\left(\frac{u}{s^2} + \frac{u^3}{s^4}\right)\right]$$

$$z_1(x) = -N^{-1}\left[\left(\frac{u^2}{s^3}+\frac{u^4}{s^5}\right)\right]$$

$$z_1(x) = -\left(\frac{x^2}{2!}+\frac{x^4}{4!}\right)$$

Thus, the exact solution is given by.

$$z(x) = z_0(x) + z_1(x) + \cdots$$

$$z(x) = x + \frac{x^3}{3!} - \left(\frac{x^2}{2!}+\frac{x^4}{4!}\right) + \cdots$$

$$z(x) = \left(x + \frac{x^3}{3!} + \cdots\right) - \left[1 - 1 + \left(\frac{x^2}{2!}+\frac{x^4}{4!}\right) + \cdots\right]$$

$$z(x) = \sin x - 1 + \left(1 + \frac{x^2}{2!}+\frac{x^4}{4!} + \cdots\right)$$

$$z(x) = \sin x - 1 + \cos x$$

3. Discussion and Conclusion

Natural decomposition method gives reliable results in supplying exact solutions. The mathematical approximate solutions converge rapidly to the exact solution. In this paper, the natural decomposition method was used for solving non-linear ordinary differential equation. We successfully obtained to find exact solution of all application. Our aim in the future to apply the natural decomposition method to other type of differential equation that arises in the other areas of science and technology.

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