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To study some applications of Banach fixed point theorem

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Abstract
In this paper, I analyze some important applications of Banach Fixed Point Theorem. This theorem has enormous applications to confirm existence and uniqueness of solution of an Initial value problem by Picard- Lindelöf Theorem, existence and uniqueness of solution of Differential Equation by using Newton’s method of successive approximation, existence and uniqueness of solution of an Integral Equation and existence and uniqueness of solution of Digital Image Processing. We will prove existence and uniqueness of solution of an Initial value problem, Differential Equation, Integral Equation, and Digital Image Processing by using Banach Fixed Point Theorem.

Keywords: Banach fixed point theorem, complete metric space, Cauchy sequence, integral equation, digital image

Introduction
Fixed point theory consists of many branches of Mathematics such as Mathematical Analysis, Functional Analysis, Topology and Computer Science. There are various applications of fixed point theory in Mathematics, Computer Science and Image processing. Banach Fixed point theorem plays a significant role for existence and uniqueness of some problems in Mathematics and Engineering.

Def: Metric Space: A metric space is a set X together with a function d (called a metric or "distance function") which assigns a real number d(x, y) to every pair $x, y \in X$ satisfying the properties:
1. $d(x, y) \geqq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) + d(y, z) \geqq d(x, z)$

Def: Cauchy Sequence: A sequence $\{A_n\}$ in a metric space X is called a Cauchy sequence if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$ we have $d(A_m, A_n) < \varepsilon$.

Def: Complete Metric Space: A complete metric space is a metric space in which every Cauchy sequence is convergent.

Def: Lipschitz Condition: Let X be a metric space equipped with a distance d. A map $f: X \to X$ is said to be Lipschitz continuous if there is $\lambda \geq 0$ such that $d(f(x_1), f(x_2)) \leq \lambda\, d(x_1, x_2)$, $\forall x_1, x_2 \in X$. The smallest λ for which the above inequality holds is the Lipschitz constant off. If $\lambda \leq 1$ f is said to be non-expansive, if $\lambda < 1$, f is said to be a contraction.

Banach fixed point theorem: Let f be a contraction on a complete metric space X. Then f has a unique fixed point $x \in X$.

Applications of Banach Fixed point Theorem

1. Picard-Lindelöf Theorem (Existence and Uniqueness of solution of Initial value problem)

Let a function f is defined as f: I x $R^n \to R^n$, where I is a closed interval [a, b].

Let x′(t) = f(x, x(t)) be an associated ordinary differential equation. If f is Lipschitz continuous in the second argument, then this ODE possesses a unique solution on [a, a+ε] for each possible initial value x(0) = $x_0 \in R^n$ where $\varepsilon < \frac{1}{L}$, where L is Lipschitz constant of the second argument off.

Proof: By the fundamental theorem of Calculus, the ordinary differential equation

x′(t) = f(x, x(t)), t $\in$ [a, a+ε] with initial condition x(0) = x_0 can be written as

x(t) = $x_0 + \int_a^t f(s, x(s))ds$, $\forall t \in$ [a, a+ε].

This implies that the function x(t) is a fixed point of T[a, a+ε]⟶ [a, a+ε].

T(x)(t) = $x_0 + \int_a^t f(s, x(s))ds$.

Now T satisfies Lipschitz condition as follows:

$$\| T(x)(t) - T(y)(t) \| = \|\int_a^t f(s, x(s))ds - \int_a^t f(s, y(s))ds \|$$
$$\leq \int_a^t ||f(s, x(s) - f(s, y(s)\| ds$$
$$\leq \int_a^t L||x(s) - y(s)||ds$$
$$\leq (t - a) L||x - y||_\infty \leq \varepsilon L||x - y||_\infty$$

Where norm of [a, a+ε] is the supremum norm. If $\varepsilon < \frac{1}{L}$, then T is a contraction.

Hence, Banach Fixed point Theorem confirms existence and uniqueness of a fixed point.

2. Newton's method of successive approximation:

Newton's method of successive approximation is used to find roots of a differential equation f(x) by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

By writing the above equation as g(x) = x - $\frac{f(x)}{f'(x)}$, it becomes equivalent to x_{n+1} = g(x_n).

If this iteration converges to a fixed point x of g, then

x = g(x)

$\Rightarrow$ x = x - $\frac{f(x)}{f'(x)}$

$\Rightarrow \frac{f(x)}{f'(x)} = 0$

$\Rightarrow$f(x) = 0.

Hence x is a root off.

Therefore it concludes that Newton's method of successive approximation demonstrates linear convergence in the context of Banach Fixed point Theorem.

3. Existence and Uniqueness of Solutions of an Integral Equation

Def: Integral Equation: If an unknown function appears in integral sign, then this type of equation is called an Integral equation. The general type of linear integral equation is as follows:

y(x) = F(x) + $\lambda \int_a^b K(x,t)y(t)dt$ where F(x) and K(x, t) are known functions and y(x) is unknown. K(x, t) is called kernel of the integral equation.

If a and b are constants, then the equation is called Fredholm Integral equation.

If a is constant and b is variable, then the equation is called Volterra Integral equation.

Theorem: Let K(x, t) is a measurable continuous function on L^2[a, b]xL^2[a, b] with $\iint_a^b |K(x,t)|^2 dx\, dt < \infty$ and y, F $\in L^2$[a, b], then the Fredholm Integral equation has a unique solution if |λ| ||K(x, t)|| < 1.

Proof: It is given that K(x, t) is a measurable continuous function on L^2[a, b]xL^2[a, b] with $\iint_a^b |K(x,t)|^2 dx\, dt < \infty$ and y, F $\in L^2$[a, b], we have to show that $\int_a^b K(x,t)y(t)dt \in L^2$[a, b].

By Schwarz's inequality,

$$|\int_a^b K(x,t)y(t)dt| \leq \int_a^b |K(x,t)y(t)|dt$$
$$\leq (\int_a^b |K(x,t)|^2\, dt)^{1/2} (\int_a^b |y(t)|^2\, dt)^{1/2})$$

Squaring both sides, we get,

$$| \int_a^b K(x,t)y(t)dt|^2 \leq (\int_a^b |K(x,t)|^2\, dt)\, (\int_a^b |y(t)|^2\, dt)$$
$$\Rightarrow \int_a^b | \int_a^b K(x,t)y(t)dt|^2 dx \leq \int_a^b (\int_a^b |K(x,t)|^2\, dt)\, (\int_a^b |y(t)|^2\, dt) dx$$
$$\Rightarrow \int_a^b | \int_a^b K(x,t)y(t)dt|^2 dx \leq (\int_a^b \int_a^b |K(x,t)|^2 dt\, dx)\, (\int_a^b |y(t)|^2\, dt)$$

Since, $\int_a^b \int_a^b |K(x,t)|^2$ dt dx $< \infty$ and $\int_a^b |y(t)|^2\, dt < \infty$, we have $\int_a^b K(x,t)y(t)dt < \infty$.

Therefore

$\int_a^b K(x,t)y(t)dt \in L^2$[a, b].

Let T be an operator defined as T: L^2[a, b]$\to L^2$[a, b] by T(y) = y, where d standard metric in L^2. For y_1, $y_2 \in L^2$[a, b], we get,

$$d(Ty_1, Ty_2) = \left(\int_a^b |y_1 - y_2|^2\, dx\right)^{1/2}$$

$$= |\lambda| \left(\int_a^b \left| \int_a^b K(x,t)y_1(t)dt - \int_a^b K(x,t)y_2(t)dt\right|^2 dx\right)^{1/2}$$

$$= |\lambda| \left(\int_a^b \left| \int_a^b K(x,t)(y_1(t) - y_2(t))dt\right|^2 dx\right)^{1/2}$$

$$\le |\lambda| \left(\int_a^b \left| \int_a^b K(x,t)|(y_1(t) - y_2(t)|dt\right|^2 dx\right)^{1/2}$$

$$\Rightarrow d(Ty_1, Ty_2) \le |\lambda| \left(\left(\int_a^b \int_a^b |K(x,t)|^2\, dt\right) \left(\int_a^b (y_1(t) - y_2(t)|^2\right) dx\right)^{1/2}$$

$$= |\lambda| \left(\int_a^b \int_a^b |K(x,t)|^2\, dt\, dx\right)^{1/2} d(y_1, y_2)$$

$$\Rightarrow d(Ty_1, Ty_2) \le |\lambda| \|K(x, t)\|\, |\, d(y_1, y_2)|.$$

Since $|\lambda| < \|K(x,t)\|^{-1}$, T is a contraction mapping.

Hence, by the Banach Fixed point Theorem, it possesses unique solution.

4. Existence and Uniqueness of Fixed Point in Digital Metric Space

Def: Cauchy sequence of a complete metric space: A sequence $\{x_n\}$ of points of a digital metric space (X, d, κ) is a Cauchy sequence if for all ε > 0, there exists α ϵ N such that for all n, m > α, then $d(x_n, x_m) < \varepsilon$ [2].

Def: Complete digital metric space: A digital metric space (X, d, κ) is a complete digital metric space if any Cauchy sequence $\{x_n\}$ of points of (X, d, κ) converges to a point a of (X, d, κ).

Let (X, d, κ) be a complete digital metric space which has a usual Euclidean metric in Z^n.

Let f: X → X be a digital contraction map.

Then there exists a unique c ϵ X such that f(c) = c.

Proof. Let a, b ϵ X are fixed points off.

⇒ d(a, b) = d(f(a), f(b)) ≤ λ d(a, b)

⇒ (1 − λ)d(a, b) ≤ 0

⇒ a = b.

Let x_0 be any point of X.

Let us consider the iterate sequence f(x_n) = x_{n+1}. Using induction on n,

We get $d(x_{n+1}, x_n) \le \lambda d(x_n, x_{n-1}) \le \ldots \le \lambda^n d(f(x_0), x_0)$.

For natural numbers n ϵ N and m ≥ 1, we conclude that

$d(x_{n+m}, x_n) \le d(x_{n+m}, x_{n+m-1}) + \ldots + d(x_{n+1}, x_n)$

$\le [\lambda^{n+m} + \ldots + \lambda^n] d(f(x_0), x_0)$

$\le \frac{\lambda^n}{1-\lambda} d(f(x_0), x_0)$.

⇒ x_n is a Cauchy sequence. Then there must be a limit point of $x_n$because (X, d, κ) is digital complete metric space.

Let c be the limit of x_n. From the (κ, κ)-continuity of f, we get,

$f(c) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = c$.

Therefore, f has a unique fixed point.

Conclusion

In this paper, we have introduced the concepts of different applications of Banach Fixed Point Theorem. There are numerous applications of Banach Fixed Point Theorem in various branches of Mathematics and Science. We have given different theorems that confirm existence and uniqueness of solution in different areas of mathematical problems. Our results can open the door for further research activity in the field for other areas or other iterative processes.

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