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Bayesian approximation of the reliability function of the three parameter generalized compound Rayleigh distribution under the squared error loss function

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Abstract

Life testing and reliability theory is a very important branch of Engineering Statistics. Over the years reliability estimation methods based on sampling theory have been found to be extremely useful for a wide variety of problems. There are, however, many instances in which the classical methods have been found to be less than satisfactory. The main objective of the paper is to obtain the Bayes estimate of reliability function of Generalized Compound Rayleigh distribution (re-parameterized) assuming all the parameters unknown under the squared error loss function. Then by using Lindley approximation procedure we have obtained the Approximate Bayes estimate of reliability function of the Generalized Compound Rayleigh distribution under SELF. We also analyze the sensitivity of Approximate Bayes Estimate of model and presented a numerical study to illustrate the above technique on generated observations. The comparison is done by using R-programming.

Keywords: Bayesian approximation, reliability function, parameter generalized, Rayleigh distribution, squared error loss function

1. Introduction

Reliability engineering concerned with the ability of a system or component to perform its required functions under stated conditions for a specified time. When a manufacturer floats a new product in the market, he would like his customer to have some information the reliability of product with regards to its performance. In this process he would like to know the answers of the questions that what is the probability that the item will fail in the time interval $[t, t + \Delta t]$ given that it was working at the time t , or when customer purchases some items e.g. refrigerator, an air conditioner or a LED bulb etc., he thinks about the life of the item means some prior knowledge about its mean life and variance etc. Since the failure of the item may occur at any time and is affected by random causes, it may assumed the failure time (T) is a random variable.

Over the years reliability estimation methods based on sampling theory have been found to be extremely useful for a wide variety of problems. There are, however, many instances in which the classical methods have been found to be less than satisfactory.

Increasing instance of cost-effectiveness in reliability testing programs has had a decreasing effect on the case for consideration of sampling theory methods. If one were to consider only the use of sampling theory methods, one would be extremely limited, because of cost and time constraints, to a very small number of samples. Such a limited sample size would result in either very low level of confidence in the reliability estimate or imprecise estimates. Grohowski, Hausman, and Lamberson (1976) [5]. For the reason such as these, clauses in military and government reliability contracts permit other methods such as Bayesian procedures, with the onus of justification put on the producer.

There are several benefits in using Bayesian methods in reliability. First of all, it is important to recognize that all the inferential theories whether the sampling theory, Bayesian theory likelihood or otherwise, requires some degree of subjectivity in their use. And only Bayesian method provides a satisfactory way of explicitly introducing and organizing assumptions regarding prior knowledge or ignorance. Evans (1969) the distribution function of failure time is given by $F(t)=[T < t <]$ which is the probability that the item will survive till time 't', and

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Let $f(t)$ be the probability density function of the random variable 'T' (Sinha, S.K, 1986) ^[12]. Consider the function

$$h(t) = \lim_{\Delta t \rightarrow 0} P \frac{[t \leq T \leq t + \Delta t | T > t]}{\Delta t}$$

$$\frac{f(t)}{1-F(t)} = \frac{f(t)}{R(t)} \tag{1.1}$$

The function $h(t)$ is called the hazard rate or instantaneous failure rate. Let

$$R(t) = 1-F(t) \tag{1.2}$$

Denote the failure free operation till time 't', then it is quite evident that the stochastic behavior of time may be studied through either of the four functions, $f(t)$, $F(t)$, $R(t)$ or $h(t)$. Here,
Reliability Function

$$R(t) = 1 - F(t) \tag{1.3}$$

Failure Rate density Function

$$F(t) = d \frac{F(t)}{dt} \tag{1.4}$$

Hazard Rate Function

$$H(t) = \frac{d}{dt} \log\{1 - F(t)\} \tag{1.5}$$

$$F(t) = h(t) \exp\left\{-\int_0^t h(x)dx\right\} \tag{1.6}$$

Now, since Failure Rate distribution function

$$F(t) = 1 - \exp\left\{-\int_0^t h(x)dx\right\} \tag{1.7}$$

Various life time distributions may be classified according to behavior of the function $h(t)$. If $h(t)$ is increasing, decreasing or constant function of 't' it has the direct physical meaning that more the chronological age of the item more, less or equal are the chances of failure of an item. (Sinha, S.K, 1986) ^[12].

The Bayesian inference procedures have been developed generally under squared error loss function

$$L(\hat{R}(t), R(t)) = (\hat{R}(t) - R(t))^2 \tag{1.8}$$

The Generalized Compound Rayleigh Distribution is a special case of the three-parameter Burr type XII distribution. Considered it as a gamma mixture of Rayleigh distribution and obtained the compound Rayleigh model with unimodal hazard function. This unimodal hazard function is generalized and a flexible parametric model is thus constructed, which embeds the compound Rayleigh model, by adding shape parameter. Bain and Engelhardt (1991) ^[1] studied this distribution (also known as the Compound Weibull distribution from a Poisson perspective). The Generalized Compound Rayleigh Distribution is a special case of the three-parameter Burr type XII distribution with probability density function (p.d.f.) re-parameterized γ as $\frac{1}{\gamma}$

$$f(x; \alpha, \beta, \gamma) = \frac{\alpha}{\gamma} \beta^{\frac{1}{\gamma}} x^{\alpha-1} (\beta + x^\alpha)^{-(\gamma+1)}; x, \alpha, \beta, \gamma > 0 \tag{1.9}$$

With Probability Distribution Function

$$F(x) = 1 - (1 - \beta x^\alpha)^{\frac{1}{\gamma}}; x, \alpha, \beta, \gamma > 0 \tag{1.10}$$

Reliability function

$$R(t) = \left(\frac{\beta}{\beta+t^\alpha}\right)^{\frac{1}{\gamma}} \tag{1.11}$$

Hazard rate function

$$H(t) = \frac{\alpha t^{\alpha-1}}{\gamma \beta + t^\alpha} \tag{1.12}$$

2. Estimators

Let $x_1 \leq x_2 \leq \dots \leq x_n$ be the n failures in complete sample case. The likelihood function is given by

$$L(\underline{x} | \alpha, \beta, \gamma) = \prod_{j=1}^n f(x_j, \alpha, \beta, \gamma) \tag{2.1}$$

$$\text{Log } L = n \log \alpha + \frac{n}{\gamma} \log \beta - n \log \gamma + (\alpha - 1) \sum_{j=1}^n \log x_j - \left(\frac{1}{\gamma} + 1\right) \sum_{j=1}^n \log(\beta + x_j^\alpha) \tag{2.2}$$

And differentiation of equation (2.2) with respect to α, β and γ yields respectively, we get

$$\frac{\partial \text{Log } L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{j=1}^n \log x_j - \left(\frac{1}{\gamma} + 1\right) \sum_{j=1}^n \frac{x_j^\alpha \log x_j}{(\beta + x_j^\alpha)} \tag{2.3}$$

$$\frac{\partial \text{Log } L}{\partial \beta} = \frac{n}{\gamma \beta} - \left(\frac{1}{\gamma} + 1\right) \sum_{j=1}^n \frac{1}{(\beta + x_j^\alpha)} \tag{2.4}$$

$$\frac{\partial \text{Log } L}{\partial \gamma} = \frac{-n \log \beta}{\gamma^2} - \frac{n}{\gamma} + \frac{n}{\gamma^2} \sum_{j=1}^n \log(\beta + x_j^\alpha) \tag{2.5}$$

Setting the expressions for the derivatives in (2.3) and (2.5) equal to zero and solving α, β and γ yield. The maximum likelihood estimators (MLE) of the parameters namely $\hat{\alpha}_{MLE}, \hat{\beta}_{MLE}$ and $\hat{\gamma}_{MLE}$.

However, no closed form solutions exist in this case the elimination of γ in $\frac{\partial \text{Log } L}{\partial \beta}$ and $\frac{\partial \text{Log } L}{\partial \gamma}$ and in $\frac{\partial \text{Log } L}{\partial \alpha}$ and $\frac{\partial \text{Log } L}{\partial \gamma}$ yield a set of equations in terms of β and γ .

$$\frac{\sum_{j=1}^n \frac{1}{\beta + x_j^\alpha}}{\sum_{j=1}^n \frac{x_j^\alpha}{\beta + x_j^\alpha}} - \frac{n}{\sum_{j=1}^n \log \left[1 + \frac{x_j^\alpha}{\beta} \right]} = 0 \tag{2.6}$$

$$\frac{n}{\alpha} + \sum_{j=1}^n \log x_j - \sum_{j=1}^n \frac{x_j^\alpha \log x_j}{\beta + x_j^\alpha} - \frac{n \sum_{j=1}^n \frac{x_j^\alpha \log x_j}{\beta + x_j^\alpha}}{\sum_{j=1}^n \log \left[1 + \frac{x_j^\alpha}{\beta} \right]} = 0 \tag{2.7}$$

Respectively. Applying the Newton-Raphson method $\hat{\alpha}_{MLE}$ and $\hat{\beta}_{MLE}$ can be derived and then from them $\hat{\gamma}_{MLE}$ can be obtained. The MLE's of $R(t)$ are given respectively by replacing α, β and γ by $\hat{\alpha}_{ML}, \hat{\beta}_{ML}$ and $\hat{\gamma}_{ML}$ on the value obtained by solving the equation (1.3).

Bayes estimated for γ with known parameter α and β under SELF

If $\hat{\alpha}$ and $\hat{\beta}$ is known we assume $\gamma(a, b)$ as conjugate prior for γ as

$$g(\gamma | \underline{x}) = \frac{b^a}{\Gamma(a)} \left(\frac{1}{\gamma}\right)^{(a+1)} e^{-b\gamma}; (a, b, \gamma) > 0 \tag{2.8}$$

Combining the likelihood function equation (2.1) and prior density equation (2.8), we obtain the posterior density of γ in the form;

$$h(\gamma | \underline{x}) = \frac{\frac{\alpha^n}{\gamma^n} \beta^{\frac{n}{\gamma}} \prod_{j=1}^n x_j^{(\alpha-1)} \prod_{j=1}^n (\beta + x_j^\alpha)^{-\left(\frac{1}{\gamma}+1\right)} \frac{b^a}{\Gamma(a)} \left(\frac{1}{\gamma}\right)^{(a+1)} e^{-b/\gamma}}{\int_0^\infty \frac{\alpha^n}{\gamma^n} \beta^{\frac{n}{\gamma}} \prod_{j=1}^n x_j^{(\alpha-1)} \prod_{j=1}^n (\beta + x_j^\alpha)^{-\left(\frac{1}{\gamma}+1\right)} \frac{b^a}{\Gamma(a)} \left(\frac{1}{\gamma}\right)^{(a+1)} e^{-b/\gamma} d\gamma} \tag{2.9}$$

$$h(\gamma | \underline{x}) = \frac{\gamma^{-(n+a-1)} e^{-\frac{(b+T)}{\gamma}} (b+T)^{(n+a)}}{\Gamma(n+a)} \tag{2.9}$$

Bayes Estimate of Reliability function of Generalized Compound Rayleigh distribution

The posterior mean of the Reliability function of the Generalized Compound Rayleigh distribution is obtained by substituting

$R = \left(\frac{\beta}{\beta + t^\alpha}\right)^{\frac{1}{\gamma}}$ in equation (2.9) given below

$$h(\gamma | \underline{x}) = \frac{\gamma^{-(n+a-1)} e^{-\frac{(b+T)}{\gamma}} (b+T)^{(n+a)}}{\Gamma(n+a)}$$

$$dR(t) = \frac{(b+T)^{(n+a)}}{\Gamma(n+a)} \frac{(-\log R)^{(n+a-1)}}{C^{(n+a-1)}} R^{\left(\frac{b+T}{C}-1\right)} C dR; \tag{2.10}$$

$$h(R(t)) = \frac{\left(\frac{b+t}{c}\right)^{(n+a)}}{\Gamma(n+a)} (-\log R)^{(n+a-1)} R^{\left(\frac{b+T}{c}-1\right)}; 0 < R < 1 \tag{2.11}$$

This is the posterior mean of the reliability function of Generalized Compound Rayleigh distribution.

Bayes Estimate of R under Squared Error Loss Function (SELF)

The Bayes estimator of reliability function of generalized Compound Rayleigh Distribution under squared error loss function is given by

$$\hat{R}_{BS} = \int_0^1 R \frac{\left(\frac{b+t}{c}\right)^{(n+a)}}{\Gamma(n+a)} (-\log R)^{(n+a-1)} R^{\left(\frac{b+T}{c}-1\right)} dR ; 0 < R < 1 \tag{2.13}$$

$$\hat{R}_{BS} = \left(\frac{b+T}{b+T+c}\right)^{(n+a)} ; \tag{2.14}$$

3. Bayes Estimators with unknown parameters α, β and γ

The joint prior density of α, β, γ is given by

$$\begin{aligned} G(\alpha, \beta, \gamma) &= g_1(\alpha)g_2(\beta)g_3\left(\frac{\gamma}{\beta}\right) \\ &= \frac{c}{\delta\Gamma\xi} \beta^{-\xi} \gamma^{(\xi+1)} \exp\left[-\left(\frac{\gamma}{\beta} + \frac{\beta}{\delta}\right)\right] \end{aligned} \tag{3.1}$$

Where

$$g_1(\alpha) = c \tag{3.2}$$

$$g_2(\beta) = \frac{1}{\delta} e^{-\frac{\beta}{\delta}} \tag{3.3}$$

$$g_3(\gamma) = \frac{1}{\Gamma\xi} \beta^{-\xi} \gamma^{(\xi+1)} e^{-\frac{1}{\beta\gamma}} \tag{3.4}$$

The joint posterior with likelihood equation (2.1) and (3.1) is given by

$$h(\alpha, \beta, \gamma | \underline{x}) = \frac{\beta^{-\xi} \gamma^{(\xi+1)} \exp\left[-\left(\frac{\gamma}{\beta} + \frac{\beta}{\delta}\right)\right] L(\underline{x} | \alpha, \beta, \gamma)}{\int_{\alpha} \int_{\beta} \int_{\gamma} \beta^{-\xi} \gamma^{(\xi+1)} \exp\left[-\left(\frac{\gamma}{\beta} + \frac{\beta}{\delta}\right)\right] L(\underline{x} | \alpha, \beta, \gamma) d\alpha d\beta d\gamma} \tag{3.5}$$

Approximate Bayes estimators

$$U(\theta) = U(\alpha, \beta, \gamma) \tag{3.6}$$

$$U_{BS} = E(U | \underline{x}) = \frac{\int_{\alpha} \int_{\beta} \int_{\gamma} U(\alpha, \beta, \gamma) G^*(\alpha, \beta, \gamma) d\alpha d\beta d\gamma}{\int_{\alpha} \int_{\beta} \int_{\gamma} G^*(\alpha, \beta, \gamma) d\alpha d\beta d\gamma} \tag{3.7}$$

Lindley Approximation Procedures

The ratio of integrals in equation (3.7) does not seem to take a closed form so we must consider the Lindley approximation procedure as

$$E(\mu(\theta, p) | \underline{x}) = \frac{\int \mu(\theta).e^{(l(\theta)+\rho(\theta))} d\theta}{\int e^{(l(\theta)+\rho(\theta)).} d\theta} \tag{3.7a}$$

Lindley developed approximate procedure for evaluation of posterior expectation of $\mu(\theta)$. Several other authors have used this technique to obtain Bayes estimators (see Sinha (1986) [12], Sinha and sloan (1988) [13], Soliman (2001)) [15]. The posterior expectation of Lindley approximation procedure to evaluate of $\mu(\theta)$ in equation (3.7a) under SELF, where where $\rho(\theta) = \log g(\theta)$, and $g(\theta)$ is an arbitrary function of θ and $l(\theta)$ is the logarithm likelihood function (Lindley (1980)) [6].

$$\begin{aligned} E(U(\alpha, \beta, \gamma | \underline{x})) &= U(\theta) + \frac{1}{2} [A(U_1 \sigma_{11} + U_2 \sigma_{12} + U_3 \sigma_{13}) + B(U_1 \sigma_{21} + U_2 \sigma_{22} + U_3 \sigma_{23}) + P(U_1 \sigma_{31} + U_2 \sigma_{32} + U_3 \sigma_{33})] + \\ & (U_1 a_1 + U_2 a_2 + U_3 a_3 + a_4 + a_5) + 0\left(\frac{1}{n^2}\right) \end{aligned} \tag{3.8}$$

$$E(U | \underline{x}) = U + \varphi_1 + \varphi_2 \tag{3.9}$$

Where

$$\varphi_1 = u_1 a_1 + u_2 a_2 + u_3 a_3 + a_4 + a_5 \tag{3.10}$$

$$\varphi_2 = \frac{1}{2} [(A\sigma_{11} + B\sigma_{21} + P\sigma_{31}) \cdot U_1 + (A\sigma_{12} + B\sigma_{22} + P\sigma_{32}) \cdot U_2 + (A\sigma_{13} + B\sigma_{23} + P\sigma_{33}) U_3] \tag{3.11}$$

Evaluated at the MLE $\hat{U} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ where

$$a_1 = \rho_1 \sigma_{11} + \rho_2 \sigma_{12} + \rho_3 \sigma_{13} \tag{3.12}$$

$$a_2 = \rho_1 \sigma_{21} + \rho_2 \sigma_{22} + \rho_3 \sigma_{23} \tag{3.13}$$

$$a_3 = \rho_1 \sigma_{31} + \rho_2 \sigma_{32} + \rho_3 \sigma_{33} \tag{3.14}$$

$$a_4 = U_{12} \sigma_{12} + U_{13} \sigma_{13} + U_{23} \tag{3.15}$$

$$a_5 = \frac{1}{2} (U_{11} \sigma_{11} + U_{22} \sigma_{22} + U_{33} \sigma_{33}) \tag{3.16}$$

And

$$A = [\sigma_{11} l_{111} + 2\sigma_{12} l_{121} + 2\sigma_{13} l_{131} + 2\sigma_{23} l_{231} + \sigma_{22} l_{221} + \sigma_{33} l_{331}] \tag{3.17}$$

$$B = [\sigma_{11} l_{112} + 2\sigma_{12} l_{122} + 2\sigma_{13} l_{132} + 2\sigma_{23} l_{232} + \sigma_{22} l_{222} + \sigma_{33} l_{332}] \tag{3.18}$$

$$P = [\sigma_{11} l_{113} + 2\sigma_{13} l_{133} + 2\sigma_{12} l_{123} + 2\sigma_{23} l_{233} + \sigma_{22} l_{223} + \sigma_{33} l_{333}] \tag{3.19}$$

To apply Lindley approximation on equation (3.7), we first obtain

$$\sigma_{ij} = [-l_{ijk}]^{-1} i, j, k = 1, 2, 3$$

Likelihood function from equation (2.1) is

$$L = \frac{\alpha^n}{\gamma^n} \beta^\gamma \prod_{j=1}^n x_j^{\alpha-1} \prod_{j=1}^n (\beta + x_j^\alpha)^{-\left(\frac{1}{\gamma}+1\right)}; (x, \alpha, \beta, \gamma > 0)$$

$$\text{Log } L = n \log \alpha + n \log \gamma + \frac{n}{\gamma} \log \beta + (\alpha - 1) \sum_{j=1}^n x_j - \left(\frac{1}{\gamma} + 1\right) \sum_{j=1}^n \frac{x_j^\alpha \log x_j}{(\beta + x_j^\alpha)}$$

$$l_1 = \frac{n}{\alpha} + \sum_{j=1}^n \log x_j - \left(\frac{1}{\gamma} + 1\right) \omega_{11} \text{ Where } \omega_{11} = \sum_{j=1}^n \frac{x_j^\alpha \log x_j}{(\beta + x_j^\alpha)} \tag{3.20}$$

$$l_2 = \frac{n}{\gamma \beta} - \left(\frac{1}{\gamma} + 1\right) \delta_{11} \text{ Where } \delta_{11} = \sum_{j=1}^n \frac{1}{(\beta + x_j^\alpha)} \tag{3.21}$$

$$l_3 = -\frac{n}{\gamma} - \frac{n \log \beta}{\gamma^2} + \frac{1}{\gamma^2} \delta_{10} \text{ Where } \delta_{10} = \sum_{j=1}^n \log (\beta + x_j^\alpha) \tag{3.22}$$

$$l_{11} = \frac{-n}{\alpha^2} - \left(\frac{1}{\gamma} + 1\right) \beta \omega_{122} \text{ Where } \omega_{122} = \sum_{j=1}^n \frac{x_j^\alpha (\log x_j)^2}{(\beta + x_j^\alpha)^2} \tag{3.23}$$

$$l_{12} = \left(\frac{1}{\gamma} + 1\right) \omega_{14} = l_{21} \text{ where } \omega_{14} = \sum_{j=1}^n \frac{x_j^\alpha \log x_j}{(\beta + x_j^\alpha)^2} \tag{3.24}$$

$$l_{13} = \frac{\omega_{11}}{\gamma^2} = l_{31} \text{ Where } \omega_{11} = \sum_{j=1}^n \frac{x_j^\alpha \log x_j}{(\beta + x_j^\alpha)} \tag{3.25}$$

$$l_{22} = -\frac{n}{\gamma \beta^2} + \left(\frac{1}{\gamma} + 1\right) \delta_{12} \tag{3.26}$$

$$l_{23} = -\frac{1}{\gamma^2} \left(\frac{n}{\beta} - \delta_{11}\right) = l_{32} \text{ Where } \delta_{11} = \sum_{j=1}^n \frac{1}{(\beta + x_j^\alpha)} \tag{3.27}$$

$$l_{33} = \frac{n}{\gamma^2} + \frac{2n \log \beta}{\gamma^3} - \frac{2}{\gamma^3} \delta_{10} \text{ Where } \delta_{10} = \sum_{j=1}^n \log (\beta + x_j^\alpha) \tag{3.28}$$

$$l_{111} = \frac{2n}{\alpha^3} - \left(\frac{1}{\gamma} + 1\right) \omega_{133} \text{ Where } \omega_{133} = \sum_{j=1}^n \frac{x_j^\alpha (\beta - x_j^\alpha) (\log x_j)^3}{(\beta + x_j^\alpha)^3} \tag{3.29}$$

$$l_{222} = \frac{2n}{\gamma\beta^3} - 2\left(\frac{1}{\gamma} + 1\right)\delta_{13} \text{ Where } \delta_{13} = \sum_{j=1}^n \frac{1}{(\beta+x_j^\alpha)} \tag{3.30}$$

$$l_{333} = -\frac{2n}{\gamma^3} - \frac{6n \log \beta}{\gamma^4} + \frac{6}{\gamma^4}\delta_{10} \text{ Where } \delta_{10} = \sum_{j=1}^n \log(\beta + x_j^\alpha) \tag{3.31}$$

$$l_{112} = -\left(\frac{1}{\gamma} + 1\right)\omega_{123} = l_{121} \text{ Where } \omega_{123} = \sum_{j=1}^n \frac{x_j^\alpha(\beta-x_j^\alpha)(\log x_j)^2}{(\beta+x_j^\alpha)^3} \tag{3.32}$$

$$l_{113} = \frac{\beta}{\gamma^2}\omega_{122} = l_{131} \text{ Where } \omega_{122} = \sum_{j=1}^n \frac{x_j^\alpha(\log x_j)^2}{(\beta+x_j^\alpha)^2} \tag{3.33}$$

$$l_{221} = -2\left(\frac{1}{\gamma} + 1\right)\omega_{113} = l_{212} \text{ Where } \omega_{113} = \sum_{j=1}^n \frac{x_j^\alpha \log x_j}{(\beta+x_j^\alpha)^3} \tag{3.34}$$

$$l_{223} = \frac{n}{(\gamma\beta)^2} - \frac{1}{(\gamma)^2}\delta_{12} \text{ Where } \delta_{12} = \sum_{j=1}^n \frac{1}{(\beta+x_j^\alpha)^2} \tag{3.35}$$

$$l_{331} = -\frac{2}{\gamma^3}\omega_{11} = l_{313} \text{ Where } \omega_{11} = \sum_{j=1}^n \frac{x_j^\alpha \log x_j}{(\beta+x_j^\alpha)} \tag{3.36}$$

$$l_{332} = \frac{2}{\gamma^3}\left(\frac{n}{\beta} - \delta_{11}\right) = l_{323} \tag{3.37}$$

$$l_{231} = -\frac{1}{\gamma^2}\sum_{j=1}^n \frac{x_j^\alpha \log x_j}{(\beta+x_j^\alpha)^2} = l_{213} \text{ Where } \omega_{14} = \sum_{j=1}^n \frac{x_j^\alpha \log x_j}{(\beta+x_j^\alpha)^2} \tag{3.38}$$

$$l_{123} = -\frac{\omega_{14}}{\gamma^2} = l_{132} \text{ And } l_{133} = -\frac{2}{\gamma^3}\omega_{11} \tag{3.39}$$

$$l_{122} = -2\left(\frac{1}{\gamma} + 1\right)\omega_{113} \text{ Where } \omega_{113} = \sum_{j=1}^n \frac{x_j^\alpha \log x_j}{(\beta+x_j^\alpha)^3} \tag{3.40}$$

$$l_{233} = \frac{2}{\gamma^3}\left(\frac{n}{\beta} - \delta_{11}\right) \tag{3.41}$$

Now

$$[-l_{ijk}] = -\begin{bmatrix} l_{111} & l_{112} & l_{113} \\ l_{221} & l_{222} & l_{223} \\ l_{331} & l_{332} & l_{333} \end{bmatrix} \tag{3.42}$$

$$= \begin{bmatrix} \frac{2n}{\alpha^3} - \left(\frac{1}{\gamma} + 1\right)\omega_{133} & , \left(\frac{1}{\gamma} + 1\right)\omega_{123}, & -\frac{\beta}{\gamma^2}\omega_{122} \\ -2\left(\frac{1}{\gamma} + 1\right)\omega_{113} & , \frac{2n\gamma}{\gamma\beta^3} - 2\left(\frac{1}{\gamma} + 1\right)\delta_{13}, & \frac{n}{(\gamma\beta)^2} - \frac{1}{\gamma^2}\delta_{12} \\ \frac{-2}{\gamma^3}\omega_{11} & , \frac{2}{\gamma^3}\left(\frac{n}{\gamma} - \delta_{11}\right) & , -\frac{2n}{\gamma^3} - \frac{6n \log \beta}{\gamma^4} + \frac{6}{\gamma^4}\delta_{10} \end{bmatrix}$$

$$[-l_{ijk}] = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

Determinant of $[-l_{ijk}]$

$$D = -\{M_{11}[M_{22}M_{33} - M_{23}M_{32}] - M_{12}[M_{21}M_{33} - M_{31}M_{23}] + M_{13}[M_{21}M_{32} - M_{22}M_{33}]\}$$

Transpose of Adjoint of $[-l_{ijk}]$

$$= \begin{bmatrix} M_{23}M_{32} - M_{22}M_{33} & M_{12}M_{33} - M_{32}M_{13} & M_{12}M_{22} - M_{12}M_{23} \\ M_{21}M_{33} - M_{31}M_{23} & M_{31}M_{13} - M_{11}M_{33} & M_{11}M_{23} - M_{13}M_{21} \\ M_{22}M_{31} - M_{21}M_{32} & M_{11}M_{32} - M_{31}M_{12} & M_{12}M_{21} - M_{11}M_{22} \end{bmatrix}$$

$$[-l_{ijk}]^{-1} = \frac{(\text{Adjoint of } [-l_{ijk}])'}{D}$$

$$[-l_{ijk}]^{-1} = \begin{bmatrix} \frac{Y_{11}}{D} & \frac{Y_{12}}{D} & \frac{Y_{13}}{D} \\ \frac{Y_{21}}{D} & \frac{Y_{22}}{D} & \frac{Y_{23}}{D} \\ \frac{Y_{31}}{D} & \frac{Y_{32}}{D} & \frac{Y_{33}}{D} \end{bmatrix}$$

$$[-l_{ijk}]^{-1} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \tag{3.43}$$

Evaluated from equation number (3.8) and from joint prior density equation (3.1) we have;

$$G(\alpha, \beta, \gamma) = g(\alpha)g_2(\beta)g_3(\gamma|\beta)$$

$$= \frac{c}{\delta\Gamma\xi} \beta^{-\xi} \gamma^{\xi-1} \exp\left[-\left(\frac{\gamma}{\beta} + \frac{\beta}{\delta}\right)\right] \tag{3.44}$$

$$\log G = \log C - \log\delta - \log\Gamma\xi + (\xi - 1)\log\gamma - \xi\log\beta - \left(\frac{\gamma}{\beta} + \frac{\beta}{\delta}\right) \tag{3.45}$$

$$\log G = \text{constant} - \xi\log\beta + (\xi - 1)\log\gamma - \frac{\gamma}{\beta} - \frac{\beta}{\delta}$$

$$\rho_1 = \frac{\delta\rho}{\delta\alpha} = 0 \tag{3.46}$$

$$\rho_2 = \frac{\delta\rho}{\delta\beta} = \frac{-\xi}{\beta} + \frac{\gamma}{\beta^2} - \frac{1}{\delta} \tag{3.47}$$

$$\rho_3 = \frac{\delta\rho}{\delta\gamma} = \frac{\xi-1}{\gamma} - \frac{1}{\beta} \tag{3.48}$$

Substituting the above values from equation (3.12)-(3.16) in (3.17)-(3.19), we get the values of A, B, and P as

$$A = [\sigma_{11}l_{111} + 2\sigma_{12}l_{121} + 2\sigma_{13}l_{131} + 2\sigma_{23}l_{231} + \sigma_{22}l_{221} + \sigma_{33}l_{331}]$$

$$= \frac{1}{D} \left[Y_{11} \left(\frac{2n}{\alpha^3} - \left(\frac{1}{\gamma} + 1 \right) \omega_{133} \right) + 2Y_{12} \left(\frac{1}{\gamma} + 1 \right) \omega_{123} + 2Y_{13} \frac{\beta}{\gamma^3} \omega_{122} - 2Y_{23} \frac{\omega_{14}}{\gamma^2} - 2Y_{22} \left(\frac{1}{\gamma} + 1 \right) \omega_{113} - \frac{2}{\gamma^3} Y_{33} \omega_{11} \right] \tag{3.49}$$

$$B = [\sigma_{11}l_{112} + 2\sigma_{12}l_{122} + 2\sigma_{13}l_{132} + 2\sigma_{23}l_{232} + \sigma_{22}l_{222} + \sigma_{33}l_{332}]$$

$$= \frac{1}{D} \left[\left(\frac{1}{\gamma} + 1 \right) \omega_{123} Y_{11} - 4Y_{12} \left(\frac{1}{\gamma} + 1 \right) \omega_{113} - 2Y_{13} \left(\frac{\omega_{14}}{\gamma^2} \right) + (2Y_{23} + Y_{22}) \left(\frac{n}{(\gamma\beta)^2} - \frac{1}{\gamma^2} \delta_{12} \right) + Y_{33} \left(\frac{2}{\gamma^3} \left(\frac{n}{\beta} - \delta_{11} \right) \right) \right] \tag{3.50}$$

$$P = [\sigma_{11}l_{113} + 2\sigma_{12}l_{123} + 2\sigma_{13}l_{133} + 2\sigma_{23}l_{233} + \sigma_{22}l_{223} + \sigma_{33}l_{333}]$$

$$= \frac{1}{D} \left[\frac{Y_{11}\beta}{\gamma^2} \omega_{122} - \frac{2Y_{12}\omega_{14}}{\gamma^4} - \frac{4Y_{13}\omega_{11}}{\gamma^3} + \frac{4Y_{23}}{\gamma^3} \left(\frac{n}{\beta} - \delta_{11} \right) + Y_{22} \left(\frac{n}{\gamma^2\beta^2} - \frac{1}{\gamma^2} \delta_{12} \right) + Y_{33} \left(-\frac{2n}{\gamma^3} - \frac{6n \log\beta}{\gamma^4} + \frac{6}{\gamma^4} \delta_{10} \right) \right] \tag{3.51}$$

$$\hat{U}_{AB} = E(U | \mathbf{x})$$

$$= u + (u_1 a_1 + u_2 a_2 + u_3 a_3 + a_4 + a_5) \frac{1}{2} [A(u_1 \sigma_{11} + u_2 \sigma_{12} + u_3 \sigma_{13}) + P(u_1 \sigma_{21} + u_2 \sigma_{22} + u_3 \sigma_{23}) P(u_1 \sigma_{31} + u_2 \sigma_{32} + u_3 \sigma_{33})]$$

$$+ 0 \left(\frac{1}{n^2} \right)$$

$$E(U | \mathbf{x}) = U + \varphi_1 + \varphi_2 \tag{3.52}$$

Where

$$\varphi_1 = u_1 a_1 + u_2 a_2 + u_3 a_3 + a_4 + a_5 \tag{3.53}$$

$$\varphi_2 = \frac{1}{2} [(A\sigma_{11} + B\sigma_{21} + P\sigma_{31}) \cdot U_1 + (A\sigma_{12} + B\sigma_{22} + P\sigma_{32}) \cdot U_2 + (A\sigma_{13} + B\sigma_{23} + P\sigma_{33}) U_3] \tag{3.54}$$

Evaluated at the MLE $\hat{U} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ where

$$a_1 = \rho_1 \sigma_{11} + \rho_2 \sigma_{12} + \rho_3 \sigma_{13} = \left(\frac{-\xi}{\beta} + \frac{\gamma}{\beta^2} - \frac{1}{\delta} \right) \frac{Y_{12}}{D} + \left(\frac{\xi - 1}{\gamma} - \frac{1}{\beta} \right) \frac{Y_{13}}{D}$$

$$a_2 = \rho_1\sigma_{21} + \rho_2\sigma_{22} + \rho_3\sigma_{23} = \left(\frac{-\xi}{\beta} + \frac{\gamma}{\beta} - \frac{1}{\delta}\right)\frac{Y_{22}}{D} + \left(\frac{\xi-1}{\gamma} - \frac{1}{\beta}\right)\frac{Y_{23}}{D}$$

$$a_3 = \rho_1\sigma_{31} + \rho_2\sigma_{32} + \rho_3\sigma_{33} = \left(\frac{-\xi}{\beta} + \frac{\gamma}{\beta^2} - \frac{1}{\delta}\right)\frac{Y_{32}}{D} + \left(\frac{\xi-1}{\gamma} - \frac{1}{\beta}\right)\frac{Y_{33}}{D}$$

$$a_4 = U_{12}\sigma_{12} + U_{13}\sigma_{13} + U_{23}\sigma_{23} = \frac{Y_{12}}{D}U_{12} + \frac{Y_{13}}{D}U_{13} + \frac{Y_{23}}{D}U_{23}$$

$$a_5 = \frac{1}{2}(U_{11}\sigma_{11} + U_{22}\sigma_{22} + U_{33}\sigma_{33}) = \frac{1}{2D}(Y_{11}U_{11} + Y_{22}U_{22} + Y_{33}U_{33})$$

5. Approximate Bayes Estimator of R under squared Error Loss Function

$$\hat{U}_{ABS} = E(U) = R \tag{5.1}$$

Where

$$E_u(U|\underline{x}) = \frac{\int_{\alpha} \int_{\beta} \int_{\gamma} U G^*(\alpha, \beta, \gamma) \partial \alpha \partial \beta \partial \gamma}{\int_{\alpha} \int_{\beta} \int_{\gamma} G^*(\alpha, \beta, \gamma) \partial \alpha \partial \beta \partial \gamma} \tag{5.2}$$

The above equation (5.2) is evaluated by method of Lindley approximation, whose simplified form is equation (3.8) replace U by R in equation (3.52),

$$\hat{R}_{ABS} = E(R|\underline{x}) \tag{5.3}$$

$$E(R|\underline{x}) = R + \varphi_1 + \varphi_2 \tag{5.4}$$

Now $U = \left(\frac{\beta}{\beta + t^\alpha}\right)^{\frac{1}{\gamma}}$

$$\log u = \frac{1}{\gamma} \log \frac{\beta}{(\beta + t^\alpha)}$$

Now differentiating u with respect to α

$$\frac{1}{u} u_1 = -\frac{1}{\gamma} \frac{t^\alpha}{(\beta + t^\alpha)} \log t$$

$$u_1 = -\frac{R}{\gamma} \frac{t^\alpha}{(\beta + t^\alpha)} \log t; \tag{5.5}$$

Now differentiating u with respect to α, β and γ respectively we get;

$$u_{11} = \frac{R}{\gamma} t^\alpha \frac{(\log t)^2}{(\beta + t^\alpha)^2} \left(\frac{t^\alpha}{\gamma} - \beta\right) \tag{5.6}$$

$$u_{12} = -\frac{R}{\gamma} \frac{t^\alpha \log t}{(\beta + t^\alpha)^2} \left[\frac{t^\alpha}{\gamma \beta} - 1\right] \tag{5.7}$$

$$u_{13} = \frac{R}{\gamma^3} \frac{t^\alpha}{(\beta + t^\alpha)} \log t (\delta^* + \gamma) \tag{5.8}$$

Again differentiating u with respect to β

$$u_2 = \frac{R}{\gamma} \frac{t^\alpha}{\beta(\beta + t^\alpha)} \tag{5.9}$$

Again differentiating u_2 with respect to α, β and γ respectively we get;

$$\log u_2 = \log R + \alpha \log t - \log \gamma - \log \beta - \log(\beta + t^\alpha)$$

$$u_{21} = -\frac{R}{\gamma} \frac{t^\alpha \log t}{(\beta + t^\alpha)^2} \left[\frac{t^\alpha}{\gamma \beta} - 1\right] \tag{5.10}$$

$$u_{22} = \frac{R}{\gamma} \frac{t^\alpha}{(\beta(\beta + t^\alpha))^2} \left(\frac{t^\alpha}{\gamma} - (2\beta + t^\alpha)\right) \tag{5.11}$$

$$u_{23} = -\frac{R}{\gamma^3} \frac{t^\alpha}{\beta(\beta+t^\alpha)} (\delta^* + \gamma) \tag{5.12}$$

Again differentiating u with respect to γ

$$u_3 = -\frac{\delta^*}{\gamma^2} R \tag{5.13}$$

Again differentiating u_3 with respect to α, β and γ respectively we get;

$$\log u_3 = \log(-1) + \log R + \log \left(\log \left(\frac{\beta}{(\beta + t^\alpha)} \right) \right) - 2 \log \gamma$$

$$u_{31} = \frac{R}{\gamma^3} \frac{t^\alpha}{(\beta+t^\alpha)} \log t (\delta^* + \gamma) \tag{5.14}$$

$$u_{32} = -\frac{R}{\gamma^3} \frac{t^\alpha}{\beta(\beta+t^\alpha)} (\delta^* + \gamma) \tag{5.15}$$

$$u_{33} = \frac{R}{\gamma^4} \delta^* (\delta^* + 2\gamma) \tag{5.16}$$

Substituting the above values of $u_1, u_2, u_3, u_{11}, u_{12}, u_{13}, u_{22}, u_{23}$ and u_{33} in the equations (3.53) and (3.54), we get

$$\varphi_1 = u_1 a_1 + u_2 a_2 + u_3 a_3 + a_4 + a_5$$

$$\varphi_1 = \frac{R}{\gamma} \left[-K a_1 + \frac{K}{(\beta+t^\alpha)} a_2 - \frac{1}{\gamma} \delta^* a_3 - \frac{Y_{12}}{D} \frac{K}{(\beta+t^\alpha)} \left(\frac{t^\alpha}{\gamma} - \beta \right) + \frac{Y_{13}}{D \gamma^2} - \frac{Y_{23}}{D \gamma^2 \beta \log t} (K(\delta^* + \gamma)) + \frac{Y_{11}}{2D t^\alpha} K^2 \left(\frac{t^\alpha}{\gamma} - \beta \right) + \frac{Y_{22}}{2D (\beta \log t)^2 t^\alpha} \left(\frac{t^\alpha}{\gamma} - (2\beta + t^\alpha) \right) + \frac{Y_{33}}{2D \gamma^3} \delta^* (\delta^* + 2\gamma) \right]$$

$$\varphi_1 = \frac{R}{\gamma} \varphi_3 \tag{5.17}$$

And

$$\varphi_2 = \frac{1}{2} \left[U_1 ((A\sigma_{11} + B\sigma_{21} + P\sigma_{31}) \cdot U_1 + (A\sigma_{12} + B\sigma_{22} + P\sigma_{32}) \cdot U_2 + (A\sigma_{13} + B\sigma_{23} + P\sigma_{33}) U_3) \right]$$

$$\varphi_2 = \frac{R}{\gamma} \left[-K \left(\frac{AY_{11} + BY_{21} + PY_{31}}{2D} \right) + \frac{K}{\beta \log t} \left(\frac{AY_{12} + BY_{22} + PY_{32}}{2D} \right) - \frac{\delta^*}{\gamma^2} \left(\frac{AY_{13} + BY_{23} + PY_{33}}{2D} \right) \right]$$

$$\varphi_2 = \frac{R}{\gamma} \tag{5.18}$$

Substituting φ_1 and φ_2 in equation (5.4) we get Approximate Bayes Estimator under Squared Error Loss Function as

$$E(R|\underline{x}) = R + \varphi_1 + \varphi_2$$

$$E(R|\underline{x}) = R + \frac{R}{\gamma} (\varphi_3 + \varphi_4)$$

$$\hat{R}_{ABS} = R [1 + \gamma^{-1} \varphi_5] \tag{5.19}$$

Where

$$\varphi_5 = \varphi_3 + \varphi_4 \text{ and } K = \frac{t^\alpha}{(\beta+t^\alpha)} \log t \tag{5.20}$$

$$\varphi_3 = \left[-K a_1 + \frac{K}{(\beta+t^\alpha)} a_2 - \frac{1}{\gamma} \delta^* a_3 - \frac{Y_{12}}{D} \frac{K}{(\beta+t^\alpha)} \left(\frac{t^\alpha}{\gamma} - \beta \right) + \frac{Y_{13}}{D \gamma^2} - \frac{Y_{23}}{D \gamma^2 \beta \log t} (K(\delta^* + \gamma)) + \frac{Y_{11}}{2D t^\alpha} K^2 \left(\frac{t^\alpha}{\gamma} - \beta \right) + \frac{Y_{22}}{2D (\beta \log t)^2 t^\alpha} \left(\frac{t^\alpha}{\gamma} - (2\beta + t^\alpha) \right) + \frac{Y_{33}}{2D \gamma^3} \delta^* (\delta^* + 2\gamma) \right] \tag{5.21}$$

$$\varphi_4 = \left[-K \left(\frac{AY_{11} + BY_{21} + PY_{31}}{2D} \right) + \frac{K}{\beta \log t} \left(\frac{AY_{12} + BY_{22} + PY_{32}}{2D} \right) - \frac{\delta^*}{\gamma^2} \left(\frac{AY_{13} + BY_{23} + PY_{33}}{2D} \right) \right] \tag{5.22}$$

6. Simulations and Numerical Comparison

The simulations and numerical calculations are done by using R Language programming and results are presented in form of table in table (1).

1. The Random variable of Generalized Compound Rayleigh Distribution is generated by R-Language programming by taking the values of the parameters α, β, γ , taken as $\alpha = 1, \beta = 0.5$ and $\gamma = 0.8$ in the equations [(2.3)-(2.5) and equation (1.9)].
2. Taking the different sizes of samples $n=10(10)80$ with complete sample, MLE's, the Approximate Bayes estimators, and their respective MSE's (in parenthesis) are obtained by repeating the steps 500 times, are presented in the tables from (2.1- 2.2), for $t=0.5, R(t)=0.42, H(t)=0.625$ and parameters of prior distribution $a = 2$ and $b = 3$.
3. Table(1) presents the MLE of $R(t)$ and Approximate Bayes estimators of reliability function $R(t)$ of reparametrized Generalized Compound Rayleigh Distribution under SELF (for α, β and γ unknown) with their MSE. The all three estimators are efficient for the sample size $n=50$ but as sample approaches to 70 their MSE's started increasing. The Approximate Bayes estimator \hat{R}_{ABS} performs better among other estimators.

Table 1: Mean and MSE's of $R(t)$ ($\alpha = 1, \beta = 0.5, \gamma = 0.8, k = -10$)

n	\hat{R}_{ML}	\hat{R}_{BS}	\hat{R}_{ABS}
10	0.5496425 [1.7239x10 ⁻⁴]	0.5497635 [4.0544x10 ⁻⁴]	0.5497635 [4.0544x10 ⁻⁴]
20	0.513952 [1.6316x10 ⁻⁴]	0.550967 [3.67188x10 ⁻⁴]	0.550967 [3.67188x10 ⁻⁴]
30	0.490252 [2.7167x10 ⁻⁴]	0.555524 [4.0074x10 ⁻⁴]	0.555524 [4.0074x10 ⁻⁴]
40	0.468581 [7.1028x10 ⁻⁵]	0.4682655 [7.0887 x10 ⁻⁵]	0.4682655 [7.0887 x10 ⁻⁵]
50	0.449611 [1.7239x10 ⁻⁵]	0.449769 [4.0544x10 ⁻⁵]	0.449769 [4.0544x10 ⁻⁵]
60	0.4012582 [1.6316x10 ⁻⁵]	0.455967 [3.67188x10 ⁻⁵]	0.455967 [3.67188x10 ⁻⁵]
70	0.400252 [2.7167x10 ⁻⁵]	0.425524 [4.0074x10 ⁻⁵]	0.425524 [4.0074x10 ⁻⁵]
80	0.364581 [6.1028x10 ⁻⁴]	0.3682655 [6.0887 x10 ⁻⁴]	0.3682655 [6.0887 x10 ⁻⁴]

7. References

1. Bain LJ, Engelhardt M. Statistical analysis of reliability and life testing models: Theory and Methods. New York: M. Dekker, 1991.
2. Burr IW. Cumulative Frequency Distribution, Annals of Mathematical Statistics. 1942; 13:215-232.
3. Dubey S, Pradhan B, Kundu D. Parameter Estimation of the Generalized Rayleigh Distribution, 2010.
4. George W. Evans; Simulation Using Digital Computers, 2010, 60-67.
5. Grohowski G, Hausman W, Lamberson LR. (ASQC) General Motor Corporation, Warren.MI; Wayne State University, Detroit, MI. QICID: 5240 October, 1976, 197-208.
6. Lindley DV. Approximate Bayesian Method. Trab. Esta. 1980; 31:223-237.
7. Mostert J, Roux J, Bekker J. A Generalized Compound Rayleigh Distribution Using Bayesian Method. 2000; 29(7):1419-1433'
8. Patil GP, Boswell MT, Joshi SW, Ratnaparkhi V. Dictionary and classified Bibliography of Statistical Distributions in Scientific Work. International Cooperative publishing house, Fairland, MD, 1984, 1.
9. Satya D, Dubey; june, 1968.
10. <https://doi.org/10.1002/nav.3800150205>.
11. Sinha SK. Robustness of Posteriors under Improper Priors in Some Life Testing Models, J Indian Statist. Assoc. 1980; 18:127-144.
12. Sinha SK. Reliability and Life Testing, Wiley Eastern Ltd. Delhi, India, 1986.
13. Sinha SK, Sloan JA. Bayesian Estimation of Parameters and Reliability Function of the 3-Parameters Weibull Distrib. IEEE Tra. Rel. V-R-37, 1988, 364-369.
14. Siddiqui J, Weiss G. Families of Distributions for received power of radio wave, J Research of the NBS, D. Radio Propagation, 1963, 753-762.
15. Soliman AA. Linex and Quadratic Approximate Bayesian Estimators Applied to Pareto Model", Communication in Statistics-Simulation. 2001; 30(1):47-62.
16. Varian HR. A Bayesian Approach to Real Estate Assessment, in Studies in Bayesian Econometrics and Statistics in Honor of Leonard J. Savage, SE. Fienberg and A. Zellner, Eds. North-Holland Amsterdam, the Netherlands., 1975, 195-208.
17. Zellner A, Giesel MS. Sensitivity of Control to Uncertainty and Form of Criterion Function in the Future of Statistics. Ed. Donal G. Watts. N Y Academic Press, 1968.