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MSM fractional integration and differentiation operators of multi-parametric K-Mittag Leffler function and generalized multi-index Bessel function

Mridula Purohit and Aditi Badguzer

Abstract
In this paper, we aim at finding image of product of Multi-Parameter k-Mittag-Leffler Function and Generalized Multi-Index Bessel Function under the Marichev-Saigo-Maida (MSM) Fractional Integration Operators in terms of generalized k-Wright Function. Some useful special cases of our main results are also obtained.

Keywords: Multi-Parameter k-mittag-leffler function, multi-index Bessel function, generalized k-Wright function, MSM fractional integration and differentiation operators

Introduction

Generalized Multi-Index Bessel Function

A new generalization of Bessel function called Multi-Index Bessel Function is introduced by Nisar. K.S. [8] as

$$J^{(v_t)_s,\lambda,C}_{(\mu_t)_s,\omega,d}(Z) = \sum_{l=0}^{\infty} \frac{c^l (\lambda)_{\omega l}\, Z^l}{\prod_{t=1}^{s}\Gamma\left(v_t\, l+\mu_t+\frac{d+1}{2}\right)}\, \frac{1}{l\,!} \tag{1.1}$$

For $v_t, \mu_t, \lambda, d, c \,\epsilon\, C$ (t = 1, 2 …s) be such that $\sum_{t=1}^{s}$ Re (v_t) > max$\{0, Re(\omega)-1\}$; $\omega > 0$, Re$(\mu_t) > 0$ and Re $(\lambda) > 0$, & s ϵ N

Here $(\lambda)_{\omega l}$ denotes the Pochhammer symbol defined (for $\lambda, \omega l \,\epsilon\, C$) in terms of Gamma function Γ (see [12], section 1.1]) by

$$(\lambda)_{\omega l} = \frac{\Gamma(\lambda+\omega l)}{\Gamma(\lambda)} = \begin{Bmatrix} 1 & (\omega l = 0; \lambda \epsilon c \setminus \{0\}) \\ \lambda(\lambda+1)\ldots\ldots(\lambda+n-1) & (\omega l = n \epsilon N, \lambda \epsilon C) \end{Bmatrix} \tag{1.2}$$

Multi-Parameter K- Mittag-Leffler Function

Gehlot K.S. introduced Multi-Parameter k- Mittag-Leffler function [4] as

$${}_pE^{(\xi,\eta)_m}_{q\,k}(Z) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{P}(a_j)_{n,k}\, Z^n}{\prod_{r=1}^{q}(b_r)_{n,k} \prod_{i=1}^{m}\Gamma_k(n\,\eta_i+\xi_i)n!} \tag{1.3}$$

Where $\Gamma_k(x)$ is the generalized k - Gamma function [2] given by

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n!\,k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}, \text{ K} > 0, x \,\epsilon\, c \setminus k\bar{Z} \tag{1.4}$$

And $(x)_{n,k}$ is the k-pochhammer symbol defined as

$$(x)_{n,k} = x(x+\text{k})(x+2\text{k})\ldots..(x+(n-1)k), x\epsilon C, k\epsilon R, n\epsilon N^+ \tag{1.5}$$

For Re(x) > 0, Γ_k (x) is defined as the integral

$$\Gamma_k(x) = \int_0^\infty t^{x-1} \, e^{\frac{-t^k}{k}} dt \tag{1.6}$$

From (1.4) it follows that $\Gamma_k(x) = k^{\frac{x}{k}-1}\, \Gamma\left(\frac{x}{k}\right)$ (1.7)

Generalized K-Wright Function

For Z, $a_i, b_j \epsilon c$ and $\alpha_j, \beta_j \epsilon$ R ($\alpha_j, \beta_j \neq 0$; $i = 1, 2, .. p$; j = 1,2, …q) the generalized k-Wright Function [3] is represented as

$${}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{matrix}(a_i, \alpha_j)_{1,P} \\ (b_j, \beta_j)_{1,q}\end{matrix} \mid Z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{P}\Gamma(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j n)} \frac{Z^n}{n!} \tag{1.8}$$

Gehlot. K.S. and Prajapati [5] defined the generalized k- Wright Function as ${}_p\Psi_q^k(z)$ for a_i,b_j, z ϵ C, k ϵ R^+, $\alpha_i, \beta_j \in R\big(\alpha_i, \beta_j \neq 0; i = 1,2, \ldots .p \,; j = 1, 2, \ldots . q\big)$ and $(a_i + \alpha_i n)$, $(b_j + \beta_j n)$ ϵ $C \setminus k\bar{Z}$, as

$${}_p\Psi_q^k(z) = {}_p\Psi_q^k(z)\left[\begin{matrix}(a_i, \alpha_i)_{1,P} \\ (b_j, \beta_j)_{1,q}\end{matrix} \mid Z\right] \sum_{n=o}^{\infty} \frac{\prod_{i=1}^{P}\Gamma_k(a_i+\alpha_i n)}{\prod_{j=1}^{q}\Gamma_k(b_j+\beta_j n))} \frac{Z^n}{n!} \tag{1.9}$$

Marichev-Saigo-Maida (Msm) Fractional Integration and Differential Operators

Saigo operators [3] which is an useful generalization of the hyper-geometric fractional operators has been introduced by Marichev [7] (see details in samko *et al.* [11], P-194, (10.47) and whole section 10.3) and later extended and studied by Saigo and Maeda [10] in terms of any complex order with Appell function $F_3(.)$ in the kernel as follows:

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)}\int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3\left(\alpha, \alpha', \beta, \beta', \gamma, 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)dt, Re(\gamma) > 0 \tag{1.10}$$

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)}\int_x^\infty (t-x)^{\gamma-1}\, t^{-\alpha} F_3\left(\alpha, \alpha', \beta, \beta', \gamma, 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)dt, \mathrm{Re}(\gamma) > 0 \tag{1.11}$$

$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{0,+}^{-\alpha',-\alpha,\beta',-\beta,-\gamma} f\right)(x)\ (Re(\gamma) > 0) \tag{1.12}$$

$$= \left(\frac{d}{dx}\right)^k \left(I_{0,+}^{-\alpha',-\alpha,\beta'+k,-\beta,-\gamma+k} f\right)(x), (\mathrm{Re}\,(\gamma) > 0), k = [\mathrm{Re}(\gamma) + 1]$$

And

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{-}^{-\alpha',\alpha,\beta',-\beta,-\gamma} f\right)(x)\ (Re(\gamma) > 0) \tag{1.13}$$

$$= \left(\frac{d}{dx}\right)^k \left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta+k,-\gamma+k} f\right)(x), (\mathrm{Re}\,(\gamma) > 0), k = [\mathrm{Re}(\gamma) + 1]$$

These operators (1.10)-(1.13) reduce to the Saigo fractional calculus operators [9] as

$$\left(I_{0,+}^{\alpha,0,\beta,\beta',\gamma} f\right)(x) = \left(I_{0,+}^{\gamma,\alpha-\gamma,-\beta,} f\right)(x)\ (\gamma \epsilon c) \tag{1.14}$$

$$\left(I_{-}^{\alpha,0,\beta,\beta',\gamma} f\right)(x) = \left(I_{-}^{\gamma,\alpha-\gamma,-\beta,} f\right)(x)\ (\gamma \epsilon c) \tag{1.15}$$

$$\left(D_{0,+}^{0,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(D_{0,+}^{\gamma,\alpha'-\gamma,\beta'-\gamma,} f\right)(x)\ Re(\gamma) > 0 \tag{1.16}$$

$$\left(D_{-}^{0,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(D_{-}^{\gamma,\alpha'-\gamma,\beta'-\gamma,} f\right)(x)\ Re(\gamma) > 0 \tag{1.17}$$

The left-hand sided and right-hand sided generalized integration of the type (1.10) and (1.11) for a power function formula | see [11], P-394, eq (4.18) and (4.19) are given by

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\sigma-1}\right)(x) = \frac{\Gamma(\sigma)\Gamma(\sigma+\gamma-\alpha-\alpha'-\beta)\Gamma(\sigma+\beta'-\alpha')}{\Gamma(\sigma+\beta)\Gamma(\sigma+\gamma-\alpha-\alpha')\Gamma(\sigma+\gamma-\alpha'-\beta)} x^{\sigma-\alpha-\alpha'+\gamma-1} \tag{1.18}$$

where $\mathrm{Re}(\gamma) > 0$, $\mathrm{Re}(\sigma) > \max\{0, \mathrm{Re}(\alpha+\alpha'+\beta-\gamma), Re(\alpha'-\beta')\}$

And $\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\sigma-1}\right)(x) = \frac{\Gamma(1-\sigma-\gamma+\alpha+\alpha')\Gamma(1-\sigma+\alpha+\beta'-\gamma)\Gamma(1-\sigma-\beta)}{\Gamma(1-\sigma)\Gamma(1-\sigma+\alpha+\alpha'+\beta'-\gamma)\Gamma(1-\sigma+\alpha-\beta)} x^{\sigma-\alpha-\alpha'+\gamma-1}$ (1.19) where Re$(\gamma) > 0$,Re$(\sigma) < 1+$ min { Re$(-\beta)$, Re$(\alpha+\alpha'-\gamma), Re(\alpha+\beta'-\gamma)\}$.

2. Generalized Fractional Integration and Differentiation of Product of Multi-Parameter K-Mittag-Leffler Function and Generalized Multi-Index Bessel Function

Here, in this section we find images of product of Multi-Parameter k-Mittag Leffler Function and Generalized Multi-Index Bessel Function under the MSM operators in terms of generalized k-Wright Function. These formulas are given by the following theorem.

Theorem 1

If $\alpha,\alpha',\beta,\beta',\gamma,\sigma,v_t,u_t,\lambda,d,c,a_j,b_j,\epsilon C, k>0,\alpha_j,\beta_j,\in R$ and $x > 0$ be such that $\sum_{t=1}^{s} Re(v_t) > \max\{0, Re(k)-1\}$; $k > 0, Re(\mu_t) > 0$, Re$(\lambda) > 0$, Re (γ) >0 and Re $\left(\frac{\sigma}{k}\right) > max[0, Re(\alpha+\alpha'+\beta-\gamma), Re\ (\alpha'-\beta')]$

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[Z^{\frac{\sigma}{k}-1}\ {}_pE_{q\,k}^{(\xi,\eta)m}\ \left(Z^{\frac{\rho}{k}}\right).J_{(\mu_t)_s,\omega,d}^{(v_t)_s,\lambda,c}\ \left(Z^{\frac{\rho'}{k}}\right)\right]\right)(x)$$

$$= \frac{x^{\frac{\sigma}{k}-\alpha-\alpha'+\gamma-1}\prod_{r=1}^{q}\Gamma_k(b_r)}{k^{\gamma}\prod_{j=1}^{p}\Gamma_k(a_j)}\sum_{l=o}^{\infty}\frac{C^l(\lambda)_{\omega l}}{\prod_{t=1}^{s}(v_t l+\mu_t+\frac{d+1}{2})}\frac{x^{\frac{\rho' l}{k}}}{l!}$$

$${}_{p+3}\Psi_{q+m+3}^{k}\left[\begin{matrix}(a_i,k)_{1,P};(\sigma+\rho' l,\rho);(\sigma+\rho' l+k\gamma-k\alpha-k\alpha'-k\beta,\rho);\\(b_r,k)_{1,q}\,;(\xi_i,\eta_i)_{1,m};(\sigma+\rho' l+k\beta',\rho);(\sigma+\rho' l+k\gamma-k\alpha-k\alpha',\rho);\end{matrix}\right.$$

$$\left.\begin{matrix}(\sigma+\rho' l+k\beta'-k\alpha',\rho)\\(\sigma+\rho' l+k\gamma-k\alpha'-k\beta,\rho)\end{matrix}\ ;\ x^{\frac{\rho}{k}}\right] \tag{2.1}$$

Proof: Using definitions (1.1), (1.3) & (1.10) and changing the order of integration and summation in LHS of (2.1) it reduces to

$$\sum_{n,l=0}^{\infty}\frac{\prod_{j=1}^{P}(a_j)_{n,k}}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=1}^{m}\Gamma_k\,(n\,\eta_i+\ \xi_i)n!}.\frac{C^l(\lambda)_{\omega l}}{\prod_{t=1}^{s}\Gamma\left(v_t\ l+\mu_t+\frac{d+1}{2}\right)}\frac{1}{l\,!}\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[Z^{\frac{\sigma+\rho n+\rho' l}{k}-1}\right]\right)(x)$$

Using (1.18) we get

$$= \sum_{n,l=0}^{\infty}\frac{\prod_{j=1}^{P}(a_j)_{n,k}}{\prod_{r=1}^{q}(b_r)_{n,k}\prod_{i=1}^{m}\Gamma_k\,(n\,\eta_i+\xi_i)n!}.\frac{C^l(\lambda)_{\omega l}}{\prod_{t=1}^{s}\Gamma\left(v_t\ l+\mu_t+\frac{d+1}{2}\right)}\frac{1}{l\,!}\,x^{\frac{\sigma+\rho n+\rho' l}{k}-\alpha-\alpha^1+\gamma-1}$$

$$.\frac{\Gamma\left(\frac{\rho n+\rho' l+\sigma}{k}\right)\Gamma\left(\frac{\sigma+\rho n+\rho' l+\gamma}{k}-\alpha-\alpha'-\beta\right)\Gamma\left(\frac{\sigma+\rho n+\rho' l}{k}+\beta'-\alpha'\right)}{\Gamma\left(\frac{\sigma+\rho n+\rho' l}{k}+\beta'\right)\Gamma\left(\frac{\sigma+\rho n+\rho' l}{k}+\gamma-\alpha-\alpha'\right)\Gamma\left(\frac{\sigma+\rho n+\rho' l}{k}+\gamma-\alpha'-\beta\right)}$$

Using (1.7) and definition of generalized k-Wright Function from (1.9) we get the desired result.

Theorem 2

Let $\alpha,\alpha',\beta,\beta',\gamma,\sigma,v_t,u_t,\lambda,d,c,a_j,b_j,\epsilon\ C,k>0,\alpha_j,\beta_j,\in R$ and $x > 0$ be such that

$\sum_{t=1}^{s} Re(v_t) >$ max $\{0, Re(k)-1\}; k>0, Re(\mu_t)>0$, Re$(\lambda) > 0$, Re $(\gamma) > 0$ and

Re $\left(\frac{\sigma}{k}\right) > max[0, Re(\alpha+\alpha'+\beta-\gamma), Re\ (\alpha'-\beta')]$ then

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left[Z^{\frac{\sigma}{k}-1}\ {}_pE_{q\,k}^{(\xi,\eta)m}\ \left(Z^{\frac{\rho}{k}}\right).J_{(\mu_t)_s,\omega,d}^{(v_t)_s,\lambda,c}\ \left(Z^{\frac{\rho'}{k}}\right)\right]\right)(x)$$

$$= \frac{x^{\frac{\sigma}{k}-\alpha-\alpha'+\gamma-1}\prod_{r=1}^{q}\Gamma_k(b_r)}{k^{\gamma}\prod_{j=1}^{p}\Gamma_k(a_j)}\sum_{l=o}^{\infty}\frac{C^l(\lambda)_{\omega l}}{\prod_{t=1}^{s}(v_t l+\mu_t+\frac{d+1}{2})}\frac{x^{\frac{\rho' l}{k}}}{l!}.$$

$${}_{p+3}\Psi_{q+m+3}^{k}\left[\begin{matrix}(a_j,k)_{1,P};(1-\sigma-\rho' l-k\beta,\rho);(1-\sigma-\rho' l-k\gamma+k\alpha+k\alpha',\rho);\\(b_r,k)_{1,q};(\xi_i,\eta_i)_{1,m};(1-\sigma-\rho' l,\rho);(1-\sigma-\rho' l-k\gamma+k\alpha+k\alpha'+k\beta',\rho);\end{matrix}\right.$$

$$\left.\begin{matrix}(1-\sigma-\rho' l+k\gamma+k\alpha+k\beta',\rho);\\(1-\sigma-\rho' l-k\beta+k\alpha,\rho);\end{matrix}\ x^{\frac{\rho}{k}}\right] \tag{2.2}$$

Proof: Using (1.1), (1.3) & (1.11) and changing the order of integration and summation in LHS of (2.2), and preceding as same as Theorem 1, we arrive at RHS of (2.2).

Theorem 3
Let $\alpha,\alpha',\beta,\beta',\gamma,\sigma,v_t,u_t,\lambda,d,c,a_j,b_j,\epsilon\, C, k>0,\alpha_j,\beta_j,\in R$ and $x>0$ be such that $\sum_{t=1}^{s} Re(v_t) > \max\{0, Re(k)-1\}; k>0, Re(\mu_t)>0, \mathrm{Re}(\lambda) > 0$ Re $(\gamma)>0$ and Re $\left(\frac{\sigma}{k}\right) > max[0, Re(\alpha+\alpha'+\beta-\gamma), Re\,(\alpha'-\beta')]$

Then there holds the following formula

$$\left(D_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[Z^{\frac{\sigma}{k}-1}\ {}_pE_{q\,k}^{(\xi,\eta)m}\left(Z^{\frac{\rho}{k}}\right).J_{(\mu_t)_s,\omega,d}^{(v_t)_s,\lambda,c}\left(Z^{\frac{\rho'}{k}}\right)\right]\right)(x)$$

$$= \frac{x^{\frac{\sigma}{k}+\alpha+\alpha'-\gamma-1}\ \prod_{r=1}^{q}\Gamma_{\mathrm{K}}(\mathrm{b}_r)}{k^{\gamma}\ \prod_{j=1}^{p}\Gamma_{\mathrm{K}}(\mathrm{a}_j)}\sum_{l=o}^{\infty}\frac{C^l(\lambda)_{\omega l}}{\prod_{t=1}^{s}(v_t l+\mu_t+\frac{d+1}{2})}\frac{x^{\frac{\rho' l}{k}}}{l!}$$

$${}_{p+3}\Psi_{q+m+3}^{k}\left[\begin{matrix}(a_j,k)_{1,P};(\sigma+\rho' l,\rho);(\sigma+\rho' l-k\gamma+k\alpha+k\alpha'+k\beta',\rho);\\(b_r,k)_{1,q;}(\xi_i\ \eta_i)_{1,m};(\sigma+\rho' l-k\beta,\rho);(\sigma+\rho' l-k\gamma+k\alpha+k\alpha',\rho);\end{matrix}\right.$$

$$\left.\begin{matrix}(\sigma+\rho' l+k\alpha-k\beta,\rho);\\(\sigma+\rho' l-k\gamma+k\alpha+k\beta',\rho);\end{matrix}\ x^{\frac{\rho}{k}}\right] \tag{2.3}$$

Proof: Similar to as of Theorem 1 & 2.

Theorem 4
Let $\alpha,\alpha',\beta,\beta',\gamma,\sigma,v_t,u_t,\lambda,d,c,a_j,b_j,\epsilon\, C, k>0,\alpha_j,\beta_j,\in R$ and $x>0$ be such that $\sum_{t=1}^{s} Re(v_t) > \max\{0, Re(k)-1\}; k>0, Re(\mu_t)>0, \mathrm{Re}(\lambda) > 0$ Re $(\gamma)>0$ and Re $\left(\frac{\sigma}{k}\right) > max[0, Re(\alpha+\alpha'+\beta-\gamma), Re\,(\alpha'-\beta')]$

Then the following formula holds true

$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left[Z^{\frac{\sigma}{k}-1}\ {}_pE_{q,k}^{(\xi,\eta)m}\left(Z^{\frac{\rho}{k}}\right)J_{(\mu_t)_s,\omega,d}^{(v_t)_s,\lambda,c}\left(Z^{\frac{\rho'}{k}}\right)\right]\right)(x)$$

$$= \frac{x^{\frac{\sigma}{k}+\alpha+\alpha'-\gamma-1}\ \prod_{r=1}^{q}\Gamma_{\mathrm{k}}(\mathrm{b}_r)}{k^{\gamma}\ \prod_{j=1}^{p}\Gamma_{\mathrm{k}}(\mathrm{a}_j)}\sum_{l=o}^{\infty}\frac{C^l(\lambda)_{\omega l}}{\prod_{t=1}^{s}\left(v_t l+\mu_t+\frac{d+1}{2}\right)}\frac{x^{\frac{\rho' l}{k}}}{l!}.$$

$${}_{p+3}\Psi_{q+m+3}^{k}\left[\begin{matrix}(a_j,k)_{1,P};(1-\sigma-\rho' l+k\beta',\rho);(1-\sigma-\rho' l+k\gamma-k\alpha-k\alpha',\rho)\\(b_r,k)_{1,q};(\xi_i\ ,\eta_i)_{1,m};(1-\sigma-\rho' l,\rho);(1-\sigma-\rho' l+k\gamma-k\alpha-k\alpha'-k\beta,\rho)\end{matrix}\right.$$

$$\left.\begin{matrix}(1-\sigma-\rho' l+k\gamma-k\alpha'-k\beta,\rho);\\(1-\sigma-\rho' l+k\beta'-k\alpha',\rho);\end{matrix}\ x^{\rho/k}\right] \tag{2.4}$$

Proof: Similar to as of Theorem 1 & 2.

Special Cases of Theorems
(i) Assuming k = 1, p = q = 1, $a_1=\tau$ and $b_1 = 1$ in the main result (2.1) multi-parameter k-Mittag Leffler function reduces to generalized Mittag- Leffler studied by [6], thus we have

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[Z^{\sigma-1}\ {}_1E_{1,1}^{(\xi,\eta)m}\left(Z^{\rho}\right).J_{(\mu_t)_s,\omega,d}^{(v_t)_s,\lambda,c}\left(Z^{\rho'}\right)\right]\right)(x) = \frac{x^{\sigma-\alpha-\alpha'+\gamma-1}}{\Gamma(\rho)}\sum_{l=o}^{\infty}\frac{C^l(\lambda)_{\omega l}\,x^{\rho' l}}{\prod_{t=1}^{s}\Gamma\left(v_t\,l+\mu_t+\frac{d+1}{2}\right)l!}.\ {}_4\Psi_{\mathrm{m}+3}\left[\begin{matrix}(\tau,-);(\sigma+\rho' l,\rho);(\sigma+\rho' l+\gamma-\alpha-\alpha'-\beta,\rho)\quad(\sigma+\rho' l+\beta'-\alpha',\rho)\\(\xi_i\ \eta_i)_{1,m};(\sigma+\rho' l+\beta'\rho);(\sigma+\rho' l+\gamma-\alpha-\alpha',\rho)(\sigma+\rho' l+\gamma-\alpha'-\beta,\rho)\end{matrix};x^{\rho}\right] \tag{2.5}$$

(ii) Assuming k = 1, p = q = m= 1, $a_1 = b_1 = 1$ in our main result (2.1), we get

$$\left(I_{0,+}^{\propto,\propto',\beta,\beta',\gamma}\left[Z^{\sigma-1}\ {}_1E_{1,1}^{(\xi,\eta)1}\left(Z^{\rho}\right)J_{(\mu_t)_s,\omega,d}^{(v_t)_s,\lambda,c}\left(Z^{\rho'}\right)\right]\right)(x) = = x^{\sigma-\propto-\propto'+\gamma-1}\sum_{l=o}^{\infty}\frac{C^l(\lambda)_{\omega l}\,x^{\rho' l}}{\prod_{t=1}^{s}\Gamma\left(v_t\ l+\mu_t+\frac{d+1}{2}\right)l!}.$$

$${}_3\Psi_4\left[\begin{matrix}(\sigma+\rho' l,\rho);(\sigma+\rho' l+\gamma-\propto-\propto'-\beta,\rho);(\sigma+\rho' l+\beta'-\propto',\rho)\\(\xi,\eta);(\sigma+\rho' l+\beta'\rho);(\sigma+\rho' l+\gamma-\propto-\propto',\rho);(\sigma+\rho' l+\gamma-\propto'-\beta,\rho)\end{matrix};x^{\rho}\right] \tag{2.6}$$

(iii) If we reduce multi-parameter k-Mitteg- Leffler function set to unity and takes

$$\mathrm{k} = 1\,, v_1 = 1, u_1 = v, \omega = 0, d = 1, C = -1, s = 1, Z^{\rho'} \to \frac{Z^2}{4}$$

We get

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[Z^{\sigma-1}\, J_{v,0,1}^{1,\lambda,-1}\left(\frac{Z^2}{4}\right)\right]\right)(x) =$$

$$x^{\sigma-\alpha-\alpha'+\gamma-1}\;{}_3\Psi_4\left[\begin{matrix}(\sigma+v,2);(\sigma+v+\gamma-\alpha-\alpha'-\beta,2);(\sigma+v+\beta'-\alpha',2)\\(\sigma+v+\beta',2);(\sigma+v+\gamma-\alpha-\alpha',2);(\sigma+v+\gamma-\alpha'-\beta,2)\end{matrix};\,-\frac{Z^2}{4}\right] \tag{2.7}$$

Where (2.7) is a known result obtained by S.D. Purohit, D.L. Suthar and S.L. Kalla in [14]

Special Cases of Theorem 2

(i) Taking k = 1, p = q = 1, $a_1 = \tau$ and $b_1 = 1$ in the main result (2.2) Multi-Parameter k-Mittag-Leffler Function reduced to generalized Mittag-Leffler studied by [6], thus we have

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma}\left[Z^{\sigma-1}\;{}_1E_{1,1}^{(\xi,\eta)m}\left(Z^{\rho}\right).J_{(\mu_t)_s,\omega,d}^{(v_t)_s,\lambda,c}\left(Z^{\rho'}\right)\right]\right)(x)$$

$$= \frac{x^{\sigma-\alpha-\alpha'+\gamma-1}}{\Gamma(\rho)}\sum_{l=o}^{\infty}\frac{C^l(\lambda)_{\omega l}\,x^{\rho' l}}{\prod_{t=1}^{s}\Gamma\left(v_t\, l+\mu_t+\frac{d+1}{2}\right)l!}.\;{}_4\Psi_{m+3}\left[\begin{matrix}(\tau,-);(1-\sigma-\rho' l-\beta',\rho);(1-\sigma-\rho' l-\gamma+\alpha+\alpha',\rho);\;\;(1-\sigma-\rho' l-\gamma+\alpha+\beta',\rho);\\(\xi_i\,\eta_i)_{1,m};(1-\sigma-\rho' l,\rho);(1-\sigma-\rho' l-\gamma+\alpha+\alpha'+\beta',\rho)\;\;(1-\sigma-\rho' l-k)\beta+k\alpha,\rho):\end{matrix}\;x^{\rho}\right] \tag{2.8}$$

(ii) If we put k = 1, p = q = m= 1, $a_1 = b_1 = 1$ in our main result (2.2) Multi- Parameter k-Mittag-Leffler Function reduced to Mittag-Leffler studied by [13], thus we have

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma}\left[Z^{\sigma-1}\;{}_1E_{1,1}^{(\xi,\eta)1}\left(Z^{\rho}\right)J_{(\mu_t)_s,\omega,d}^{(v_t)_s,\lambda,c}\left(Z^{\rho'}\right)\right]\right)(x)$$
$$= x^{\sigma-\alpha-\alpha'+\gamma-1}\sum_{l=o}^{\infty}\frac{C^l(\lambda)_{\omega l}\,x^{\rho' l}}{\prod_{t=1}^{s}\Gamma\left(v_t\, l+\mu_t+\frac{d+1}{2}\right)l!}.\;{}_3\Psi_4\left[\begin{matrix}(1-\sigma-\rho' l-\beta',\rho);(1-\sigma-\rho' l-\gamma+\alpha+\alpha',\rho);\\(\xi\,\eta);(1-\sigma-\rho' l);(1-\sigma-\rho' l-\gamma+\alpha+\alpha'+\beta',\rho);\end{matrix}\right.$$

$$\left.\begin{matrix}(1-\sigma-\rho' l-\gamma+\alpha+\beta',\rho);\\(1-\sigma-\rho' l-k\beta+k\alpha,\rho);\end{matrix}\;x^{\rho}\right] \tag{2.9}$$

(iii) If we put C = -1 & d = 1 in our main result (2.2) then Multi-Index Bessel Function reduced to Multi-Index Bessel Function obtained by Choi & Agarwal [1], so we have

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma}\left[Z^{\frac{\sigma}{k}-1}\;{}_pE_{q\,k}^{(\xi,\eta)m}\left(Z^{\frac{\rho}{k}}\right).J_{(\mu_t)_s,\omega,1}^{(v_t)_s,\lambda,-1}\left(Z^{\frac{\rho'}{k}}\right)\right]\right)(x)$$

$$= \frac{x^{\frac{\sigma}{k}-\alpha-\alpha'+\gamma-1}\prod_{r=1}^{q}\Gamma_{\mathrm{k}}(\mathrm{b}_r)}{k^{\gamma}\prod_{j=1}^{p}\Gamma_{\mathrm{k}}(\mathrm{a}_j)}\sum_{l=o}^{\infty}\frac{(-1)^l(\lambda)_{\omega l}}{\prod_{t=1}^{s}(v_t l+\mu_t+1)}\,\frac{x^{\frac{\rho' l}{k}}}{l!}.$$

$${}_{p+3}\Psi_{q+m+3}^{k}\left[\begin{matrix}(a_j,k)_{1,p};(1-\sigma-\rho' l-k\beta,\rho);(1-\sigma-\rho' l-k\gamma+k\alpha+k\alpha',\rho);\\(b_r,k)_{1,q};(\xi_i,\eta_i)_{1,m};(1-\sigma-\rho' l,\rho);(1-\sigma-\rho' l-k\gamma+k\alpha+k\alpha'+k\beta',\rho);\end{matrix}\right.$$

$$\left.\begin{matrix}(1-\sigma-\rho' l+k\gamma+k\alpha+k\beta',\rho);\\(1-\sigma-\rho' l-k\beta+k\alpha,\rho);\end{matrix}\;x^{\frac{\rho}{k}}\right] \tag{2.10}$$

Special Case of Theorem 3

(i) If we put C = -1, & d =1 in our main result (2.3) then Multi-index Bessel Function reduced to Multi-Index Bessel Function obtained by Choi & Agarwal [1], so we have

$$\left(D_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[Z^{\frac{\sigma}{k}-1}\;{}_pE_{q\,k}^{(\xi,\eta)m}\left(Z^{\frac{\rho}{k}}\right).J_{(\mu_t)_s,\omega,1}^{(v_t)_s,\lambda,-1}\left(Z^{\frac{\rho'}{k}}\right)\right]\right)(x)$$

$$= \frac{x^{\frac{\sigma}{k}+\alpha+\alpha'-\gamma-1}\prod_{r=1}^{q}\Gamma_{\mathrm{k}}(\mathrm{b}_r)}{k^{\gamma}\prod_{j=1}^{p}\Gamma_{\mathrm{k}}(\mathrm{a}_j)}\sum_{l=o}^{\infty}\frac{(-1)^l(\lambda)_{\omega l}}{\prod_{t=1}^{s}(v_t l+\mu_t+1)}\,\frac{x^{\frac{\rho' l}{k}}}{l!}$$

$$ {}_{p+3}\Psi^{k}_{q+m+3}\left[\begin{matrix}(a_j,k)_{1,P};(\sigma+\rho' l,\rho);(\sigma+\rho' l-k\gamma+k\alpha+k\alpha'+ks,\rho)\\(b_r,k)_{1,q;}(\xi_i\,\eta_i)_{1,m};(\sigma+\rho' l-k\beta,\rho);(\sigma+\rho' l-k\gamma+k\alpha+k\alpha',\rho)\end{matrix}\right. $$

$$ \left.\begin{matrix}(\sigma+\rho' l+k\alpha-k\beta,\rho);\\(\sigma+\rho' l-k\gamma+k\alpha+k\beta',\rho);\end{matrix}\ x^{\frac{\rho}{k}}\right] \tag{2.11} $$

Special Case of Theorem 4

(i) Taking k = 1, P = q = 1, $a_1 = \tau$ and $b_1 = 1$ in the main result (2.4) we get

$$ \left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left[Z^{\sigma-1}{}_{1}\,E_{1,1}^{(\xi,\eta)m}\,(Z^{\rho}).J_{(\mu_t)_s,\omega,d}^{(v_t)_s,\lambda,c}\left(Z^{\rho'}\right)\right]\right)(x) $$

$$ =\frac{x^{\sigma+\alpha+\alpha'-\gamma-1}}{\Gamma(\rho)}\sum_{l=o}^{\infty}\frac{C^{l}(\lambda)_{\omega l}}{\prod_{t=1}^{s}\left(v_t l+\mu_t+\frac{d+1}{2}\right)}\frac{x^{\rho' l}}{l!}. $$

$$ {}_{4}\Psi_{m+3}\left[\begin{matrix}(\tau,-);(1-\sigma-\rho' l+\beta,\rho);(1-\sigma-\rho' l+\gamma-\alpha-\alpha',\rho)\\(\xi_i\,\eta_i)_{1,m};(1-\sigma-\rho' l,\rho);(1-\sigma-\rho' l+\gamma-\alpha-\alpha'-\beta,\rho)\end{matrix}\right. $$

$$ \left.\begin{matrix}(1-\sigma-\rho' l+\gamma-\alpha'-\beta,\rho);\\(1-\sigma-\rho' l-k\alpha'+k\beta',\rho);\end{matrix}\ x^{\rho}\right] \tag{2.12} $$

3. Conclusion

In this paper, we studied and obtained images of product of two generalized function namely Multi-Parameter k-Mittag-Leffler Function and Multi-Index Bessel Function under MSM Fractional Integral and Derivatives formulas in terms of generalized k-Wright Function. We obtained many results of earlier work as special cases of our main results.

4. References

1. Choi J, Agarwal P. A note on fractional integral operator associated multiindex Mittag-Leffler functions, Filomat. 2016; 30(7):1931-1939.
2. Diaz R, Pariguan E. On hypergeometric functions and Pochhammer k-symbol. Divulgaciones Mathematicas. 2007; 15(2):179-192.
3. Wright EM. The asymptotic expansion of generalized hypergeometric Function. J London Math. Soc. 1935; 10:286-293. Doi: 10.1112/jlms/s1-10.40.286.
4. Gehlot KS. Multiparameter K-Mittag-Leffler Function,International Mathematical Forum. 2013; 8(34):1691-1702.
5. Gehlot Prajapati KS. JC. On generalization of K-Wright function andit's properties, International Journal of Pure and Applied Mathematics. 2014; 93(5):715-728.
6. Luchko Yu F, Yakubovich SB. An operational method for solving some class of intro differential equations, (Russian). Differ. Uravn, translation in Differential Equation, 30, 269-280, 247-256.
7. Marichev OI. Volterra equation of Mellin Convolution type with a Horn function in the kernel (In Russian), Izv. ANBSSR Ser. Fiz.-Mat. Nauk. 1974; 1:128-129.
8. Nisar KS. Fractional calculus and certain integrals of generalized multiindex bessel function, Mathematics Subject Classification, 2017, 33C20; 33B15.
9. Saigo M. A remark on integral operators involving the Gauss hypergeometric functions, Math. Rep. Kyushu Univ. 1978; 11:135-143.
10. Saigo M, Maeda N. More generalization of fractional calculus, Transform Methods and Special Functions, Varna, Bulgaria, 1996, 386-400.
11. Samko SG, Kilbas AA, Marichev OI. Fractional Integrals and Derivatives. Theory and Applications, Gordan and Breach Sci. Publ. New York, 1993.
12. Srivastava HM, Choi Zeta J. q-Zeta Functions and Associated Series and Integrals, Elsevier Science Publishers, Amsterdam, London and New York.
13. Wiman A. Uber den fundamental Satz in der Theories der Funktionen *Eα* (*z*), Acta Math. 1905; 29:191-201.
14. Purohit SD, Suthar DL, Kalla SL. Marichev-Saigo-Maeda Fractional Integration Operators of the Bessel Functions LE Matematiche. 2012; LXVII(I):21-32