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Size-biased poisson-weighted Lindley distribution and its applications

Rama Shanker and Kamlesh Kumar Shukla

Abstract
A size-biased Poisson-weighted Lindley distribution has been suggested and its moments and moments based measures have been derived and discussed. Maximum likelihood estimation has been discussed for estimating the parameters of the distribution. Some applications of the proposed distribution have been explained and the goodness of fit shows much closer fit over size-biased Poisson distribution and size-biased Poisson-Lindley distribution.

Keywords: Size-biased distribution, Poisson-weighted Lindley distribution, compounding, moments, Skewness, kurtosis, maximum likelihood estimation, applications

Introduction

Suppose a random variable X has probability distribution $P_0\left(x;\theta\right)\; ; x = 0,1,2,\ldots,\theta > 0$.

If sample units are weighted or selected with probability proportional to x^{α}, then the corresponding size-biased distribution of order α is defined by its probability mass function (pmf)

$$P\left(x;\theta\right) = \frac{x^{\alpha}\cdot P_0\left(x;\theta\right)}{\mu'_{\alpha}} \tag{1.1}$$

Where $\mu'_{\alpha} = E\left(X^{\alpha}\right) = \sum_{x=0}^{\infty} x^{\alpha} P_0\left(x;\theta\right)$. When $\alpha = 1$, (1.1) is known as simple size-biased distribution and is applicable for size-biased sampling and for $\alpha = 2$, (1.1) is known as area-biased distribution and is applicable for area-biased sampling. Size-biased distributions are a particular class of weighted distributions which arise naturally in practice when observations from a sample are recorded with probability proportional to some measure of unit size. In field applications, size-biased distributions can arise either because individuals are sampled with unequal probability by design or because of unequal detection probability. Size-biased distributions come into play when organisms occur in groups, and group size influences the probability of detection. Fisher (1934) [5] firstly introduced these distributions to model ascertainment biases which were later formalized by Rao (1965) [14] in a unifying theory for problems where the observations fall in non-experimental, non-replicated and non-random categories. Size-biased distributions have applications in environmental science, econometrics, social science, biomedical science, human demography, ecology, geology, forestry etc. Much works seem be done by different researchers on size-biased distributions and their applications including Patil and Ord (1976) [11], Patil and Rao (1977, 1978) [12, 13], Patil (1981) [10], Ducey and Gove (2015), some among others.
Using (1.1) and the mean of Poisson-Lindley distribution introduced by Sankaran (1970) [15], Ghitany and Al-Mutairi (2008) [6] proposed a size-biased Poisson Lindley distribution (SBPLD) having pmf

$$P_1(x;\theta)=\frac{\theta^3}{\theta+2}\frac{x(x+\theta+2)}{(\theta+1)^{x+2}}\ ;x=1,2,3,\ldots,\ \theta>0 \tag{1.2}$$

The mean and the variance of the SBPLD are given by

$$\mu_1'=\frac{\theta^2+4\theta+6}{\theta(\theta+2)}\ \text{And}\ \mu_2=\frac{2(\theta^3+6\theta^2+12\theta+6)}{\theta^2(\theta+2)^2}.$$

Shanker *et al.* (2015) [16] have a detailed study on SBPLD and its applications to model data relating to thunderstorms and observed that SBPLD is a suitable model for thunderstorms data.

The pmf of Poisson-weighted Lindley distribution (P-WLD) introduced by Abd El-Monsef and Sohsah (2014) [1] is given by

$$P_2(x;\theta,\alpha)=\frac{\Gamma(x+\alpha)}{\Gamma(x+1)\Gamma(\alpha)}\frac{\theta^{\alpha+1}}{(\theta+\alpha)}\frac{x+\theta+\alpha+1}{(\theta+1)^{x+\alpha+1}};x=0,1,2,\ldots,\theta>0,\alpha>0. \tag{1.3}$$

Shanker and Shukla (2017) [18] have detailed study on various descriptive statistics, estimation of parameters and applications of P-WLD. The mean and the variance of P-WLD obtained by Shanker and Shukla (2017) [18] are

$$\mu_1'=\frac{\alpha(\theta+\alpha+1)}{\theta(\theta+\alpha)} \tag{1.4}$$

$$\mu_2=\frac{\alpha\left\{\theta^3+2(\alpha+1)\theta^2+(\alpha^2+3\alpha+2)\theta+(\alpha^2+\alpha)\right\}}{\theta^2(\theta+\alpha)^2} \tag{1.5}$$

Note that P-WLD is a Poisson mixture of a two-parameter weighted Lindley distribution, introduced by Ghitany *et al.* (2011) [7] is defined by its pdf

$$f_1(x;\theta,\alpha)=\frac{\theta^{\alpha+1}}{(\theta+\alpha)}\frac{x^{\alpha-1}}{\Gamma(\alpha)}(1+x)e^{-\theta x}\ ;x>0,\ \theta>0,\ \alpha>0 \tag{1.6}$$

Where, $\Gamma(\alpha)=\int_0^\infty e^{-y}y^{\alpha-1}dy;\ \alpha>0$ (1.7)

Is the complete gamma function.

It can be easily shown that Lindley distribution introduced by Lindley (1958) [9] is a particular case of WLD at $\alpha=1$. Shanker *et al.* (2016) [17] discussed various moments based properties including coefficient of variation, coefficient of skewness, coefficient of kurtosis and index of dispersion of weighted Lindley distribution and its applications to model lifetime data from biomedical sciences and engineering.

The main motivation of this paper is to propose a two-parameter size-biased Poisson- weighted Lindley distribution (SBP-WLD), a size-biased version of Poisson-Weighted Lindley distribution (P-WLD), to model count data excluding zero counts because two-parameter SBP-WLD have enough flexibility than one parameter size-biased Poisson distribution and SBPLD. Various moments and moments based measures have been obtained. The nature and behavior of coefficients of variation, skewness, kurtosis, index of dispersion have been explained graphically. Maximum likelihood estimation has been discussed for estimating the parameters of the distribution. Applications and the goodness of fit of the proposed distribution have been explained through datasets relating to some real count datasets and compared with other size-biased distributions.

2. Size-Biased Poisson-Weighted Lindley Distribution

Using the pmf of P-WLD (1.3) and its mean from (1.4) in (1.1), the pmf of size-biased Poisson-weighted Lindley distribution (SBP-WLD) can be obtained as

$$P_3(x;\theta,\alpha)=\frac{\Gamma(x+\alpha)}{\Gamma(x)\Gamma(\alpha+1)}\frac{\theta^{\alpha+2}}{(\theta+\alpha+1)}\frac{x+\theta+\alpha+1}{(\theta+1)^{x+\alpha+1}};x=1,2,3,\ldots,\theta>0,\alpha>0 \tag{2.1}$$

At $\alpha = 1$, SBP-WLD reduces to SBPLD (1.2). Since it is difficult and complicated to obtain the moments of SBP-WLD directly, an attempt has been made to derive the pmf of SBP-WLD as a size-biased Poisson mixture of size-biased weighted Lindley distribution (SB-WLD) which is very much helpful in deriving the moments. Suppose the parameter λ of size-biased Poisson distribution (SBPD) with pmf

$$g(x|\lambda) = \frac{e^{-\lambda}\lambda^{x-1}}{\Gamma(x)} \quad ; x = 1,2,3,...,; \lambda > 0 \tag{2.2}$$

Follows size-biased weighted Lindley distribution (SB-WLD) with pdf

$$h(\lambda;\theta,\alpha) = \frac{\theta^{\alpha+2}}{(\theta+\alpha+1)} \frac{\lambda^{\alpha}}{\Gamma(\alpha+1)}(1+\lambda)e^{-\theta\lambda} \quad ; \lambda > 0,\ \theta > 0, \alpha > 0 \tag{2.3}$$

Thus the SBPD mixture of SB-WLD can be obtained as

$$P(X = x) = \int_0^\infty g(x|\lambda)\cdot h(\lambda;\theta,\alpha)\, d\lambda$$

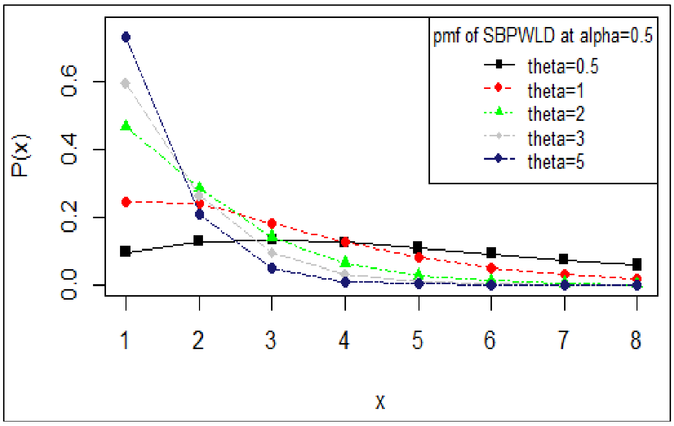
$$= \int_0^\infty \frac{e^{-\lambda}\lambda^{x-1}}{\Gamma(x)} \frac{\theta^{\alpha+2}}{(\theta+\alpha+1)} \frac{\lambda^{\alpha}}{\Gamma(\alpha+1)}(1+\lambda)e^{-\theta\lambda}\, d\lambda \tag{2.4}$$

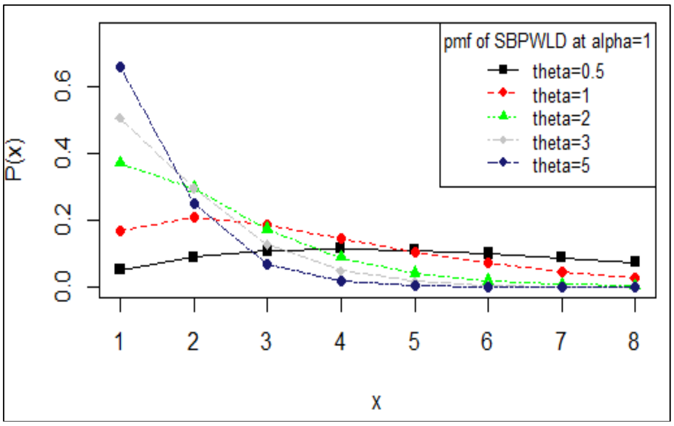
$$= \frac{\theta^{\alpha+2}}{(\theta+\alpha+1)\Gamma(\alpha+1)\Gamma(x)} \int_0^\infty e^{-(\theta+1)\lambda}\lambda^{x+\alpha-1}(1+\lambda)\, d\lambda$$

$$= \frac{\theta^{\alpha+2}}{(\theta+\alpha+1)\Gamma(\alpha+1)\Gamma(x)} \left[\frac{\Gamma(x+\alpha)}{(\theta+1)^{x+\alpha}} + \frac{\Gamma(x+\alpha+1)}{(\theta+1)^{x+\alpha+1}}\right]$$

$$= \frac{\Gamma(x+\alpha)}{\Gamma(x)\Gamma(\alpha+1)} \frac{\theta^{\alpha+2}}{(\theta+\alpha+1)} \frac{x+\theta+\alpha+1}{(\theta+1)^{x+\alpha+1}}; x = 1,2,3,...$$

Which is the pmf of SBP-WLD obtained in (2.1). The nature and behavior of SBP-WLD has been shown graphically for varying values of parameters θ and α in figure 1.





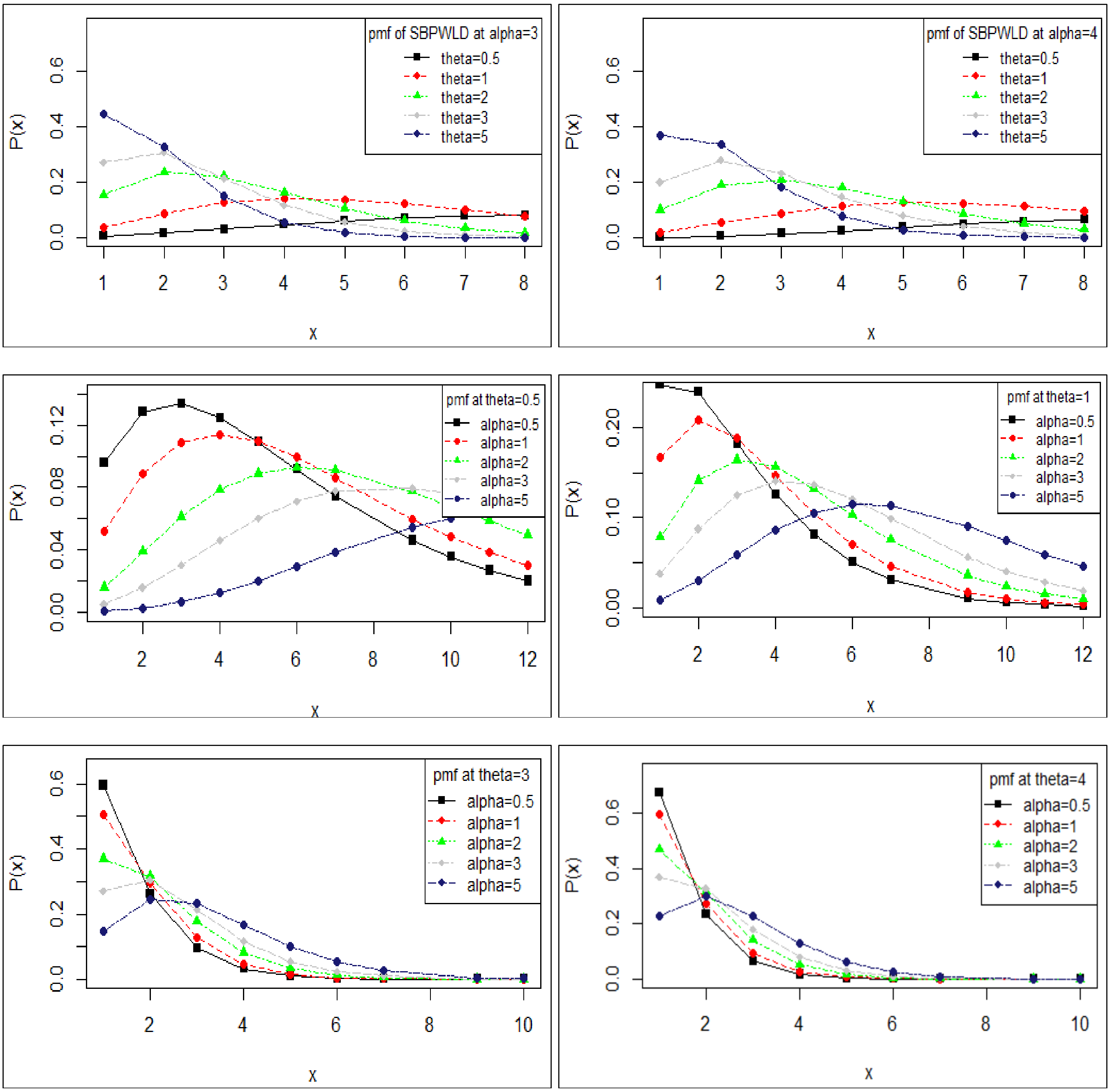


Fig 1: Probability mass function of SBP-WLD for varying values of parameters θ and α .

3. Statistical Constants

Using (2.4), the r th factorial moment about origin of the SBP-WLD (2.1) can be obtained as

$$\mu_{(r)}' = E\left[E\left(X^{(r)} \mid \lambda\right)\right], \text{ where } X^{(r)} = X(X-1)(X-2)\ldots(X-r+1)$$

$$= \int_0^{\infty}\left[\sum_{x=1}^{\infty} x^{(r)} \frac{e^{-\lambda}\lambda^{x-1}}{\Gamma(x)}\right]\frac{\theta^{\alpha+2}}{(\theta+\alpha+1)}\frac{\lambda^{\alpha}}{\Gamma(\alpha+1)}(1+\lambda)e^{-\theta\lambda}d\lambda$$

$$= \int_0^{\infty}\left[\lambda^{r-1}\left\{\sum_{x=r}^{\infty} x\frac{e^{-\lambda}\lambda^{x-r}}{(x-r)!}\right\}\right]\frac{\theta^{\alpha+2}}{(\theta+\alpha+1)}\frac{\lambda^{\alpha}}{\Gamma(\alpha+1)}(1+\lambda)e^{-\theta\lambda}d\lambda$$

Taking $x-r=y$, we get

$$\mu_{(r)}' = \int_0^\infty \left[\lambda^{r-1} \left\{ \sum_{y=0}^{\infty} (y+r) \frac{e^{-\lambda} \lambda^y}{y!} \right\} \right] \frac{\theta^{\alpha+2}}{(\theta+\alpha+1)} \frac{\lambda^\alpha}{\Gamma(\alpha+1)} (1+\lambda) e^{-\theta\lambda} d\lambda$$

$$= \int_0^\infty \lambda^{r-1} (\lambda+r) \frac{\theta^{\alpha+2}}{(\theta+\alpha+1)} \frac{\lambda^\alpha}{\Gamma(\alpha+1)} (1+\lambda) e^{-\theta\lambda} d\lambda$$

$$= \frac{\theta^{\alpha+2}}{(\theta+\alpha+1)\Gamma(\alpha+1)} \int_0^\infty \lambda^{r-1} (\lambda+r) \lambda^\alpha (\alpha+\lambda) e^{-\theta\lambda} d\lambda$$

$$= \frac{\theta^{\alpha+2}}{(\theta+\alpha+1)\Gamma(\alpha+1)} \int_0^\infty \lambda^{r+\alpha-1} \left[\lambda^2 + (r+1)\lambda + r \right] e^{-\theta\lambda} d\lambda$$

Using gamma integral and a little algebraic simplification, the r th factorial moment about origin of SBP-WLD (2.1) can be obtained as

$$\mu_{(r)}' = \frac{\Gamma(\alpha+r)}{\Gamma(\alpha+1)} \frac{\left\{ r\theta^2 + (r+1)(\alpha+r)\theta + (\alpha+r)(\alpha+r+1) \right\}}{\theta^r (\theta+\alpha+1)} \quad ; r = 1, 2, 3, \ldots. \tag{3.1}$$

Taking $r = 1, 2, 3, \text{and } 4$ in (3.1), the first four factorial moments about origin of SBP-WLD (2.1) can be obtained

$$\mu_{(1)}' = \frac{\theta^2 + 2(\alpha+1)\theta + (\alpha+1)(\alpha+2)}{\theta(\theta+\alpha+1)}$$

$$\mu_{(2)}' = \frac{(\alpha+1)\left\{ 2\theta^2 + 3(\alpha+2)\theta + (\alpha+2)(\alpha+3) \right\}}{\theta^2(\theta+\alpha+1)}$$

$$\mu_{(3)}' = \frac{(\alpha+1)(\alpha+2)\left\{ 3\theta^2 + 4(\alpha+3)\theta + (\alpha+3)(\alpha+4) \right\}}{\theta^3(\theta+\alpha+1)}$$

$$\mu_{(4)}' = \frac{(\alpha+1)(\alpha+2)(\alpha+3)\left\{ 4\theta^2 + 5(\alpha+4)\theta + (\alpha+4)(\alpha+5) \right\}}{\theta^4(\theta+\alpha+1)}.$$

Now using the relationship between factorial moments about origin and the moments about origin, the first four moments about origin of SBP-WLD (2.1) can be obtained as

$$\mu_1' = \frac{\theta^2 + 2(\alpha+1)\theta + (\alpha+1)(\alpha+2)}{\theta(\theta+\alpha+1)}$$

$$\mu_2' = \frac{\theta^3 + 4(\alpha+1)\theta^2 + 4\left(\alpha^2+3\alpha+2\right)\theta + \left(\alpha^3+6\alpha^2+11\alpha+6\right)}{\theta^2(\theta+\alpha+1)}$$

$$\mu_3' = \frac{\left\{ \begin{array}{l} \theta^4 + 8(\alpha+1)\theta^3 + 13\left(\alpha^2+3\alpha+2\right)\theta^2 + 7\left(\alpha^3+6\alpha^2+11\alpha+6\right)\theta \\ + \left(\alpha^4+10\alpha^3+35\alpha^2+50\alpha+24\right) \end{array} \right\}}{\theta^3(\theta+\alpha+1)}$$

$$\mu_4' = \frac{\left\{\begin{array}{l}\theta^5 + 16(\alpha+1)\theta^4 + 40(\alpha^2+3\alpha+2)\theta^3 + 35(\alpha^3+6\alpha^2+11\alpha+6)\theta^2 \\ +11(\alpha^4+10\alpha^3+35\alpha^2+50\alpha+24)\theta + (\alpha^5+15\alpha^4+85\alpha^3+225\alpha^2+274\alpha+120)\end{array}\right\}}{\theta^4(\alpha\theta+\alpha+1)}$$

Now, using the relationship $\mu_r = E\left(Y-\mu_1'\right)^r = \sum_{k=0}^{r}\binom{r}{k}\mu_k'\left(-\mu_1'\right)^{r-k}$ between moments about mean and the moments about origin, the moments about mean of the SBP-WLD (2.1) can be obtained as

$$\mu_2 = \frac{(\alpha+1)\theta^3 + 2(\alpha^2+3\alpha+2)\theta^2 + (\alpha^3+6\alpha^2+11\alpha+6)\theta + (\alpha^3+4\alpha^2+5\alpha+2)}{\theta^2(\theta+\alpha+1)^2}$$

$$\mu_3 = \frac{\left\{\begin{array}{l}(\alpha+1)\theta^5 + (3\alpha^2+10\alpha+7)\theta^4 + (3\alpha^3+20\alpha^2+39\alpha+22)\theta^3 \\ +(\alpha^4+14\alpha^3+51\alpha^2+70\alpha+32)\theta^2 + (3\alpha^4+21\alpha^3+51\alpha^2+51\alpha+18)\theta \\ +(2\alpha^4+10\alpha^3+18\alpha^2+14\alpha+4)\end{array}\right\}}{\theta^3(\theta+\alpha+1)^3}$$

$$\mu_4 = \frac{\left\{\begin{array}{l}(\alpha+1)\theta^7 + (7\alpha^2+22\alpha+15)\theta^6 + (18\alpha^3+97\alpha^2+166\alpha+87)\theta^5 \\ +(22\alpha^4+175\alpha^3+500\alpha^2+605\alpha+258)\theta^4 \\ +(13\alpha^5+148\alpha^4+628\alpha^3+1246\alpha^2+1159\alpha+406)\theta^3 \\ +(3\alpha^6+55\alpha^5+348\alpha^4+1034\alpha^3+1567\alpha^2+1167\alpha+338)\theta^2 \\ +(6\alpha^6+72\alpha^5+336\alpha^4+780\alpha^3+954\alpha^2+588\alpha+144)\theta \\ +(3\alpha^6+30\alpha^5+114\alpha^4+216\alpha^3+219\alpha^2+114\alpha+24)\end{array}\right\}}{\theta^4(\theta+\alpha+1)^4}$$

The coefficient of variation $(C.V)$, coefficient of Skewness $\left(\sqrt{\beta_1}\right)$, coefficient of Kurtosis (β_2) and index of dispersion (γ) of the SBP-WLD (2.1)) are thus obtained as

$$C.V = \frac{\sigma}{\mu_1'} = \frac{\sqrt{(\alpha+1)\theta^3 + 2(\alpha^2+3\alpha+2)\theta^2 + (\alpha^3+6\alpha^2+11\alpha+6)\theta + (\alpha^3+4\alpha^2+5\alpha+2)}}{\theta^2 + 2(\alpha+1)\theta + (\alpha+1)(\alpha+2)}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\left\{\begin{array}{l}(\alpha+1)\theta^5 + (3\alpha^2+10\alpha+7)\theta^4 + (3\alpha^3+20\alpha^2+39\alpha+22)\theta^3 \\ +(\alpha^4+14\alpha^3+51\alpha^2+70\alpha+32)\theta^2 + (3\alpha^4+21\alpha^3+51\alpha^2+51\alpha+18)\theta \\ +(2\alpha^4+10\alpha^3+18\alpha^2+14\alpha+4)\end{array}\right\}}{\left\{(\alpha+1)\theta^3 + 2(\alpha^2+3\alpha+2)\theta^2 + (\alpha^3+6\alpha^2+11\alpha+6)\theta + (\alpha^3+4\alpha^2+5\alpha+2)\right\}^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^{\,2}} = \frac{\left\{\begin{array}{l}(\alpha+1)\theta^7 + (7\alpha^2+22\alpha+15)\theta^6 + (18\alpha^3+97\alpha^2+166\alpha+87)\theta^5 \\ +(22\alpha^4+175\alpha^3+500\alpha^2+605\alpha+258)\theta^4 \\ +(13\alpha^5+148\alpha^4+628\alpha^3+1246\alpha^2+1159\alpha+406)\theta^3 \\ +(3\alpha^6+55\alpha^5+348\alpha^4+1034\alpha^3+1567\alpha^2+1167\alpha+338)\theta^2 \\ +(6\alpha^6+72\alpha^5+336\alpha^4+780\alpha^3+954\alpha^2+588\alpha+144)\theta \\ +(3\alpha^6+30\alpha^5+114\alpha^4+216\alpha^3+219\alpha^2+114\alpha+24)\end{array}\right\}}{\left\{(\alpha+1)\theta^3+2(\alpha^2+3\alpha+2)\theta^2+(\alpha^3+6\alpha^2+11\alpha+6)\theta+(\alpha^3+4\alpha^2+5\alpha+2)\right\}^2}$$

$$\gamma = \frac{\sigma^2}{\mu_1'} = \frac{(\alpha+1)\theta^3+2(\alpha^2+3\alpha+2)\theta^2+(\alpha^3+6\alpha^2+11\alpha+6)\theta+(\alpha^3+4\alpha^2+5\alpha+2)}{\theta(\theta+\alpha+1)\left\{\theta^2+2(\alpha+1)\theta+(\alpha+1)(\alpha+2)\right\}}.$$

Nature and behavior of coefficient of variation, coefficient of skewness, coefficient of kurtosis and index of dispersion of SBP-WLD for varying values of parameters θ and α are shown in figure-2.

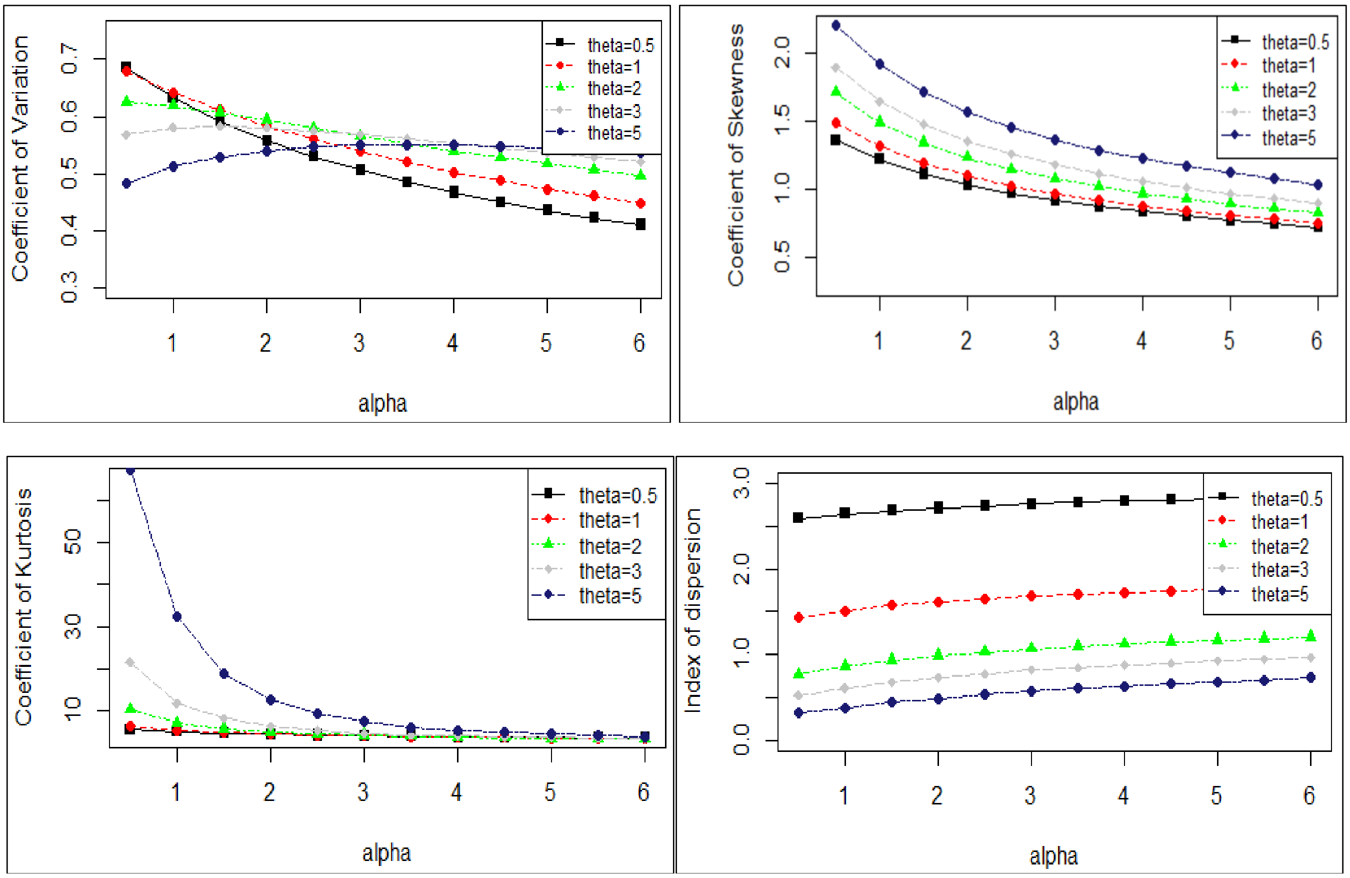


Fig 2: Nature of coefficient of variation, coefficient of skewness, coefficient of kurtosis and index of dispersion of SBP-WLD for varying values of parameters θ and α

4. Maximum Likelihood Estimation of Parameters

Suppose $(x_1, x_2, \ldots, x_n)$ be a random sample of size n from the SBP-WLD (2.1). Let f_x be the observed frequency in the sample corresponding to $X = x\ (x = 1, 2, 3, \ldots, k)$ such that $\sum_{x=1}^{k} f_x = n$, where k the largest is observed value having non-zero frequency. Thus, the log likelihood function of SBP-WLD can be obtained as

$$\log L = n\left[(\alpha+2)\log\theta - \log(\theta+\alpha+1) - \log\Gamma(\alpha+1)\right] - \sum_{x=1}^{k} f_x (x+\alpha+1)\log(\theta+1)$$
$$+\sum_{x=1}^{k} f_x\left[\log\Gamma(x+\alpha) - \log\Gamma(x)\right] + \sum_{x=1}^{k} f_x \log(x+\theta+\alpha+1)$$

The maximum likelihood estimates $(\hat{\theta},\hat{\alpha})$ of (θ,α) of SBP-WLD (2.2) is the solutions of the following log likelihood equations

$$\frac{\partial \log L}{\partial \theta} = \frac{n(\alpha+2)}{\theta} - \frac{n}{\theta+\alpha+1} - \sum_{x=1}^{k}\frac{(x+\alpha+1) f_x}{\theta+1} + \sum_{x=1}^{k}\frac{f_x}{x+\theta+\alpha+1} = 0$$

$$\frac{\partial \log L}{\partial \alpha} = n\log\theta - \frac{n}{\theta+\alpha+1} - n\psi(\alpha) + \sum_{x=1}^{k} f_x \psi(x+\alpha) - \sum_{x=1}^{k} x f_x \log(\theta+1) + \sum_{x=1}^{k}\frac{f_x}{x+\theta+\alpha+1} = 0,$$

Where $\psi(\alpha) = \frac{d}{d\alpha}\ln\Gamma(\alpha)$ is the digamma function.

These two log likelihood equations do not seem to be solved directly. However, the Fisher's scoring method can be applied to solve these equations. We have

$$\frac{\partial^2 \log L}{\partial \theta^2} = -\frac{n(\alpha+2)}{\theta^2} + \frac{n}{(\theta+\alpha+1)^2} + \sum_{x=1}^{k}\frac{(x+\alpha+1) f_x}{(\theta+1)^2} - \sum_{x=1}^{k}\frac{f_x}{(x+\theta+\alpha+1)^2}$$

$$\frac{\partial^2 \log L}{\partial \theta\,\partial \alpha} = \frac{n}{\theta} - \frac{n}{(\theta+\alpha+1)^2} - \sum_{x=1}^{k}\frac{f_x}{\theta+1} - \sum_{x=1}^{k}\frac{f_x}{(x+\theta+\alpha+1)^2} = \frac{\partial^2 \log L}{\partial \alpha\,\partial \theta}$$

$$\frac{\partial^2 \log L}{\partial \alpha^2} = \frac{n}{(\theta+\alpha+1)^2} - \psi'(\alpha) + \sum_{x=1}^{k} f_x \psi'(x+\alpha) - \sum_{x=1}^{k}\frac{f_x}{(x+\theta+\alpha+1)^2},$$

Where $\psi'(\alpha) = \frac{d}{d\alpha}\psi(\alpha)$ is the trigamma function.

The maximum likelihood estimates $(\hat{\theta},\hat{\alpha})$ of (θ,α) of SBP-WLD (2.2) is the solution of the following equations

$$\begin{bmatrix} \frac{\partial^2 \log L}{\partial \theta^2} & \frac{\partial^2 \log L}{\partial \theta\,\partial \alpha} \\ \frac{\partial^2 \log L}{\partial \theta\,\partial \alpha} & \frac{\partial^2 \log L}{\partial \alpha^2} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}} \begin{bmatrix} \hat{\theta}-\theta_0 \\ \hat{\alpha}-\alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial \log L}{\partial \theta} \\ \frac{\partial \log L}{\partial \alpha} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}}$$

Where θ_0 and α_0 are the initial values of θ and α, respectively. These equations are solved iteratively till sufficiently close values of $\hat{\theta}$ and $\hat{\alpha}$ are obtained.

5. Applications

In this section the goodness of fit of SBP-WLD has been presented for three real count datasets. The first dataset is immunogold assay data of Cullen *et al.* (1990) [3] regarding the distribution of number of counts of sites with particles from immunogold assay data, the second dataset is animal abundance data of Keith and Meslow (1968) [8] regarding the distribution of snowshoe hares captured over 7 days, and the third dataset is the data relating to the size distribution of freely-forming small groups at various public places available in Coleman and James (1961) [2]. The goodness of fit by SBPD and SBPLD has also been given for ready comparison. The fit of these distributions are based on maximum likelihood estimates of the parameters. The standard errors of parameters, values of $-2\log L$ and AIC (Akaike Information criterion) for all the considered distributions have also been given. Note that AIC has been calculated using the formula $AIC = -2\log L + 2k$, where k is the number of parameters involved in the respective distribution. Based on the values of chi-square (χ^2), $-2\log L$ and AIC, it is obvious that in three datasets SBP-

WLD gives much closer fit over SBPD and SBPLD, and hence it can be considered an important size-biased distributions. The probability plots of the fitted distributions for the three datasets have also been shown in figure 3.These plots also shows that SBP-WLD is a better model for the considered datasets.

Table 1: Distribution of number of counts of sites with particles from Immunogold data, available in Cullen *et al.* (1990) [3]

No. of sites with particles	Observed Frequency	Expected Frequency		
		SBPD	SBPLD	SBP-WLD
1	122	111.3	119.0	121.9
2	50	64.1	53.8	50.2
3	18	18.5	18.0	17.6
4	4	3.5	5.3	5.7
5	4	0.6	1.9	2.6
Total	198	198.0	198.0	198.0
ML estimate		$\hat{\theta} = 0.57576$	$\hat{\theta} = 4.05098$	$\hat{\theta} = 2.88584$ $\hat{\alpha} = 0.34390$
Standard Errors of parameters		0.053920	0.390586	1.16277 0.62969
$-2 \log L$		1009.87	409.28	355.30
AIC		1011.87	411.28	359.30
χ^2		4.64	0.43	0.01
D.F		1	2	1
p-value		0.0312	0.8065	0.9203

Table 2: Distribution of snowshoe hares captured over 7 days, available in Keith and Meslow (1968) [8]

No. times hares caught	Observed Frequency	Expected Frequency		
		SBPD	SBPLD	SBP-WLD
1	184	170.6	177.3	184.3
2	55	72.5	62.5	52.9
3	14	15.4	16.4	16.3
4	4	2.2	3.8	5.0
5	4	0.3	1.0	2.5
Total	261	261.0	261.0	261.0
ML estimate		$\hat{\theta} = 0.425287$	$\hat{\theta} = 5.351256$	$\hat{\theta} = 2.49343$ $\hat{\alpha} = -0.18668$
Standard Errors of parameters		0.04036	0.512568	0.81784 0.31548
$-2 \log L$		1185.80	457.10	405.54
AIC		1187.80	459.10	409.54
χ^2		6.22	1.18	0.44
D.F		1	1	1
p-value		0.0126	0.2773	0.5071

Table 3: Pedestrians-Eugene, spring, Morning, available in Coleman and James (1961) [2].

Group Size	Observed frequency	Expected frequency		
		SBPD	SBPLD	SBP-WLD
1	1486	1452.4	1532.5	1485.9
2	694	743.3	630.6	694.6
3	195	190.2	191.9	192.4
4	37	32.4	51.3	41.0
5	10	4.1	12.8	7.5
6	1	0.6	3.9	3.6
Total	2423	2423.0	2423.0	2423.0
ML Estimate		$\hat{\theta} = 0.5118$	$\hat{\theta} = 4.5082$	$\hat{\theta} = 10.98587$ $\hat{\alpha} = 4.29684$
Standard errors of parameters		0.014533	0.13050	3.82287 1.94498
$-2 \log L$		10445.34	4622.36	4152.56
AIC		10447.34	4624.36	4156.56
χ^2		7.370	13.760	0.42
D.F		2	3	2
p-value		0.0251	0.003	0.8081

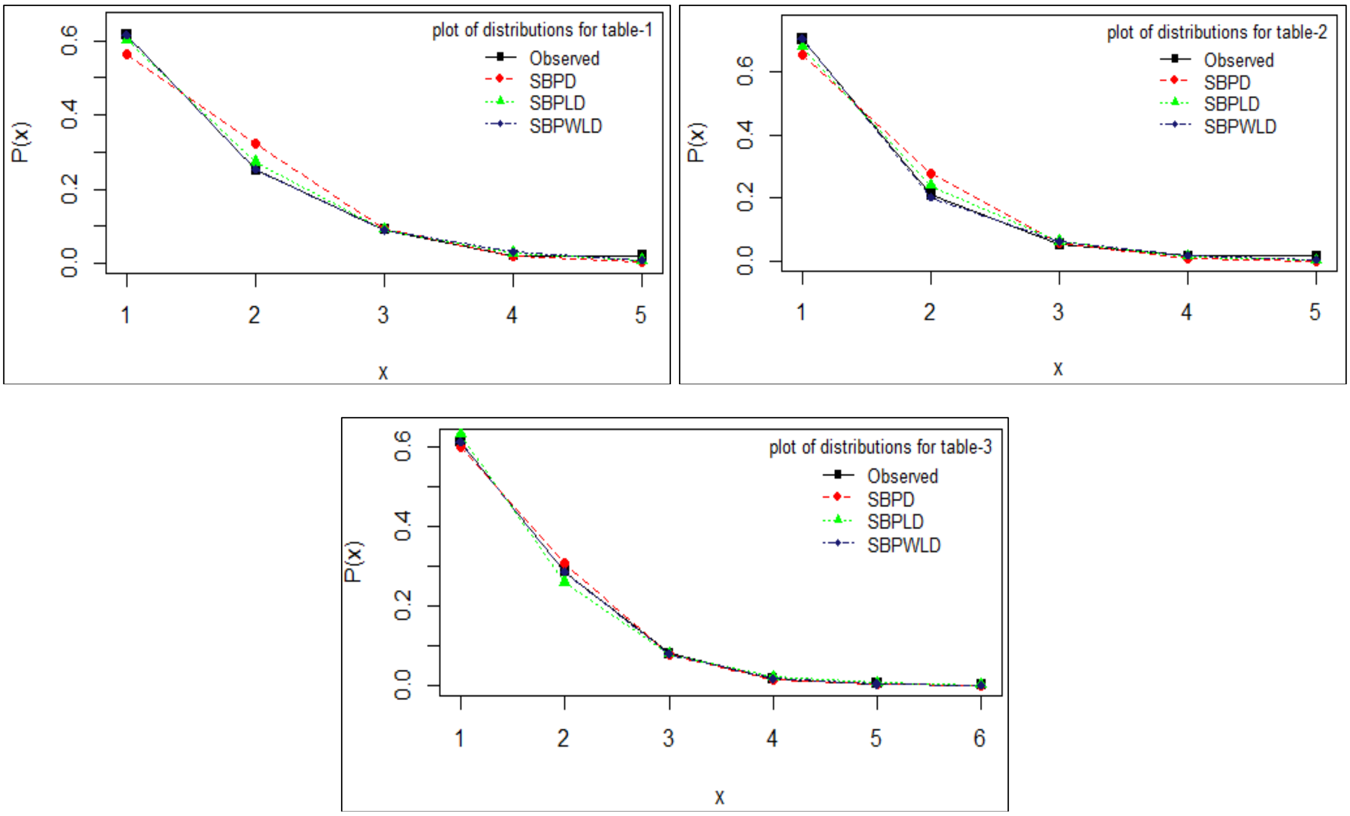


Fig 3: Fitted probability plot of distributions for datasets 1, 2, and 3

6. Concluding Remarks

In this paper a size-biased Poisson-weighted Lindley distribution (SBP-WLD) has been proposed and their nature for varying values of parameters have been studied. Its moments and moments based measures have been derived and their nature and behavior have been discussed graphically. Estimation of the parameters has been discussed using maximum likelihood. Applications of the proposed distribution have been explained through three examples of observed datasets and the goodness of fit shows much closer fit over SBPD and SBPLD.

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