International Journal of Statistics and Applied Mathematics 2018; 3(3): 270-273

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2018; 3(3): 270-273

www.mathsjournal.com
Received: 18-03-2018
Accepted: 23-04-2018

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To study common fixed point theorems in complete Hausdroff uniform spaces

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DOI: https://doi.org/10.22271/maths.2018.v3.i3c.606

Abstract
There are several fixed point theorems in Mathematics having numerous applications. In this paper, we will discuss some fixed point theorems that are applicable in Complete Hausdroff uniform space. Some common fixed points of point-valued mappings on a sequentially complete Hausdroff uniform space satisfying contractive type conditions have been obtained.

Keywords: Pseudometric, Hausdroff space

Introduction
This paper analyzes some common fixed point theorems in sequentially complete Hausdroff uniform spaces in such a way that they generalize some theorems which have already been proved in metric spaces. Let (X, υ) is a uniform space.
A family $\{d_\lambda$: λ ϵ Γ, Γ is an index set$\}$ of pseudometrics is called an associated family for the uniformity u if the family β = {V(I, r): i ϵ Γ, r > 0}, where V(I, r) = {(x, y): x, y ϵ X, d(x, y) < r} is a sub-base for the uniformity u.
A family $\{d_\lambda\}$, λ ϵ Γ, Γ is an index set of pseudometrics on X is called an augmented associated family for u if $\{d_\lambda$: λ ϵ Γ$\}$ is an associated family for u and has the additional property:
Given a, b ϵ Γ, there is a v ϵ Γ such that d_v(x, y) ≥ max $\{d_a$(x, y), d_b(x, y)$\}$.
An associated family and an augmented family will be denoted by P and P^*respectively. For details one can see Kelley [2], Thorn [5], etc.

Definitions
Let S and T be self-mapping of a sequentially complete Hausdroff uniform space (X, υ) defined by $\{d_\lambda$: λ ϵ Γ$\}$ = P^*. S and Tare said to be weakly commutative on X if
d_λ((STx, TSx) ≤ d_λ(Tx, Sx) for all x ϵ X and λ ϵ Γ.
Further S and T are said to be compatible if lim d_λ(STx_n, TSx_n) = 0 for all λ ϵ Γ whenever $\{x_n\}$ is a sequence in x and that lim d_λ(Sx_n, t) = d_λ(Tx_n, t) for all λ ϵ Γ and for some t ϵ λ.
The above two definitions are analogous to the definitions as introduced by Sessa [4] and Jungck [1] in metric spaces while proving some common fixed point theorems in metric spaces. It is to be noted that weakly commutative mappings are compatible but the converse is not true by Sastry [3].
Let w: [0, ∞) →[0, ∞) be such that w is continuous and 0 < w(r) < r for r > 0.
Now, we prove the following theorems:

Theorem 1: Let (X, υ) be a sequentially complete Hausdroff uniform space defined by $\{d_\lambda$: λ ϵ Γ$\}$ = P^*. Let f, g, h, and jss are four mappings on X satisfying.
d_λ(fx, gy) ≤ max $\{d_\lambda$(jx, hy), d_λ(fx, gx), d_λ(gy, hy)
$\frac{d_\lambda(jx,gy)+ d_\lambda\,(hy,fx)}{2}$ - w[max $\{d_\lambda$ (jx, hy), d_λ (fx, gx), d_λ (gy, hy)
$\frac{d_\lambda(jx,gy)+ d_\lambda\,(hy,fx)}{2}$ for all x, y ϵ X, λ ϵ Γ.

Let h and j be continuous, h and g be compatible, f and j be compatible. If f(x) ⊂ h(x) and g(x) ⊂ j(x), then f, g, h, and j have a unique common fixed point in X.

Proof: Suppose f(x) ⊂ h(x) and g(x) ⊂ j(x),

Let $x_0 \epsilon$ X. Then there exists a sequence $\{x_n\}$ such that
y_{2n} = fx_{2n} = hx_{2n+1} and y_{2n+1} = gx_{2n+1} = jx_{2n+2} for all n = 0, 1, 2,….
Now for all x, y ϵ X, λ ϵ Γ. We have $d_\lambda(y_{2n}, y_{2n+1})$ = d_λ(fx_{2n}, gx_{2n+1})
≤ max{d_λ(jx_{2n}, hx_{2n+1}), d_λ(fx_{2n}, jx_{2n}), d_λ(gx_{2n}, hx_{2n+1}), d_λ(jx_{2n}, gx_{2n+1}), $\frac{d_\lambda(hx_{2n}, fx_{2n})}{2}$}- w[max {$d_\lambda$(j$x_{2n}$, h$x_{2n+1}$), d_λ(fx_{2n}, jx_{2n}), d_λ(gx_{2n}, hx_{2n+1}), d_λ(jx_{2n}, gx_{2n+1}), $\frac{d_\lambda(jx_{2n}, gx_{2n+1})+ d_\lambda(hx_{2n}, fx_{2n})}{2}$}]
≤ max { d_λ(gx_{2n-1}, fx_{2n}), d_λ(fx_{2n}, gx_{2n-1}), d_λ(gx_{2n+1}, fx_{2n}), $\frac{d_\lambda(gx_{2n-1}, gx_{2n+1})+ d_\lambda(fx_{2n}, fx_{2n})}{2}$} – w[max { d_λ(gx_{2n-1}, fx_{2n}), d_λ(fx_{2n}, gx_{2n-1}), d_λ(gx_{2n+1}, fx_{2n}), $\frac{d_\lambda(gx_{2n-1}, gx_{2n+1})+ d_\lambda(fx_{2n}, fx_{2n})}{2}$}]
If $d_\lambda(y_{2n-1}, y_{2n}) < d_\lambda(y_{2n}, y_{2n+1})$ i.e., if
d_λ(gx_{2n-1}, fx_{2n}) < d_λ(gx_{2n+1}, fx_{2n}), then
d_λ(gx_{2n-1}, gx_{2n+1}) ≤ d_λ(gx_{2n-1}, fx_{2n}) + d_λ(fx_{2n}, gx_{2n+1})
< d_λ(fx_{2n}, gx_{2n+1}) + d_λ(fx_{2n}, gx_{2n+1})
And, $d_\lambda(y_{2n}, y_{2n+1}) \le \{d_\lambda(y_{2n}, y_{2n+1})\}$ – w[$d_\lambda(y_{2n-1}, y_{2n+1})$] < $d_\lambda(y_{2n}, y_{2n+1})$.

Which is a contradiction and so
$d_\lambda(y_{2n}, y_{2n+1}) \le d_\lambda(y_{2n-1}, y_{2n})$.
We next show that $\lim_{n\to\infty} d_\lambda(y_{n-1}, y_n) = 0$ for each λ ϵ Γ.
Since $d_\lambda(y_{n-1}, y_n)$ is a decreasing sequence of non-negative terms, then $\lim_{n\to\infty} d_\lambda(y_{n-1}, y_n)$ = d ϵ R, say, we want to prove that d = 0.
Suppose d > 0 and since d is continuous and $\sum_{i=0}^{n} d_\lambda(y_i, y_{i+1}) \le d_\lambda(y_0, y_1)$ - $d_\lambda(y_n, y_{n+1})$
$\Rightarrow \sum_{i=0}^{n} d_\lambda(y_n, y_{n+1}) < d_\lambda(y_0, y_1)$.
Hence, the series $\sum_{i=0}^{n} d_\lambda(y_i, y_{i+1})$ is convergent.
Hence $\lim_{n\to\infty} w(d_n) = 0$.

Since, d_n is a decreasing sequence of non-negative terms,
We have $\lim_{n\to\infty} (d_n)$ = d, say R^+.

Since w is continuous, it follows that $\lim_{n\to\infty} w(d_n)$ = w(d) and therefore w(d) = 0.
But since w(r) > 0 for r > 0 and so d = 0, i.e. $\lim_{n\to\infty} d_\lambda(y_{n-1}, y_n) = 0$ for λ ϵ Γ.

We now show that $\{y_n\}$ is a Cauchy sequence. Then for every positive number ε and for every positive integer K, there exists two positive integers 2m(K) and 2n(K) such that 2m(K) > 2n(K) > K and $d_\lambda(y_{2m(K)}, y_{2n(K)}) > \varepsilon$ and $d_\lambda(y_{2m(K)-2}, y_{2n(K)}) \le \varepsilon$.
Further let 2m(K) denote the smallest even integer for which 2m(K) > 2n(K) > k,
$d_\lambda(y_{2m(K)}, y_{2n(K)}) > \varepsilon$ and $d_\lambda(y_{2m(K)-2}, y_{2n(K)}) \le \varepsilon$

Now, $\varepsilon < d_\lambda(y_{2m(K)}, y_{2n(K)})$
$\le d_\lambda(y_{2m(K)-2}, y_{2n(K)}) + d_\lambda(y_{2m(K)-2}, y_{2n(K)-1}) + d_\lambda(y_{2m(K)-1}, y_{2n(K)})$.

Letting K→∞, we get
$d_\lambda(y_{2m(K)}, y_{2n(K)}) = \varepsilon$ for each λ ϵ Γ.

By triangle inequality, we have
$|d_\lambda(y_{2m(K)}, y_{2n(K)+1}) - d_\lambda(y_{2m(K)}, y_{2n(K)})| \le d_\lambda(y_{2m(K)}, y_{2n(K)+1})$,
$|d_\lambda(y_{2m(K)+1}, y_{2n(K)+1}) - d_\lambda(y_{2n(K)}, y_{2n(K)+1})| \le d_\lambda(y_{2m(K)}, y_{2m(K)+1})$,
$|d_\lambda(y_{2m(K)+1}, y_{2n(K)+2}) - d_\lambda(y_{2m(K)+1}, y_{2n(K)+1})| \le d_\lambda(y_{2m(K)+1}, y_{2m(K)+2})$,
$|d_\lambda(y_{2m(K)+1}, y_{2n(K)+2}) - d_\lambda(y_{2m(K)+1}, y_{2n(K)+2})| \le d_\lambda(y_{2m(K)+1}, y_{2m(K)+1})$,
$\therefore \varepsilon = \lim_{k\to\infty} d_\lambda(y_{2m(K)}, y_{2n(K)+1})$
$= \lim_{k\to\infty} d_\lambda(y_{2m(K)+1}, y_{2n(K)+1})$
$= \lim_{k\to\infty} d_\lambda(y_{2m(K)+1}, y_{2n(K)+2})$
$= \lim_{k\to\infty} d_\lambda(y_{2m(K)}, y_{2n(K)+2})$ for each λ ϵ Γ.

By the given assumption, $d_\lambda(y_{2m(K)+1}, y_{2n(K)+1})$ = d_λ(g$x_{2m(K)+1}$, f$x_{2n(K)+1}$)
≤ max {d_λ(g$x_{2n(K)+2}$, h$x_{2m(K)+2}$), d_λ(f$x_{2n(K)+2}$, j$x_{2n(K)+2}$), d_λ(g$x_{2n(K)+1}$, h$x_{2m(K)+1}$), $\frac{d_\lambda(jx_{2n(K)+2}, gx_{2m(K)+1})+ d_\lambda(hx_{2m(K)}, fx_{2n(K)+2})}{2}$}

- w[max{d_λ(g$x_{2n(K)+2}$, h$x_{2m(K)+2}$), d_λ(f$x_{2n(K)+2}$, j$x_{2n(K)+2}$), d_λ(g$x_{2n(K)+1}$, h$x_{2m(K)+1}$), $\frac{d_\lambda(\text{j}x_{2n(K)+2},\text{g}x_{2m(K)+1}) + d_\lambda(\text{h}x_{2m(K)},\text{f}x_{2n(K)+2})}{2}$}]
= max{$d_\lambda(y_{2m(K)}, y_{2n(K)+1})$, $d_\lambda(y_{2m(K)+1}, y_{2n(K)+2})$, $d_\lambda(y_{2m(K)}, y_{2m(K)+1})$, $\frac{d_\lambda(y_{2n(K)+1},y_{2m(K)+1}) + d_\lambda(y_{2m(K)},y_{2n(K)+2})}{2}$}
-w [max{$d_\lambda(y_{2m(K)}, y_{2n(K)+1})$, $d_\lambda(y_{2m(K)+1}, y_{2n(K)+2})$, $d_\lambda(y_{2m(K)}, y_{2m(K)+1})$, $\frac{d_\lambda(y_{2n(K)+1},y_{2m(K)+1}) + d_\lambda(y_{2m(K)},y_{2n(K)+2})}{2}$}] for each λ ϵ Γ.

Letting K⟶∞, we get,
$\varepsilon \le \varepsilon - \omega(\varepsilon) < \varepsilon$.

Which is a contradiction. Thus $\{y_n\}$ *is a* Cauchy sequence.
Since X is sequentially complete, there is a point ξ ϵ X such that ξ = $\lim_{n\to\infty}(y_n)$. Consequently, {fx_{2n}} = { hx_{2n+1}} and {gx_{2n+1}} = { jx_{2n+2}} converges to ξ. The mapping j is continuous. Then we have for all λ ε Γ,
d_λ(fjx_{2n}, gx_{2n+1}) ≤ max {d_λ(jjx_{2n}, hx_{2n+1}), d_λ(fjx_{2n}, jjx_{2n}), d_λ(gx_{2n}, hx_{2n+1}), $\frac{d_\lambda(\text{jj}x_{2n},\text{g}x_{2n+1}) + d_\lambda(\text{h}x_{2n+1},\text{hj}x_{2n})}{2}$} – w [max d_λ(fjx_{2n}, gx_{2n+1}) ≤ max {d_λ(jjx_{2n}, hx_{2n+1}), d_λ(fjx_{2n}, jjx_{2n}), d_λ(gx_{2n}, hx_{2n+1}), $\frac{d_\lambda(\text{jj}x_{2n},\text{g}x_{2n+1}) + d_\lambda(\text{h}x_{2n+1},\text{hj}x_{2n})}{2}$}].

Since the mapping f and j are compatible, then we have,
d_λ(jξ, ξ) ≤ max {d_λ(jξ, ξ), d_λ(jξ, jξ), d_λ(ξ, ξ), d_λ(jξ, ξ)}
- w[max d_λ(jξ, ξ) ≤ max {d_λ(jξ, ξ), d_λ(jξ, jξ), d_λ(ξ, ξ), d_λ(jξ, ξ)}]
= d_λ(jξ, ξ) – w(d_λ(jξ, ξ)).

Now, we consider ξ ≠ jξ. Since(X, υ) is a Hausdroff space and ξ ≠ jξ, there is an index μ ε Γ such that d_μ(ξ, jξ) ≠ 0.

Therefore we have,
d_μ(jξ, ξ) ≤ d_μ(jξ, ξ) – w(d_μ(jξ, ξ)) < d_μ(jξ, ξ) which is a contradiction.
Hence jξ = ξ.

Further, we have for λ ϵ Γ,
d_λ(fξ, gx_{2n+1}) ≤ max {d_λ(jξ, hx_{2n+1}), d_λ(hx_{2n+1}, fξ), d_λ(gx_{2n+1}, hx_{2n+1}), $\frac{d_\lambda(\text{j}\xi,\text{g}x_{2n+1}) + d_\lambda(\text{h}x_{2n+1},\text{f}\xi)}{2}$} – w [max d_λ(jξ, hx_{2n+1}), d_λ(hx_{2n+1}, fξ), d_λ(gx_{2n+1}, hx_{2n+1}), $\frac{d_\lambda(\text{j}\xi,\text{g}x_{2n+1}) + d_\lambda(\text{h}x_{2n+1},\text{f}\xi)}{2}$}].

Taking limit as n⟶∞, we have,
d_λ(fξ, ξ) ≤ max[d_λ(ξ, ξ), d_λ(fξ, ξ), d_λ(ξ, ξ), $\frac{1}{2}d_\lambda$(ξ, ξ)}- w [max d_λ(fξ, ξ) ≤ max[d_λ(ξ, ξ), d_λ(fξ, ξ), d_λ(ξ, ξ), $\frac{1}{2}d_\lambda$(ξ, ξ)].
i.e. d_λ(fξ, ξ) ≤ d_λ(fξ, ξ) – w(d_λ(fξ, ξ)) for each λ ϵ Γ, which is a contradiction.
Hence fξ = jξ = ξ.

Since h is continuous, we can show that gξ = hξ = ξ and ξ is a common fixed point of f, g, h, and j.
To prove uniqueness of ξ, if possible let ξ' be another fixed point of f, g, h, and j and let ξ ≠ ξ'. Then there is an index ν ϵ Γ such that d_ν(ξ, ξ') = d_ν(fξ, gξ')
≤ max { d_ν(jξ, hξ'), d_ν(fξ, jξ), d_ν(gξ', hξ'), $\frac{d_\nu(\text{j}\xi,\text{g}\xi')+d_\nu(\text{h}\xi',\text{f}\xi')}{2}$}
-w[max{ d_ν(jξ, hξ'), d_ν(fξ, jξ), d_ν(gξ', hξ'), $\frac{d_\nu(\text{j}\xi,\text{g}\xi')+d_\nu(\text{h}\xi',\text{f}\xi')}{2}$}]
= max{d_ν(ξ, ξ'),d_ν(ξ, ξ),, { d_ν(ξ', ξ), d_ν(ξ, ξ')}
-w[max{d_ν(ξ, ξ'),d_ν(ξ, ξ),d_ν(ξ', ξ), d_ν(ξ, ξ')}]
= d_ν(ξ, ξ') – w[d_ν(ξ, ξ')) < d_ν(ξ, ξ') which is a contradiction.
Hence ξ = ξ' and ξ is the unique common fixed point off, g, h and j.

Corollary 1
Let (X, υ) be a sequentially complete Hausdroff Uniform Space defined by {d_λ: λ ϵ Γ} = P^*.
Let f be a mapping on X satisfying the condition
d_λ(fx, gy) ≤ max{ d_λ(x, y), d_λ(hx, x), d_λ(hy, y), $\frac{d_\lambda(\text{x,hy})+d_\lambda(\text{y,fx})}{2}$}
-w[max d_λ(fx, gy) ≤ max{ d_λ(x, y), d_λ(hx, x), d_λ(hy, y), $\frac{d_\lambda(\text{x,hy})+d_\lambda(\text{y,fx})}{2}$}] for all x, y ϵ X, λ ϵ Γ. Then f has a unique fixed point in X.

Proof: Put f = g and j = h = I, identity mapping in theorem 1, Corollary 1 follows.

Corollary 2
Let (X, υ) be a sequentially complete Hausdroff Uniform Space defined by {d_λ: λ ϵ Γ} = P^*.
Let f, h be mappings on X satisfying the condition

d_{λ}(fx, hy) ≤ max{ d_{λ}(hx, hy), d_{λ}(fx, hx), d_{λ}(fy, hy), $\frac{d_{\lambda}(hx,hy)+d_{\lambda}(hy,fx)}{2}$}
-w[max {d_{λ}(hx, hy) d_{λ}(fx, hx), d_{λ}(fy, hy), d_{λ}(fy, hy), $\frac{d_{\lambda}(hx,hy)+d_{\lambda}(hy,fx)}{2}$}] for all x, y ϵ X, λ ϵ Γ. Then f and h have a unique common fixed point in X, provided h is continuous, f and h are compatible and f(X) ⊂ h(X).

Proof: Put f = g and j = h in theorem 1, Corollary 2 follows.

Corollary 3
Let (X, υ) be a sequentially complete Hausdroff Uniform Space defined by {d_{λ}: λ ϵ Γ} = P^{*}.
Let f, g, and h be three mappings on X satisfying the condition
d_{λ}(fx, fy) ≤ max{ d_{λ}(hx, hy), d_{λ}(fx, hx), d_{λ}(fy, hy), $\frac{d_{\lambda}(hx,gy)+d_{\lambda}(hy,fx)}{2}$}
-w[max d_{λ}(fx, fy) ≤ max{d_{λ}(hx, hy), d_{λ}(fx, hy), d_{λ}(fx, hx), d_{λ}(fy, hy) $\frac{d_{\lambda}(hx,gy)+d_{\lambda}(hy,fx)}{2}$}] for all x, y ϵ X, λ ϵ Γ. Let h be continuous, h and f be compatible. If f(X) ⊂ h(X), then f, g and h have a unique common fixed point in X.

Proof: Put h = j in theorem 1, Corollary 3 follows.

Corollary 4
Let (X, υ) be a sequentially complete Hausdroff Uniform Space defined by {d_{λ}: λ ϵ Γ} = P^{*}.
Let f, j, and h be three mappings on X satisfying the condition
d_{λ}(fx, hy) ≤ max{ d_{λ}(jx, hy), d_{λ}(fx, jx), d_{λ}(fy, hy), $\frac{d_{\lambda}(jx,hy)+d_{\lambda}(hy,fx)}{2}$}
-w[max d_{λ}(fx, hy) ≤ max{d_{λ}(jx, hy), d_{λ}(fx, jx), d_{λ}(fy, hy), d_{λ}(fy, hy) $\frac{d_{\lambda}(jx,hy)+d_{\lambda}(hy,fx)}{2}$}] for all x, y ϵ X, λ ϵ Γ. Let h and j be continuous, h and f, j and f be compatible. If f(X) ⊂ h(X) ∩ j(X) then f, j and h have a unique common fixed point in X.

Proof: Put f = g in theorem 1, Corollary 4 follows.

Conclusion
In this paper, we analyzed some properties of Pseudometrics. A large number of literatures are available which deal with .fixed and common fixed points of point-valued mappings in metric spaces, Banach Spaces, Hilbert Spaces, etc. Very few literatures are available which deal with common fixed points of point-valued mappings in Uniform Hausdroff Spaces. In this paper attempt has been made to obtain some common fixed point theorems in sequentially complete Hausdroff space. Also, this new working area will be a powerful tool for the existence solution.

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