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On stability of first order linear impulsive differential equations

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Abstract
In this paper, we focus on the stability problems of first order linear impulsive differential equations. We construct an ordinary differential equation representation of the impulsive system such that it is suitable for the qualitative analysis of the later. This process is achieved by a transformation that bijectively maps the solutions of the initial value problems for impulsive differential equations to the solutions of the initial value problems for ordinary differential equations. A relationship between stability properties of impulsive differential equations and the corresponding ordinary differential equations was established.

Keywords: Stability, Impulsive differential equations, asymptotic

1. Introduction
The theory of impulsive system was developed not long ago as an independent area of mathematical analysis. The development came out of curiosity to build a mathematical framework that truly describes physical and biological process as they occur in nature. Prior to this noble development, scientists had often made an underlying assumption that the behaviour of physical and biological systems described by ordinary differential equations is continuous and integrable in some sense. The state of a system is susceptible to change. In some processes, these changes are often characterized by short-time perturbations (impulses) whose durations are negligible when compared with the total duration of their entire time of evolution [1, 4, 5, 6, 7, 8, 9, 10, 11]. Impulsive differential equations are adequate mathematical models for the description of evolution processes characterized by the combination of continuous and jump changes of their state. For the continuous change of such processes, ordinary differential equations are used, while the moments and the magnitude of the jump are given by conditions [1, 2, 3]. Impulsive systems are systems whose states are characterized by small perturbations (impulses) in the form of jumps [12, 13, 14].

In this work, we focus on the stability problems of impulsive differential equations which is an emerging area of research presently at its infancy compared to the stability analysis of ordinary differential equations. This is probably due to the nature of the impulsive processes which are momentarily exposed to harsh impacts. The aim of this study is to establish sufficient conditions for stability of impulsive differential equations from the stability conditions for the associated ordinary differential equations.

To help in our investigation, we will define some important concepts and establish some facts.

1.1 Impulsive differential System: Impulsive differential equations with fixed moments of impulsive effect are of the form

$$\begin{cases} x'(t) = f(t, x(t)), \forall\ t \in T \setminus S \\ \Delta x(t_k) = f(t_k, x(t_k)), \forall\ t_k \in S, \end{cases} \tag{1.1}$$

Where $(t, x) \in \Omega \subset R \times R^n$ and the real numerical sequence $S = \{t_k\}_{k=1}^{\infty}$ is increasing and has no finite accumulation point. In the case of unfixed moments of impulsive effects the impulsive points may be time and state dependent; that is, $t_k := t_k(t, x(t))$.

When the function t_k depends on the state of the system (1.1) it is said to have impulses at variable times. This is reflected in the fact that different solutions will tend to undergo impulses at different times. However, if the functions t_k are all constants the systems is said to have impulses at fixed times in which case all solutions undergo impulses actions at the same time, the question of existence of solution of the system (1.1) is non-trivial when impulses occur at variable times. Even the precise notion of what a solution is must be carefully stated. It is fairly clear that solutions should be piecewise continuous and in fact piecewise continuously differentiable (or piecewise absolutely differentiable when considering generalized types of solutions). A solution will undergo simple jump discontinuity when it intersects impulse hyper-surfaces. Even after focusing on a particular class of relations $t(s,x(s))=0$ given by impulse hyper-surfaces, impulsive differential equations still exhibit some unusual behaviour (Ballinger, 1999).

We shall concentrate only on those with fixed moments. Be that as it may, to obtain or discuss the solution of an impulsive differential equation, we must take into cognizance certain peculiarities of the model. We assume that for $t \in T \setminus S$, the solution *x* of equation (1.1) is determined by the ordinary differential equation $x'(t) = f\left(t_k, x(t_k)\right)$. For $t \in S$, a change by jump of the solution *x* occurs so that $x\left(t_k^-\right) = x(t_k)$ *and* $x\left(t_k^+\right) = x(t_k) + \Delta x(t_k) = x(t_k) + f\left(t_k, x(t_k)\right)$. After the jump at the moment $t = t_k$, the solution *x* of the system (2.1) coincides with the solution *y* of the initial value problem (Bainov and Simeonov, 1995):

$$\begin{cases} y'(t) = f\left(t, x(t)\right), \ t_k < t \le t_{k+1} \\ y(t_k) = x\left(t_k^+\right), t = t_k \in S. \end{cases} \tag{1.2}$$

This simply means that, after the jump at $t = t_k$, a new function *y*(*t*) takes over control from *x*(*t*). The controlling impulsive differential equation is given by

$$\begin{cases} x'(t) = f(t, x(t)), \forall \ t \in T \setminus S \\ \Delta x(t) = g(t_k, x(t_k)), \forall \ t_k \in S, \\ x(t_0) = x_0, t_0 \in T \setminus S, (t_0, x_0) \in \Omega \end{cases} \tag{1.3}$$

1.2 Construction of an associated ordinary differential equation from an impulsive differential equation

Let the impulsive differential equation be defined by equation (1.3). We will follow the construction steps of the construction of the absolute continuous trajectory of the Caratheodory type absolute continuous equation with a mapping connecting the two trajectories.

As a preparatory step, we will establish a relationship between a set

$$S := \left\{t_j \mid t_j \in R, \ t_j < t_{j+1}, \forall j \in \mathbb{N} \right\}$$

Of time point (impulse points) and sequences of intervals

$$S_c := \left\{\left[\hat{t}_j, \tau_j\right) \mid \left[\hat{t}_j, \tau_j\right) \subset R, \ \hat{t}_j < \tau_j < \hat{t}_{j+1}, \forall \ j \in \mathbb{N} \right\};$$

$$S_i := \left\{\left[\tau_j, \hat{t}_{j+1}\right) \mid \left[\tau_j, \hat{t}_{j+1}\right) \subset R, \hat{t}_j < \tau_j < \hat{t}_{j+1}, \forall j \in \mathbb{N} \right\}.$$

Notation 1.1: Let us denote by

$$\hat{S} := \left\{\hat{t}_j\right\}_{j=0}^{\infty},$$

$$S_U := \bigcup_{j=0}^{\infty} \left[\hat{t}_j, \tau_j\right) \text{ and by}$$

$$S_O := \bigcup_{j=0}^{\infty} \left[\hat{t}_j, \tau_j\right)$$

The set of images of impulse points and unions of the intervals in S_c and its interior respectively.

Definition 1.1: Let $\omega := S_U \to \left[t_0, \infty\right)$ be defined as follows:

$\omega\left(\hat{t}_k\right) := t_k$ *And* $\omega\left(\tau_k - 0\right) := t_{k+1} \ \forall k \in \mathbb{N}$

$$\omega(t) := t_k + \frac{(t - \hat{t}_k)(t_{k+1} - t_k)}{\tau_k - \hat{t}_k} \forall t \in [\tau_k, \hat{t}_k), \forall k \in \mathbb{N}.$$

Moreover let $\omega_{-}(\tau_k) := \omega(\tau_k - 0) = t_{k+1} \ \forall k \in \mathbb{N}$.

Definition 1.2: We define an ODE with right side measurable in *t*, continuous or Lipschitzian in *(x,y)* for each fixed *t*, to the impulsive differential equation (1.3) as follows:

$$\Phi(t, x, y) := \begin{cases} (f(\omega(t), x), 0), & \forall (\omega(t), x, 0) \in \Omega \times \{0\} \\ 0 & (\omega(t), x, y) \notin \Omega \times \{0\} \\ \varphi_{\varepsilon_k, g, \tau_k, \hat{t}_{k+1}}(t - \tau_k, x, y) & (t, x, y) \in [\tau_k, \hat{t}_{k+1}) \times R^n \times R^n, \\ & k \in \mathbb{N}, \ \varepsilon_k := \frac{1}{k}. \end{cases}$$

Lemma 1.1. The ordinary differential equation

$$(1, x', y')(t) = \varphi_{\varepsilon, g, \tau_k, \hat{t}_{k+1}}(t, x(t), y(t)),$$

$$(x(t_0), y(t_0)) = (x_0, y_0), \forall \ (t_0, x_0, y_0) \in [\tau_k, \hat{t}_{k+1}) \times R^n \times R^n, \ k \in \mathbb{N} \quad (1.4)$$

Has a unique solution.

Theorem 1.1. The solution of the initial value problem of the differential equation with right side

$$\Phi(t, x, y), (t, x, y) \in [t_0, \infty) \times R^n \times R^n,$$

$$(x'(t), y'(t)) = \Phi(t, x(t), y(t)),$$

$$\left(x\left(\omega^{-1}(s_0)\right), y\left(\omega^{-1}(s_0)\right)\right) = (x_0, y_0), \left(\omega^{-1}(s_0), x_0, y_0\right) \in \Omega \times R^n \quad (1.5)$$

Exists and is unique provided that the solution of the initial value problem $x(s_0) = x_0$ of the impulsive differentiable equation (1.3) exists and is unique. Moreover, the trajectory of solution (x, y) of initial value problems $s_0 \notin S, (s_0, x_0, 0) \in \Omega \times 0$ satisfies the condition that $x\left(\omega^{-1}(t)\right), t \in [s_0, \infty)$ is a solution of the impulsive differential equation (1.3).

Proof: The statement of the theorem follows from the construction, relation (1.5) and Definition 1.2.

Now, let us consider the composite system discussed in theorem 4.1 using the dynamical systems and the associated ordinary differential equation.

Condition 1.1

C.1 $\exists \ \alpha_k > 0, \|f(t, x)\| < \alpha_k \|x\|, \forall (t, x) \in [t_k, t_{k+1}) \times R^n, \forall t_k, t_{k+1} \in S;$

C.2 $\exists \ \beta_k > 0, \|g(t_k, x)\| < \beta_k \|x\|, \forall (t_k, x) \forall S \times R^n;$

C.3 $\exists \ \gamma_k, \gamma_f > 0, 0 > -\gamma_k \|x\| > \frac{(f(t, x), x)}{\|x\|}$ and $\left|\frac{(f(t, x), x)}{\|x\|}\right| >$
$\gamma_f \|f(t, x)\|, \|x\| > 0, \forall (t, x) \in [t_k, t_{k+1}) \times R^n, \forall t_k, t_{k+1} \in S;$

C.4 $\exists\ \delta_g, \delta_k > 0, 0 > -\delta_k \|x\| > \frac{(g(t_k, x), x)}{\|x\|}$ and $\left|\frac{(g(t_k, x), x)}{\|x\|}\right| >$

$\delta_g \|g(t,x)\|, \|x\| > 0, \forall (t,x) \in S \times R^n;$

We will now prove some basic inequalities useful in our stability analysis.

Lemma 1.2 Let the right side of the one dimensional ordinary differential equation defined in (1.5) fulfil condition 1.1, C.1 then the solution of the initial value problem in (1.5) satisfies the inequality:

$$|x(t)| \leq |x(t_k)| e^{\alpha(t-t_k)}, \forall (t,x) \in [t_k, t_{k+1}) \times R^n, \forall t_k, t_{k+1} \in S. \tag{1.6}$$

If it fulfils Condition 1.1, C.3 then

$$|x(t)| \leq |x(t_k)| e^{\gamma(t-t_k)}, \forall (t,x) \in [t_k, t_{k+1}) \times R^n, \forall t_k, t_{k+1} \in S. \tag{1.7}$$

Proof: Integrating both sides of the condition C.1, with $x(t) := x_0 =: \varphi_0(t)$ we get

$$\varphi_1(t) := \left| x_0 + \int_{t_0}^{t} f(s, \varphi_0(s))\, ds \right| \leq |x_0| + \int_{t_0}^{t} |\alpha \varphi_0(s)|\, ds =$$

$$|x_0| (1 + \alpha(t - t_0)).$$

By induction, we assume that with $\varphi_{j+1}(t) := x_0 + \int_{t_0}^{t} f(s, \varphi_j(s))\, ds$ and the inequality

$|\varphi_q(t)| \leq |x_0| \sum_{p=0}^{q} \frac{(\alpha(t-t_0))^p}{p!}, \forall\ 1 \leq q \leq j$ Holds.

Then using the condition C.1 we get

$$\varphi_{j+1}(t) := \left| x_0 + \int_{t_0}^{t} f(s, \varphi_j(s))\, ds \right| \leq |x_0| + \int_{t_0}^{t} |\alpha \varphi_j(s)|\, ds =$$

$$|x_0| + \int_{t_0}^{t} \left(\sum_{p=0}^{j} \frac{(\alpha(t-t_0))^p}{p!} \right) ds = |x_0| \sum_{p=0}^{j+1} \frac{(\alpha(t-t_0))^p}{p!}.$$

By the successive approximation for the solution, $\varphi_j \to x$, the upper estimate tends to $e^{\alpha(t-t_0)}$, and hence proves inequality (1.6).

The proof of inequality (1.7) is essentially the same, except that the basic estimate is as stated in condition C.3: $-\gamma |x| > f(t,x)$ holds in Ω. The same steps are repeated as in the first case and we get the statement of the lemma.

2. Statement of the Problem

This work is aimed at investigating the conditions under which a solution will remain close to another solution when at the initial time point they are sufficiently close to each other. This issue is even more emphasized for impulsive differential equations where the system is exposed to significantly harsh impacts at the impulse points. One wants to know which system can remain close to the behaviour of the original dynamical system. As earlier mentioned, these aspects of the investigations are at their infancy whereas their practical importance are highly rated. Here, we analyze stability issues of linear first order impulsive ordinary differential equations with initial conditions.

3. Main Results

We are now sufficiently equipped to formulate theorems about the composite system.

Theorem 4.3: Assume that the components of the impulsive differential equation (1.3) or equivalently of the associated ordinary differential equation (1.5) fulfil Condition 1.1, C.1 and C.4 or C.3 and C.2 or C.4 in n = 1 dimension and in addition

i) $\min\left\{e^{\alpha_k(\tau_k-\hat{t}_k)-\delta_k(\hat{t}_{k+1}-\tau_k)},\ e^{\gamma_k(\tau_k-\hat{t}_k)-\beta_k(\hat{t}_{k+1}-\tau_k)}\right\}\le 1,\forall k\in \mathbb{N}$;

then the sequence $\left\{x\left(\hat{t}_j\right)\right\}_{j=0}^{\infty}$ is stable.

ii) $\min\left\{e^{\alpha_k(\tau_k-\hat{t}_k)-\delta_k(\hat{t}_{k+1}-\tau_k)},\ e^{\gamma_k(\tau_k-\hat{t}_k)-\beta_k(\hat{t}_{k+1}-\tau_k)}\right\}<\mu<1,\forall k\in \mathbb{N}$;

then the sequence $\left\{x\left(\hat{t}_j\right)\right\}_{j=0}^{\infty}$ is asymptotically stable.

iii) If Condition 1.1, C.1 holds and C.3 does not hold then
$\exists M>0$ such that $e^{\alpha k(\tau_k-\hat{t}_k)}\le M,\forall k\in \mathbb{N}$.

If conditions (i) and (ii) are true with C.3, or condition (iii) is true then the solution $x:\left[\hat{t}_0,\infty\right)$ is stable or asymptotically stable by the stability of the sequence $\left\{x\left(\hat{t}_j\right)\right\}_{j=0}^{\infty}$ respectively.

Proof: i) By Lemma 1.2, using Condition 1.1, C.1 and condition (iii) of the theorem,

$$\left|x(t)\right|\le x\left(\hat{t}_k\right)e^{\alpha_k(t-\hat{t}_k)}\le M\left|x\left(\hat{t}_k\right)\right|,\forall t\in\left[\hat{t}_k,\tau_k\right)$$

And by condition C.2

$$\left|x(t)\right|\le x\left(\tau_k\right)e^{\beta_k(t-\hat{t}_k)},\forall t\in\left[\tau_k,\hat{t}_{k+1}\right)\forall k\in \mathbb{N}\ .$$

ii) By the second statement of the same lemma, using Condition 1.1, C.3, C.4,

$$\left|x(t)\right|\le x\left(\hat{t}_k\right)e^{\gamma_k(t-\hat{t}_k)},\forall t\in\left[\hat{t}_k,\tau_k\right)$$ and

$$\left|x(t)\right|\le x\left(\tau_k\right)e^{\delta_k(t-\tau_k)},\forall t\in\left[\tau_k,\hat{t}_{k+1}\right)\forall k\in \mathbb{N}$$

iii) From points (i) and (ii) we have that

$$\left|x(t)\right|\le \max\left\{\left|x\left(\hat{t}_k\right)\right|,M\left|x\left(\hat{t}_k\right)\right|,\left|x\left(\tau_k\right)\right|\right\},\forall t\in\left[\hat{t}_k,\hat{t}_{k+1}\right),\ k\in \mathbb{N}\ .$$

iv) By condition (i) of this theorem, the set $\left\{\left|x\left(\hat{t}_k\right)\right|\right\}_{k\in N}$ is bounded hence the same holds for the trajectory.

v) By condition (ii) of this theorem, $x\left(\hat{t}_k\right)\to 0$ if $k\to\infty$ hence by condition (iii) of this theorem, the system is asymptotically stable.

Theorem 4.4: Let equation (1.3) fulfil the following conditions and let n = 1:

i) $f(t,0)=0,\forall t\in\left[t_0,\infty\right)$ And $g(t_k,0)=0,\forall t_k\in S$. hence or otherwise, $x(t)\equiv 0$ is a solution of the initial value problem in $[t_0,\infty)$.

ii) Let $0>-\alpha>\dfrac{f(t,x)}{|x|},\forall\ |x|>0,\forall t\in\left[t_0,\infty\right)$.

iii) Let $\left|g(t_k,x)\right|<\beta|x|,\ \beta<1,\forall t_k\in S,\forall x\in R;$

If $t_{k+1}-t_k>\gamma>0,\forall t_k\in S$ and

$$\exists \;\; 0 < \theta < 1, \left(1 + \beta\right) e^{-\alpha\gamma < 1 - \theta}, \tag{3.1}$$

Then the identically zero solution is asymptotically stable.

Proof: Conditions 2. And 3. And inequality (3.1) grant the conditions of theorem 4.3 case C.3 condition 2. hence the system is stable or asymptotically stable subject to the way of fulfilment of inequality (3.1).

4. Conclusion

In this case the stability of the combined system comes from the stability of the ordinary differential equation component of the impulsive differential equation. The stability properties of impulsive systems are inherited by the associated ordinary differential equations and vice versa.

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