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Bayes estimation of the shape parameter of burr type XII distribution with generalised squared error loss and linex loss functions under some prior distributions

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Abstract
In this paper, we obtain some classical and Bayes estimators of the shape parameters of the Burr Type XII distribution. The Bayes estimation was made under generalized squared error loss function (GSELF) and linear exponential (LINEX) loss functions with a non-informative prior (Jeffreys prior) and an informative prior (Gamma prior).

Keywords: Bayes estimation, prior distribution, posterior distribution, loss function and mean square error

1. Introduction
Twelve different forms of distribution functions useful in modeling wide range of experimental and biological data (eg forestry, fracture roughness, life testing, operational risk etc), were introduced by Burr (1942). Among these distributions the Burr Types III, X and XII have over time gained more recognition and greater application in researches and literatures because of their flexibility and compatibility with other distributions. For instance, Burr types III and XII approximate several distributions in the families of non-normal distributions (see Burr, 1973; Rodriguez, 1977; Tadikamalia, 1980 and Headrick *et al.,* 2010). Burr Type X distribution is particularly important in modelling processes with gradually increasing failure rate without any bound. It is also effective in modelling strength data and life time (Surles and Padgett, 2001).
Several studies had been carried out on Burr Type XII distribution which include Wu *et al,* (2007), Makhdoom and Jafari (2011), Soliman *et al,* (2011), Panahi and Asadi (2011), Sindhu and Aslami (2013), Al-Saiari *et al,* (2014), Jang (2014), and Tasuim *et al,* (2015).This paper focused on Bayes estimation of the shape parameter of a two-parameter Burr Type XII distribution with respect to Generalized Square Error Loss Function (GSELF) and Linear Exponential (LINEX) loss function using Jeffrey's prior as a non-informative prior and gamma prior as an informative prior, with the aim of comparing the performance of the estimators under the two loss functions.

2. Classical Estimation of the Shape Parameter of Burr Type XII Distribution
Here we consider three methods of classical estimation, the Maximum Likelihood Estimation (MLE), Minimum Mean Square Error (MSE) and the Uniform Minimum Variance Unbiased Estimation (UMVUE).
Let $X \sim Burr\ Type\ XII\ (\alpha, \beta)$, then the probability density function (pdf) of X is given by:

$$f(x:\alpha,\beta) = \frac{\alpha\beta x^{\beta-1}}{(1-x^{\beta})^{\alpha+1}} \quad ; x,\alpha,\beta > 0 \tag{2.1}$$

where α is the shape parameter and β, the scale parameter.
It can easily be shown that (2.1) is a valid pdf and the log-likelihood function is given by:

$$InL(x:\alpha,\beta) = nIn\alpha + nIn\beta + (\beta-1)\sum_{i=1}^{n} Inx_i - (\alpha+1)\sum_{i=1}^{n} In(1+x^{\beta}) \tag{2.2}$$

Thus we obtain the MLE as:

$$\hat{\alpha}(\beta)_{MLE} = \frac{n}{\sum_{i=1}^{n} In(1+x^{\beta})} = \frac{n}{W} \text{ where } W = \sum_{i=1}^{n} In(1+x^{\beta}) \tag{2.3}$$

(Rasheed and Najam (2014) evaluated the Minimum Mean Square Error (MinMSE) in the class of estimators represented as $\frac{K}{W}$. and obtained K as:

$$K = \frac{\alpha E\left(\frac{1}{W}\right)}{E\left[\left(\frac{1}{W}\right)^2\right]} \tag{2.4}$$

Rasheed and Najam (2014) noted that, since (2.1) is in the family of exponential distribution; it implies that W has a Gamma distribution with parameters n and α (i.e $W \sim G(n.\alpha)$). Hence $E\left(\frac{1}{W}\right) = \frac{\alpha}{n-1}$ and $E\left[\left(\frac{1}{W}\right)^2\right] = \frac{\alpha^2}{(n-1)(n-2)}$. Substituting these expectations in (2.4) imply $K = n-2$ and thus the MinMSE of α is;

$$\hat{\alpha}(\beta)_{MinMSE} = \frac{K}{W} = \frac{n-2}{\sum_{i=1}^{n} In(1+x^{\beta})} \tag{2.5}$$

Based on Lehmann-Scheffe's theorem, $\frac{n-1}{W}$ is an unbiased estimate of α since $E\left(\frac{1}{W}\right) = \frac{\alpha}{n-1}$ and as such W is a complete sufficient statistic for α. Therefore the UMVUE of α is given by:

$$\hat{\alpha}(\beta)_{UMVUE} = \frac{n-1}{\sum_{i=1}^{n} In(1+x^{\beta})} \tag{2.6}$$

Yarmohammadi and Pizira (2010) derived the Mean Square Error (MSE) of the three classical estimators and also showed that
$MSE_{\alpha}(\hat{\alpha}(\beta)_{MinMSE}) \le MSE_{\alpha}(\hat{\alpha}(\beta)_{UMVUE}) \le MSE_{\alpha}(\hat{\alpha}(\beta)_{MLE})$

3. Bayes Estimation of the Shape Parameter of Burr Type XII Distribution

Bayes estimation involves the estimation of an unknown parameter which is regarded as a random variable from a given probability distribution. This can be carried out using a non-informative prior distribution (eg Jeffrey prior) or an informative prior distribution of the parameter of interest, in consideration of observed data say, $x_1, x_2, \ldots\ldots, x_n$. However, Jeffrey prior and gamma prior distribution were considered as non-informative and informative prior distributions of the shape parameter of Burr Type XII distribution and estimations were made under GSELF and LINEX Loss Function.

3.1 The Posterior Distributions

The non-informative prior distribution for α using Jeffrey prior is given by:

$$g_1(\alpha) \propto \sqrt{I(\alpha)} \tag{3.1}$$

where is the Fisher information defined as:

$$I(\alpha) = -nE\left(\frac{\partial^2 Inf(x:\alpha,\beta)}{\partial\alpha^2}\right) \tag{3.2}$$

Taking the natural log of (2.1), we have

$$Inf(x:\alpha,\beta) = In\alpha + In\beta + (\beta-1)Inx - (\alpha+1)In(1+x^{\beta}) \tag{3.3}$$

The second order partial derivative of (3.3) with respect to α gives

$$\frac{\partial^2 Inf(x:\alpha,\beta)}{\partial\alpha^2} = -\frac{1}{\alpha^2} \tag{3.4}$$

Substituting (3.4) into (3.2) implies:

$$I(\alpha) = \frac{n}{\alpha^2}$$

which further implies that

$$g_1(\alpha) = K\frac{\sqrt{n}}{\alpha} \quad ;\alpha > 0 \tag{3.5}$$

We obtain the posterior density function of α given Jeffrey's prior as:

$$h_1(\alpha / x_1, x_2, \ldots, x_n) = \frac{L(x;\alpha,\beta)g_1(\alpha)}{\int_0^{\infty} L(x:\alpha,\beta)g_1(a)da} = \frac{\alpha^n\beta^n e^{(\beta-1)\sum_{i=1}^{n} Inx_i} e^{-(\alpha+1)\sum_{i=1}^{n} In(1+x^{\beta})} K\frac{\sqrt{n}}{\alpha}}{\int_0^{\infty} \alpha^n\beta^n e^{(\beta-1)\sum_{i=1}^{n} Inx_i} e^{-(\alpha+1)\sum_{i=1}^{n} In(1+x^{\beta})} K\frac{\sqrt{n}}{\alpha}d\alpha}$$

$$\Rightarrow h_1(\alpha / x_1, x_2, \ldots, x_n) = \frac{W^n\alpha^{n-1}e^{-\alpha W}}{\Gamma(n)} \tag{3.6}$$

Thus the posterior distribution of α given Jeffrey's prior is Gamma with parameters n and W.
If the prior distribution of α is gamma with parameters θ and λ then the probability density function (pdf) of α is given by:

$$g_2(\alpha) = \frac{\lambda^{\theta}\alpha^{\theta-1}e^{-\alpha\lambda}}{\Gamma(\theta)} \tag{3.7}$$

Using (3.7), we obtain the posterior distribution of α as:

$$h_2(\alpha / x_1, x_2, \ldots, x_n) = \frac{L(x;\alpha,\beta)g_2(\alpha)}{\int_0^{\infty} L(x:\alpha,\beta)g_2(a)da} = \frac{\alpha^n\beta^n e^{(\beta-1)\sum_{i=1}^{n} Inx_i} e^{-(\alpha+1)\sum_{i=1}^{n} In(1+x^{\beta})} \frac{\lambda^{\theta}\alpha^{\theta-1}e^{-\alpha\lambda}}{\Gamma(\theta)}}{\int_0^{\infty} \alpha^n\beta^n e^{(\beta-1)\sum_{i=1}^{n} Inx_i} e^{-(\alpha+1)\sum_{i=1}^{n} In(1+x^{\beta})} \frac{\lambda^{\theta}\alpha^{\theta-1}e^{-\alpha\lambda}}{\Gamma(\theta)}d\alpha}$$

$$\Rightarrow h_1(\alpha / x_1, x_2, \ldots, x_n) = \frac{(W+\lambda)^{(n+\theta)}\alpha^{(n+\theta)-1}e^{-\alpha(W+\lambda)}}{\Gamma(n+\theta)} \tag{3.8}$$

Thus the posterior distribution of α given Gamma prior is Gamma with parameters $n+\theta$ and $W+\lambda$.

3.2 Bayes Estimation under GSELF

Rasheed and Al Gazi (2014) gave the Generalized Square Error Loss function (GSELF) $l(\hat{\theta},\theta)$ as $L\ (\hat{\theta},\theta) = (\sum a_j\theta^j)(\hat{\theta}-\theta)^2\,; K = 0.1.2,3,..$, and obtained an estimator for θ using the corresponding risk function as:

$$\hat{\theta} = \frac{a_0E(\theta|t)+a_1E(\theta^2|t)+\cdots+a_KE(\theta^{K+1}|t)}{a_0+a_1E(\theta|t)+\cdots+a_KE(\theta^K|t)} \tag{3.9}$$

Substituting α for θ and x for t in (3.9)

$$\rightarrow \hat{\alpha} = \frac{a_0E(\alpha|x)+a_1E(\alpha^2|x)+\cdots+a_KE(\alpha^{K+1}|x)}{a_0+a_1E(\alpha|x)+\cdots+a_KE(\alpha^K|x)} \tag{3.10}$$

The result in (3.6) implies that $E(\alpha|x) = \frac{n}{w}$ and $E(\alpha^2|x) = \frac{n(n+1)}{w^2}$, hence substituting these moments in (3.10) gives the GSELF estimator for α using Jeffrey's prior,

$$\hat{\alpha}_{GSELF(J)} = \frac{a_0\frac{n}{w}+a_1\frac{(n+1)n}{w^2}+\cdots+a_K\frac{(n+k)(n+k-1)\ldots(n+1)n}{w^{k+1}}}{a_0+a_1\frac{n}{w}+\cdots+a_K\frac{(n+k-1)(n+k-2)\ldots(n+1)n}{w^k}}$$

$$= \frac{\sum_{j=o}^{n} a_j \frac{\Gamma(n+1+j)}{\left(\sum_{i-1}^{n} In\left(1+x^{\beta}\right)\right)^{j+1}\Gamma(n)}}{\sum_{j=o}^{n} a_j \frac{\Gamma(n+j)}{\left(\sum_{i-1}^{n} In\left(1+x^{\beta}\right)\right)^{j}\Gamma(n)}} \quad (3.11)$$

From (3.8) $E(\alpha|x) = \frac{n+\theta}{W+\lambda}$ and $E(\alpha^2|x) = \frac{(n+\theta)(n+\theta+1)}{(W+\lambda)^2}$, hence the GSELF estimator for α using gamma prior,

$$\hat{\alpha}_{GSELF} = \frac{a_0\frac{n+\theta}{w+\lambda}+a_1\frac{(n+\theta+1)(n+\theta)}{(w+\lambda)^2}+\cdots+a_K\frac{(n+\theta+k)(n+\theta+k-1)\ldots(n+\theta+1)(n+\theta)}{(w+\lambda)^{k+1}}}{a_0+a_1\frac{n+\theta}{w+\lambda}+\cdots+a_K\frac{(n+\theta+k-1)(n+\theta+k-2)\ldots(n+\theta+1)(n+\theta)}{w^k}}$$

$$= \frac{\sum_{j=o}^{n} a_j \frac{\Gamma(n+\theta+1+j)}{\left(\sum_{i-1}^{n} In\left(1+x^{\beta}\right)+\lambda\right)^{j+1}\Gamma(n+\theta)}}{\sum_{j=o}^{n} a_j \frac{\Gamma(n+\theta+j)}{\left(\sum_{i-1}^{n} In\left(1+x^{\beta}\right)+\lambda\right)^{j}\Gamma(n+\theta)}} \quad (3.12)$$

3.3 Bayes Estimation under LINEX

Rasheed and Sultan (2015) applied LINEX loss function of the form:

$$L(\hat{\theta},\theta) = b\left[e^{a\left(\hat{\theta}-\theta\right)} - a(\hat{\theta}-\theta) - 1\right] \quad (3.13)$$

where $b > 0, a \neq 0$

and obtained an estimator for θ using the risk function as:

$$\hat{\theta} = \frac{-\ln E\left(e^{-a\theta}\right)}{a} \quad (3.14)$$

Substituting α for θ in (3.14), we have

$$\hat{\alpha} = \frac{-\ln E\left(e^{-a\alpha}\right)}{a} \quad (3.15)$$

Applying the result in (3.6), we obtained an estimator for α under LINEX loss function with Jeffrey's prior as follows;

$$E\left(e^{-a\alpha}\right) = \int_0^{\infty} e^{-a\alpha} h_1(\alpha : x_1, x_2, .., x_n) d\alpha = \int \frac{e^{-a\alpha} W^n \alpha^{n-1} e^{-W\alpha}}{\Gamma(n)} d\alpha = \frac{1}{n}\left(\frac{W}{a+W}\right)^n \quad (3.16)$$

Substituting (3.16) in (3.15), we have

$$\hat{\alpha}(\beta)_{LINEX\,(j)} = \frac{-\ln\left[\frac{1}{n}\left(\frac{W}{a+W}\right)^n\right]}{a} = \frac{-\ln\left[\frac{1}{n}\left(\frac{\sum_{i=1}^{n}\ln(1+x^{\beta})}{a+\sum_{i=1}^{n}\ln(1+x^{\beta})}\right)^n\right]}{a} \quad (3.17)$$

Given the posterior distribution in (3.8), the LINEX estimator for α using gamma prior is obtained as follows:

$$E\left(e^{-a\alpha}\right) = \int_0^{\infty} e^{-a\alpha} h_1(\alpha : x_1, x_2, .., x_n) d\alpha = \int \frac{e^{-a\alpha}(W+\lambda)^{(n+\theta)} \alpha^{(n+\theta)-1} e^{-(W+\lambda)\alpha}}{\Gamma(n+\alpha)} d\alpha$$

$$= \frac{1}{(n+\theta)}\left(\frac{W+\lambda}{a+W+\lambda}\right)^{n+\theta} \quad (3.18)$$

$$\Rightarrow \hat{\alpha}(\beta)_{LINEX\ (g)} = \frac{-\ln\left[\frac{1}{(n+\theta)}\left(\frac{\sum_{i=1}^{n}\ln(1+x^{\beta})+\lambda}{a+\lambda+\sum_{i=1}^{n}\ln(1+x^{\beta})}\right)^{n+\theta}\right]}{a} \qquad (3.19)$$

4. Conclusion

In this work, we have been able to derive the Bayes estimator of the shape parameter of the Burr type IX distribution under squared error loss and LINEX functions with Jeffrey's and Gamma(α, β) priors.

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