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Divorce transmission model

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Abstract
This paper is an improvement on 'Mathematical Model of Divorce Epidemic in Ghana' proposed by Gambrah and Adzadu (2018), we analyze population dynamics of divorce. Divorce free and divorce endemic equilibrium are computed. The local and global stabilities of the proposed problem are established. The numerical simulation is given to validate the transmission of population into different compartments.

Keywords: Population dynamics of divorce, basic reproduction number, local stability, global stability

1. Introduction
Marriage is the legal union between two people. Divorce on the other hand is the dissolution of marriage between two partners. Been marriage reflects a person's social status and cultural prestige. It is observed that the rate of increase of marriages nowadays is proportional to the divorce rate due to Inter-cultural marriages. There is a similarity between spread of infectious disease and divorce. In other words, divorce can be treated as a virus which diffuses among the compartments by social pressure, and many other reasons.
In this paper, we analyze quantitative model of a divorce transmission in a population similar to *SEIR* model.
The notation description, the mathematical model, basic reproduction number, the local and global stability are formulated in Section 2. Numerical simulations are illustrated in Section 3, Discussions of results is in section 4 and conclusions are given in Section 5.

2. Materials and Methods
2.1 Model Description
We formulate a mathematical model and divide the population into four compartments: Susceptible(S), Married (M), Separated (P), and Divorced (D). The interaction between the three states are shown in the schematic diagram in Fig. 1

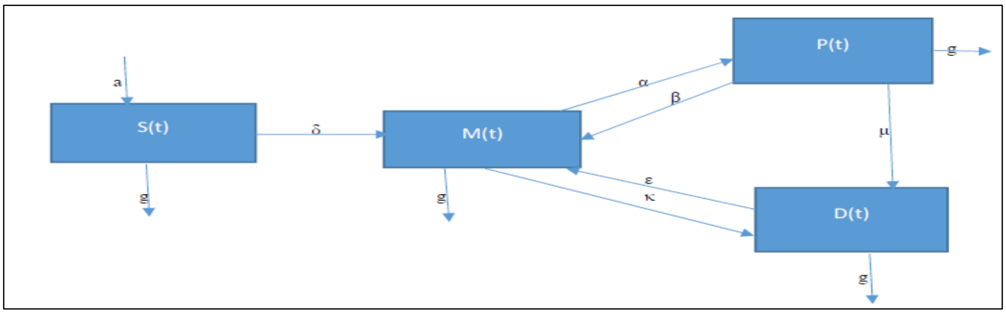

Fig 1: Schematic diagram of the four classes in the model

Anybody in the population is susceptible to marriage. When married, one can enter the divorce or separated compartment due to some external factors and also one can move to the divorce compartment from the separation compartment. Thus, this resembles the *SEIR* model.

Now, we make *SMPD* model with some assumptions. The portion of married person's who go into divorce at a rate proportional to κM with $\kappa \succ 0$. Therefore, the portion of married will decrease with the same rate. Here, ∂ is called the effective marrying rate. The portion of divorced and separated people moving back to married with a rate εD (progression rate) with $\varepsilon > 0$ and βP with $\beta > 0$. The portion of divorced with the rate κ *with* $\kappa > 0$. The portion of separated to divorced with the rate μP with $\mu > 0$ and αM with $\alpha > 0$.
The model is formulated as following system of differential equations:
A: recruitment rate of S
g: Natural death rate
∂: Rate of getting married
ε: Rate of divorced getting married again
β: Rate of separated getting married again
μ: Rate of separated being divorced
α: Rate of married becoming separated
κ: Rate of the married becoming divorced

$$\frac{dS}{dt} = a - \partial SM - gS \quad [1]$$

$$\frac{dM}{dt} = \partial SM - gM - \beta M - \varepsilon M + \alpha P + \kappa D \quad [2]$$

$$\frac{dP}{dt} = \beta M - \alpha P - \mu P - gP \quad [3]$$

$$\frac{dD}{dt} = \varepsilon M - \kappa D - gD + \mu P \quad [4]$$

With the initial conditions $S(t) \geq 0, M(t) \geq 0, P(t) \geq 0, D(t) \geq 0$
We assume that the system of nonlinear differential equations [1]-[4] has positive initial conditions, then every solution $(S(t), M(t), P(t), D(t))$ Of [1]-[4] has the positive properties, that is,
$S(t) \geq 0, M(t) \geq 0, P(t) \geq 0$ and $D(t) \geq 0$. Hence the feasible region
$X = \left\{ (S, M, P, D) \in R^{4+} : S + M + P + D \leq \frac{a}{g} \right\}$, is positively invariant set for the system [1]-[4].

This implies that:

$$N(t) = S(t) + M(t) + P(t) + D(t) \quad [5]$$

Also

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dM}{dt} + \frac{dP}{dt} + \frac{dD}{dt}$$
$$\frac{dN}{dt} = a - g(S + M + P + D) = a - gN \quad [6]$$

From [6] it follows that:

$$\lim_{t \to \infty} \sup N(t) \leq \frac{a}{g},$$

Thus, the feasible region of the system [1]-[4] is given by the set X

2.2 Divorce Free Equilibrium and Basic Reproductive Number

We now consider the divorced free equilibrium $Y_0 = \left(\frac{a}{g}, 0, 0, 0 \right)$. Thus, a situation where there is no divorce problem. We analyze the stability of the divorce free equilibrium by considering the linearized system of ODE's [1]-[4], taking the Jacobian matrix, we obtain

$$J(S, M, P, D) = \begin{bmatrix} \partial M - g & -\partial S & 0 & 0 \\ \partial M & \partial S - g - \beta - \varepsilon & \alpha & \kappa \\ 0 & \beta & -\alpha - \mu - g & 0 \\ 0 & \varepsilon & \mu & -\kappa - g \end{bmatrix} \quad [7]$$

The local stability of the equilibrium can be determined from the Jacobian matrix [7]. This implies that, the Jacobian matrix for the divorce free equilibrium is given by

$$J(Y_0) = \begin{bmatrix} -g & -\partial & 0 & 0 \\ 0 & \partial - g - \beta - \varepsilon & \alpha & \kappa \\ 0 & \beta & -\alpha - \mu - g & 0 \\ 0 & \varepsilon & \mu & -\kappa - g \end{bmatrix} \quad [8]$$

From the characteristic equation of $J(S, 0, 0, 0)$, the following eigenvalues were obtained:

$\lambda_1 = -g, \lambda_2 = \partial - g - \beta - \varepsilon, \lambda_3 = -\alpha - \mu - g$ and $\lambda_4 = -\kappa - g$. It may be seen that $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are real and negative. We know that $R_0 < 1$, this infers that $\partial < g + \beta + \varepsilon$ and hence λ_2 is real and negative. This implies that the system [1]-[4] is asymptotically stable.

The basic reproductive number R_0, is given by

$$R_0 = \frac{\partial}{g + \beta + \varepsilon} \quad [9]$$

Theorem 1: The divorce free equilibrium $Y_0 = \left(\frac{a}{g}, 0, 0, 0 \right)$, of the system [1]-[4] is asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$.

2.2.1 Equilibrium

We evaluate the equilibrium points of the ODE [1]-[4] by setting the right -hand side of equation [1]-[4] to zero and then solve for S^*, M^*, P^* and D^* we got:

$$M^* = \frac{-(g^3\mu + g^3\beta + g^3\varepsilon + g^3\alpha + g^3\kappa + g^4 - a\delta g^2 g^2 \mu\beta + g^2\mu\varepsilon + g^2\mu\kappa + g^2\beta\kappa + g^2\varepsilon\alpha + g^2\alpha\kappa - a\delta g\mu - a\delta g\alpha - a\delta g\kappa - a\delta\mu\kappa - a\delta\alpha\kappa)}{\delta g^3 + \delta g^2\mu + \delta g^2\beta + \delta g^2\varepsilon + \delta g\alpha^2 + \delta g^2\kappa + \delta g\mu\beta + \delta g\mu\varepsilon + \delta g\mu\kappa + \delta g\beta\kappa + \delta g\varepsilon\alpha + \delta g\alpha\kappa}$$

$$S^* = \frac{g^2\mu + g^2\beta + g^2\varepsilon + g^2\alpha + g^2\kappa + g^3 + g\mu\beta + g\mu\varepsilon + g\mu\kappa + g\beta\kappa + g\varepsilon\alpha + g\alpha\kappa}{\delta g^2 + \delta g\mu + \delta g\alpha + \delta g\kappa + \delta\mu\kappa + \delta\alpha\kappa}$$

$$P^* = \frac{\beta M}{\alpha + \mu + g} \quad [10]$$

And

$$D^* = \frac{\varepsilon M + \mu P}{\kappa + g}$$

We now consider the case when $R_0 > 1$. At the endemic equilibrium, all the four states are present in the population. The steady states conditions under which all four states can coexist is the equilibrium. We represent $Y^* = (S^*, M^*, P^*, D^*)$ as endemic equilibrium of the system [1]-[4] and $(S^* \neq 0, M^* \neq 0, P^* \neq 0, D^* \neq 0)$.

2.2.2 Stability of the Equilibrium

2.2.2.1 Local Stability

The divorce free equilibrium is stable if all the eigenvalues of the Jacobian matrix of the system [1]-[4] have negative real parts. For this, the Jacobian of the system [1]-[4] at $Y_0 = (1, 0, 0, 0)$ takes the form

$$J(Y_0) = \begin{bmatrix} -g & -\partial & 0 & 0 \\ 0 & \partial - g - \beta - \varepsilon & \alpha & \kappa \\ 0 & \beta & -\alpha - \mu - g & 0 \\ 0 & \varepsilon & \mu & -\kappa - g \end{bmatrix}$$

Here the $trace(J) = -(\beta - \partial + 4g + \varepsilon + \mu + \alpha + \kappa) < 0$

2.2.2.2 Global stability of the endemic equilibrium (Y^*)

We determine the global stability of the endemic equilibrium, by using the last three equations of the system [1]-[4] that is:

$$\frac{dM}{dt} = \partial SM - gM - \beta M - \varepsilon M + \alpha P + \kappa D$$

$$\frac{dP}{dt} = \beta M - \alpha P - \mu P - gP \qquad [11]$$

$$\frac{dD}{dt} = \varepsilon M - \kappa D - gD + \mu P$$

In the region $X^* = \{(M, P, D) \in R^{3+} : M + P + D \leq 1, M > 0, P \geq 0, D \geq 0\}$, X^* is positively invariant, thus every solution of the model [11], with initial conditions in X^* remains there for time (t>0).

We also consider

$$X^{**} = \left\{(M, P, D) : M + \left(\frac{\beta + g}{g}\right)P + \left(\frac{\varepsilon + g}{g}\right)D = 1, M > 0, P \geq 0, D \geq 0\right\}$$ Where $X^{**} \subset X^*$, X^{**} is positively invariant, $Y^* \in X^*$ and $a = g$.

3. Numerical Simulation

We now perform numerical simulation to show the dynamical behavior of our results, by assuming that our total population is 100% and choose S = 0.50, M = 0.23, P = 0.15 and D = 0.12. The other parameters that would be used in this section are;

$$g : 0.121, \alpha : 0.031, \beta : 0.052, \varepsilon : 0.061, \kappa : 0.022, \mu : 0.021 \text{ and } \partial : 0.101$$

We investigate the nature of the model by conducting sensitivity analysis of the reproductive number R_0.

At the divorce free equilibrium, $g : 0.121, \beta : 0.052, \varepsilon : 0.061, \text{and } \partial : 0.101$, R_0 = 0.4316<1.

(i) If the value of ∂ is increased to any figure greater than 0.2340 and the values of the other parameters remain the same, R_0 >1.

(ii) If the value of the other parameters are increased and the value of ∂ remains the same, R_0 <1.

4. Discussion of Results

We use *SMPD* model to study the dynamics of divorce as an epidemic. We discussed the existence and stability of divorce free equilibria and performed the sensitivity analysis of the reproductive number. Based on the data used, the basic reproductive number of the divorced free equilibrium is estimated to be $R_0 = 0.4316<1$. This implies that the divorce free equilibrium is asymptotically stable.

Considering the situation when $R_0 >1$, that will imply that there is an epidemic problem which shows the situation in which the Susceptible, Married, Separated and Divorced co-exist in the population

$$\left(S^*, M^*, P^*, D^*\right) = \left(2.1220, 1.0533, 0.3166, 0.4958\right).$$

This indicates the existence of divorce problem in the population since $D^* \neq 0$.

5. Conclusion

Our model shows that, divorce epidemic cannot only be controlled by reducing the contact rate between the
Married and Divorced but also increasing the number of divorced that go back to marriage and educating the susceptible not to enter into marriages they are not sure of its future and also the Married to refrain from Divorce and Separation and these can be useful in combating the epidemic.

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