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Motzkin numbers revisited

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Abstract
Motzkin numbers, like Catalan numbers, have important role in number theory and they preserve their own identity in many ways. In this note we have, as mentioned in the title, have revisited the system and tried to associate some facts with Euclidian geometry. In addition to that some algebraic results are noted and important features of Motzkin numbers with that of Catalan's are critically compared.

Keywords: Motzkin number, Motzkin path, Catalan Path

1. Introduction
In this paper we have attempted to review the system of Motzkin numbers with different approaches. In the first section we have established an algebraic relation connecting vertices, edges, and faces in a Motzkin path in a rectangular set-up of n x [n/2] unit squares which parallels to an algebraic relation in graph theory. In the second section one more relation is derived which is a lemma on the first theorem. The final section gives important points on comparison between Catalan system and Motzkin system.

1.2. All That We Need
In this section we are trying to establish a relation which parallels to known relation
$v - e + f = 2$. The relation that we have identified and established that connects vertices, edges and faces of a Motzkin graph drawn in a rectangle of $n \times \left[\frac{n}{2}\right]$ graph composed of unit squares.

1.2.1. All about algebra of Motzkin Path
We define $f(n) = \left[\frac{n}{2}\right]$ for some $n \in N$ $(n \geq 2)$, an integer in $\frac{n}{2}$ which is not greater than $\frac{n}{2}$.
Now we consider different components (a) vertices, (b) edges and (c) faces in a Motzkin path.

(a) Vertices: For a Motzkin Path in rectangle constructed using $n \times \left[\frac{n}{2}\right]$ unit squares, total number of vertices is given by, $v = (n+1)\left(\left[\frac{n}{2}\right] + 1\right)$ (1)

(b) Edges: The total number of edges is the number of line segments involved in the rectangular structure of $n \times \left[\frac{n}{2}\right]$ unit squares. In this, an edge is considered once only and it may be an edge of a square or that of Motzkin path. In this case there are two extreme cases.

1. The Motzkin path coincides with the straight horizontal path of n edges from 0 to n.
2. The Motzkin path that has maximum number of diagonal line segments which are not the part of sides of squares of unit length.

We give the formulae for edges in each case.
Case (a): (minimum Edges) Path coinciding with edges of unit square. In the case of M_n for n=4, and n=5, the corresponding paths are as shown below.

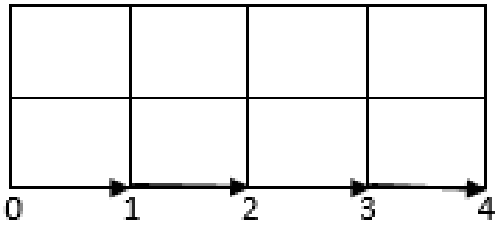

Fig 1

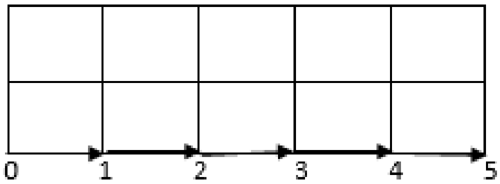

Fig 2

Total edges in these structures are

$$e = \left[\frac{n}{2}\right](2n+1) + n \tag{2}$$

Case (b): (maximum case): Path that contains maximum diagonals with minimum Horizontal lines. We consider two cases as above; for n= 4 and n= 5.

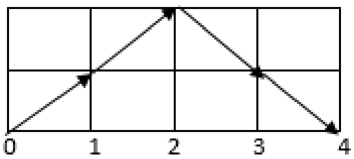

Fig 3: (For n = 4, an even integer)

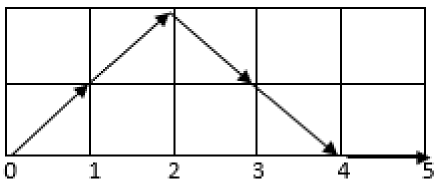

Fig 4: (For n = 5, an odd integer)

Total edges in these case is given by

$$e = \begin{cases} \left[\frac{n}{2}\right](2n+1) + 2n\,;\ if\ n\ is\ even \\ \left[\frac{n}{2}\right](2n+1) + (2n-1)\,;\ if\ n\ is\ odd \end{cases} \tag{3}$$

(c) Faces: Face in a path is a convex region which is made by a path and the squares or triangle in the structure. Number of faces (f) is given by

$$f = \begin{cases} n \times \left[\frac{n}{2}\right];\ in\ minimum\ case \\ n\left(\left[\frac{n}{2}\right] + 1\right);\ in\ maximum\ case, n\ is\ even \\ \left(n \times \left[\frac{n}{2}\right]\right) + (n-1);\ in\ maximum\ case, n\ is\ odd \end{cases} \tag{4}$$

1.2.2 Motzkin Paths and final Calculations

In accordance with above calculations we clarify the situations as follows:

In a Motzkin path constructed in a rectangle made up of $n \times \left[\frac{n}{2}\right]$ unit squares we have,

1. The number of vertices $v = (n+1)\left(\left[\frac{n}{2}\right] + 1\right)$ (5)
2. Total number of line segments including path segments

Case 2A: When the maximum number of path segments that coincide with the line segment of the unit squares, Minimum number of line segments,

$$e = \left[\frac{n}{2}\right](2n+1) + n \tag{6}$$

Case 2B: When the maximum number of path segments is different from sides of the unit square

$$e = \begin{Bmatrix} \left[\frac{n}{2}\right](2n+1) + 2n\ ;\ if\ n\ is\ even \\ \left[\frac{n}{2}\right](2n+1) + (2n-1)\ ;\ if\ n\ is\ odd \end{Bmatrix} \text{(Maximum case)} \tag{7}$$

3. Total number of Faces: Finally, in content to above cases,

Case 3A: Minimum edges case

$$f = n \times \left[\frac{n}{2}\right] \tag{8}$$

Case 3B: Maximum edges case

$$f = \begin{Bmatrix} n\left(\left[\frac{n}{2}\right] + 1\right); if\ n\ is\ even \\ \left(n \times \left[\frac{n}{2}\right]\right) + (n-1);\ if\ n\ is\ odd \end{Bmatrix} \tag{9}$$

1.3 Key Result

In this section we prove two main results in the extreme cases of a Motzkin path. In general with any Motzkin path M_n, the number of vertices in the corresponding mesh of unit squares is given by the result (1), $v = (n+1)\left(\left[\frac{n}{2}\right] + 1\right)$

On the basis of the above fact on number of vertices we prove two results.

1.3.1 Theorem: To prove that for each one of Motzkin path in a rectangular mesh made-up of n x [n/2] unit squares v- e + f = 1; where v shows number of vertices, e stands for edges, and f stands for interior faces of the mesh after construction of any one of M_n Motzkin paths. We have two cases discussed below.

Case 1: When all the edges coincide with edges of those of the unit squares,

$e = \left[\frac{n}{2}\right](2n+1) + n$

$f = n \times \left[\frac{n}{2}\right]$

In this case, v – e +f = $(n+1)\left(\left[\frac{n}{2}\right] + 1\right)$ - [$\left[\frac{n}{2}\right](2n+1) + n$] + $n \times \left[\frac{n}{2}\right]$ = 1

Case 2:

We discuss the case when maximum number of segments of the Motzkin path does not coincide with those of the sides of unit squares.

Case 2A: When 'n' is an even integer,

Number of edges = e = $\left[\frac{n}{2}\right](2n+1) + 2n$ and Number of faces = f = $n\left(\left[\frac{n}{2}\right] + 1\right)$

In this case, v – e +f = $(n+1)\left(\left[\frac{n}{2}\right] + 1\right)$ – {$\left[\frac{n}{2}\right](2n+1) + 2n$ } + $n\left(\left[\frac{n}{2}\right] + 1\right)$ = 1

Case 2B: When 'n' is an odd integer

Number of edges = e = $\left[\frac{n}{2}\right](2n+1) + (2n-1)$ and

Number of faces = f = $\left(n \times \left[\frac{n}{2}\right]\right) + (n-1)$

In this case, v – e +f = $(n+1)\left(\left[\frac{n}{2}\right] + 1\right)$ – {$\left[\frac{n}{2}\right](2n+1) + 2n - 1$ } + $\left(n \times \left[\frac{n}{2}\right]\right) + (n-1)$ = 1

So we conclude that in the case of two extreme cases

v – e + f = 1 (10)

[**Note:** We have identified two observations,

(1) For an even integer n, and the next one (n+1) both, each one of M_n can be constructed in a rectangular mesh constructed by n x [n/2] unit squares. As the number of squares remains same, the number of vertices (v) for the mesh, both M_n and M_{n+1}, remains constant.

(2) The result v – e + f = 1, though proved for extreme cases, holds true for any one of M_n paths.]

1.3.2: To prove that for each one of Motzkin paths –

M_n: (n – e + f) + (n +1) [n/2] = 0

This result can be easily proved by the help of the above result (10), i.e. v – e + f = 1
and the result (5) for number of vertices, i.e. $v = (n+1)\left(\left[\frac{n}{2}\right] + 1\right)$.

Proper substitution on the left side $v = (n+1)\left(\left[\frac{n}{2}\right] + 1\right)$ will help establish the above result.

i.e. (n – e + f) + (n + 1) [n/2] = 0 (11)

1.4 Illustrations for the two results

At this stage we illustrate the soundness of the results proved above.

Example-1: For the path that coincides with unit squares.

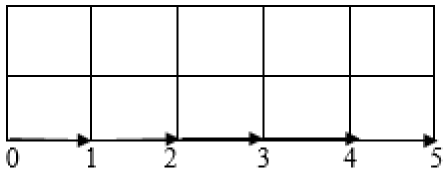

Fig 5

n = 5, [n/2] = 2, $v = (n+1)\left(\left[\frac{n}{2}\right] + 1\right)$ = (6) (3) = 18,

e = number of edges = $e = \left[\frac{n}{2}\right](2n+1) + n$ = (2) (10 + 1) + 5 = 27 and number of faces = $f = n \times \left[\frac{n}{2}\right]$ = 5 x (2) = 10

We can check both the results v – e +f =1 and (n – e + f) + (n +1) [n/2] = 0

Example-2: For the path that does not totally coincides with unit squares. We consider

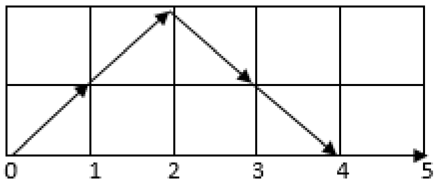

Fig 6

Here we have n=5 (an odd integer), [n/2] = 2, $v = (n+1)\left(\left[\frac{n}{2}\right] + 1\right)$ = (6) (3) = 18,

Number of edges = e = $\left[\frac{n}{2}\right](2n+1) + (2n-1)$ = (2) x (10 +1) + (10 – 1) = 31,

Number of faces = f = $\left(n \times \left[\frac{n}{2}\right]\right) + (n-1)$ = (5 x 2) + (5 -1) = 14

We can check both the results v – e +f =1 and (n – e + f) + (n +1) [n/2] = 0

Example – 3 A different case: We consider any Motzkin path which is other than the above mentioned extreme cases. We take n= 6 and draw any graph.

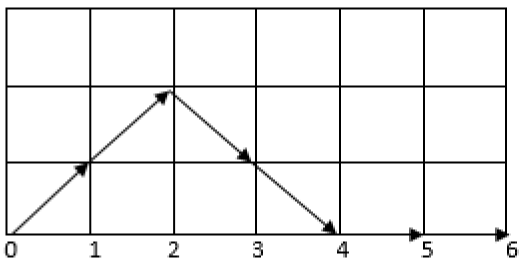

Fig 7

Number of vertices $v = (n+1)\left(\left[\frac{n}{2}\right] + 1\right)$ = (6 +1) (3 + 1) = 28

Number of Edges = e = 24 + 3x7 + 4 = 49 (As two of the six edges coincides with sides of the squares)

Number of faces = 18 + 4 = 22 (The total faces = 14 complete squares + 8 (= 4 x2))

We can check both the results v – e +f =1 and (n – e + f) + (n +1) [n/2] = 0

These three illustrations are fairly enough to explain the soundness of the results.

1.5 An overview of Motzkin and Catalan numbers

At this stage we make an overview of the various aspects of Catalan numbers and associated geometrical features and those of Motzkin numbers. These two are closely associated in most of the common features. The following table shows comparative description of prime factors.

S. No.	Description	Motzkin numbers	Catalan Numbers
1	Path Proceedings	1. Horizontal 2. Diagonally up or down (Not crossing the horizontal line)	1. Horizontal 2. Vertical (Not crossing the main diagonal of main square grid of $n \times n$ squares)
2	Interpretation	M_n shows number of path and each path is of n directed line segments *	C_nRepresents number of path and each path is of 2n directed line segments.**
3	Mesh of unit squares	$n \times \left[\frac{n}{2}\right]$	$n \times n$
4	Some initial terms corresponding to n = 1, 2, 3, 4, …	$M_n = 1,2,4,9,21, ...$	$C_n = 1,2,5,14,42, ...$
5	Recurrence Relation	$M_{n+1} = M_n + \sum_{k=o}^{n-1} M_k * M_{n-k-1}$; $n \in N$ and $M_0 = 1 = M_1$	$C_{n+1} = \sum_{i=0}^{n} C_i * C_{n-i}, n \geq 0, C_0 = 1$
6	Formula	$M_n = \sum_{k=0}^{k=\left[\frac{n}{2}\right]} \binom{n}{2k} C_k$, where C_k is Catalan number	$C_n = \frac{1}{n+1}\binom{2n}{n}$
7	Path length in structure of unit squares	Minimum length = n Maximum length = n √2 (n even) = 1 +(n-1)√2 (n odd)	Path length = 2n

*One of the M_n paths (n = 5):

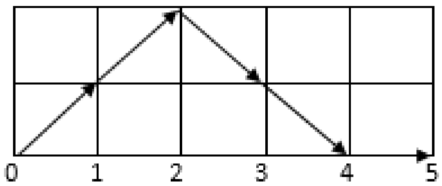

Fig 8

**One of the C_n, paths (n = 5)

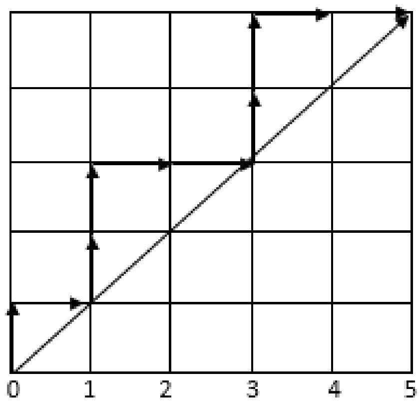

Fig 9

Conclusion

The different modules on Motzkin system discussed in the above note have inspired us to think going out of the box shading different logical approaches and search for some parallel cases that can possibly lead to some new frame work. The approach itself leaves many such problems open wide for probable research area to those opting for the branch.

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