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Oscillation criteria for second order neutral impulsive differential equations with a forcing term

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Abstract
This paper considers a class of second order nonlinear neutral impulsive differential equations with a forcing term, having fixed moments of impulse actions. Conditions for forced oscillation of all solutions are obtained and examples are given to illustrate the relevance of the results.

Keywords: Second-order, impulsive, neutral delay differential equation, forced oscillation

1. Introduction
This work aims at obtaining conditions for the oscillation of all solutions of the second-order nonlinear neutral impulsive differential equation with a forcing term of the form

$$\begin{cases} \left[y(t)+p\,y(t-\tau)\right]''+f\left(t,y(t-\sigma)\right)=Q(t),\ t\neq t_k \\ \Delta\left[y(t_k)+p\,y(t_k-\tau)\right]'+f_k\left(t_k,y(t_k-\sigma)\right)=Q(t_k),\ t=t_k. \end{cases} \tag{1.1}$$

We assume throughout this discussion and without further mention, the following conditions:

C1.1: $p,\tau>0$ and $\sigma\geq 0;$

C2.1: $f,f_k\in C\left([t_0,\infty)\,x\,R,R\right),\ y\cdot f(t,y),\ y\cdot f_k(t_k,y)>0,\ y\neq 0;$

C3.1: There exists a function $u(t)\in C^2\left([t_0,\infty),R\right)$ such that

$$\begin{cases} Q(t)=u''(t) \\ Q(t_k)=\Delta u'(t_k) \end{cases}$$

And u changes sign on $[T,\infty)$ for any $T\geq t_0$.

Set

$$u_*(t)=min\left\{\frac{\alpha(t-\tau)}{2},\ \frac{\alpha(t)}{2p}\right\},$$

$$u_*^+(t)=max\left\{u_*(t),0\right\},\quad u_*^-(t)=max\left\{-u_*(t),0\right\}.$$

Second order differential equations are generally the most important in applications. Same also applies to neutral second order delay impulsive differential equations which have been developed to model impulsive problems in physics, population dynamics, biotechnology, pharmacokinetics, industrial robotics, and so forth. A few recent results can be seen in [17-21]. The introduction of oscillation theory has further boosted the concept and particularly

helped in identifying more areas of applications both within and outside differential equations.
It is worth mentioning here that a lot of work has been done in the area of forced oscillations for neutral/non-neutral differential equations, with or without delay [13, 15, 14, 8, 9, 10, 16, 6]. Some results have been obtained for forced oscillations of differential equations, with or without delay, subject to impulse actions [11, 12, 4]. However, there seems to be a dearth of the study of forced oscillation theory of second-order nonlinear impulsive delay ordinary differential equations of neutral type. This could be due to the complications that arise as a result of the momentary perturbations or impulsive jumps. Motivated by this, we seek sufficient conditions for the oscillation of all solutions of equation (1.1). The above ideology becomes more meaningful if we define other related terms and concepts that will continue to be useful as we progress.
In ordinary differential equations, the solutions are continuously differentiable, sometimes at least once, whereas impulsive differential equations generally possess non-continuous solutions. Since the continuity properties of the solutions play an important role in the analysis of the behaviour, the techniques used to handle the solutions of impulsive differential equations are fundamentally different, including the definitions of some of the basic terms. In this section, we examine some of these changes.

Notation 1.1: Let $J=(\alpha,\beta)\subset R,\ -\infty<\alpha<\beta<+\infty$ is our domain of investigation

Definition 1.1: Let $S:=\{t_k\}_{k\in E}\subset J$ be a strictly ascending sequence of the time moments of impulse effects and let E be a subscript set which can be the set of natural numbers N or the set of integers Z such that

- $t_k\to\infty$ if $k\to\infty$ and if $E=Z$, then $t_k\to-\infty$ if $k\to-\infty$;
- $t_k\ge 0$ if $k\ge 0$.

Our equation under consideration then has the form

$$\begin{cases} \left[y(t)+p\,y(t-\tau)\right]''+f\left(t,y(t-\sigma)\right)=Q(t),\ t\ge t_0,\ t\in J/S \\ \Delta\left[y(t_k)+p\,y(t_k-\tau)\right]'+f_k\left(t_k,y(t_k-\sigma)\right)=Q(t_k),\ t_k\ge t_0,\forall\, t_k\in S, \end{cases} \tag{1.2}$$

Where $1\le k\le\infty$.
In order to simplify the statements of the assertions, we introduce the set of functions PC and PC^r which are defined as follows: Let $D:=[T,\bar{T})\subset J\subset R$ and let the set of impulse points S be fixed.

Definition 1.2: Let $PC(D,R):=\{\varphi\,|\,\varphi:D\to R,\ \varphi\in C(D\backslash S),\ \exists\ \varphi(t-0),\ \varphi(t+0),\ \forall\, t\in D\}$.
From the studies in Bainov and Simeonov (1998), Lakshmikantham *et al* (1989) and Isaac *et al.* (2011) [1, 7, 4], we define the function space $PC^r\ \forall\ r\in N$:

Definition 1.3: Let $PC^r(D,R):=\left\{\varphi\,|\,\varphi\in PC(D,R),\ \dfrac{d^j\varphi}{dt^j}\in PC(D,R),\ \forall\ 1\le j\le r\right\}$.

To specify the points of discontinuity of functions belonging to PC and PC^r, we shall sometimes
Use the symbols $PC(D,R;S)$ and $PC^r(D,R;S),\ r\in N$ [3].

Definition 1.4 The solution $y(t)$ of the impulsive differential equation (1.1) is said to be

1. finally positive (finally negative) if there exist $T\ge 0$ such that $y(t)$ is defined and is strictly positive (negative) for $t\ge T$ [3];
2. non-oscillatory, if it is either finally positive or finally negative; and
3. Oscillatory, if it is neither finally positive nor finally negative [1, 5].

Definition 1.5: An impulsive differential equation is said to be oscillatory if all its solutions are oscillatory.
The following is a basic lemma that is essential in carrying out our investigation. It is an extension of Lemma 4.7.1 of the work by Erbe *et al.* [2]. It is noteworthy to mention here that, in the sequel, all functional inequalities are assumed to hold finally for all t large enough. Again, without loss of generality, we will deal only with the positive solutions of equation (2.1).

Lemma 1.1: Assume $x(t)\in PC\left([t_0,\infty),R\right),\ \beta\in PC\left([t_0,\infty),R_+\right)$ and $x(t)+px(t-\tau)\ge\beta(t)\ge 0,\ t\ge t_0$, where $p,\tau>0$. Then for each $t^*\ge t_0+\tau$, there exists a set $A=\left\{t:t^*\le t\le t^*+2\tau,\ x(t-\tau)\ge\beta_*(t)\right\}$ with the Lebesgue measure $\lambda(A)\ge\tau$, where

$$\beta_*(t)=min\left\{\frac{\beta(t-\tau)}{2},\frac{\beta(t)}{2p}\right\}.$$

Proof: For any fixed $t^* \geq t_0 + \tau$, we define a set

$$B = \left\{ t : t \in [t^*, t^* + \tau], x(t) > \frac{\beta(t)}{2} \right\}.$$

If B is empty $(B = \Phi)$, then $p\,x(t-\tau) \geq \frac{\beta(t)}{2}$, for $t \in [t^*, t^* + \tau]$, that is, $A = [t^*, t^* + \tau]$. Now we consider the case that $B \neq \Phi$, then $\lambda(B) = \alpha \in (0, \tau)$. Let $\bar{B}$ denote the closure of B. In view of the piece-wise continuity of $x(t)$, we have $x(t) \geq \frac{\beta(t)}{2}$, $t \in \bar{B}$. Define a set $\bar{B} + \tau = \{t, t - \tau \in \bar{B}\}$. Then, $x(t-\tau) \geq \frac{\beta(t-\tau)}{2}$ for $t \in (\bar{B} + \tau)$.

Set

$$A = \{[t^*, t^* + \tau] \setminus B\} U (\bar{B} + \tau).$$

Then $\lambda(A) = \tau$ and $x(t-\tau) \geq \beta_*(\tau)$ on the set A.
This completes the proof of Lemma 1.1

2. Main Results
The following theorems are an extension of their neutral delay versions as identified in the work by Erbe *et al.* [2] on pages 273 and 274.

Theorem 2.1: Assume that conditions C1.1 – C1.3 hold. Further assume that f and f_k are non-decreasing functions in $x(t)$ and

$$\begin{aligned} &\int_E f\left(t, u_*^+(t+\tau-\sigma)\right)dt + \sum_{E \leq t_k} f_k\left(t_k, u_*^+(t_k+\tau-\sigma)\right) = \infty, \\ &\int_E f\left(t, -u_*^-(t+\tau-\sigma)\right)dt + \sum_{E \leq t_k} f_k\left(t_k, -u_*^-(t+\tau-\sigma)\right) = -\infty \end{aligned} \tag{2.1}$$

For every closed set E whose intersection with every segment of the form $[t-\tau, t+\tau]$, $t \geq t_0 + \tau$, has a Lebesgue measure not smaller then τ. Then every solution of equation (2.1) is oscillatory.

Proof: Without loss of generality, let us assume by contradiction that $y(t)$ is a finally positive solution of equation (1.1) for $t \geq t_0$. Set

$$z(t) = y(t) + p\,y(t-\tau).$$

Then $\left(z(t) + u(t)\right)'' < 0$ for $t \geq t_0 + \tau$. It is easy to show that $\left(z(t) - u(t)\right)' > 0$ finally, which implies that

$$\int_{t_0}^{\infty} f\left(t, y(t-\sigma)\right)dt + \sum_{t_0 \leq t_k < \infty} f_k\left(t_k, y(t_k - \sigma)\right) < \infty . \tag{2.2}$$

On the other hand, it is easy to show that $z(t) - u(t) > 0$ finally. Then we have
$z(t) = y(t) + p\,y(t-\tau) \geq u^+(t)$, $t \geq t_0 + \tau$.
By Lemma 1.1, for every $t^* \geq t_0 + 2\tau$, there exists a set $A = \left\{t : t^* \leq t \leq t^* + 2\tau, y(t-\tau) \geq u_*^+(t)\right\}$ with $\lambda(A) \geq \tau$. Let us consider the set $A - (\tau - \sigma) = \{t : t + (\tau - \sigma) \in A\}$. It is obvious that $\lambda\left(A - (\tau-\sigma)\right) \geq \tau$ and $y(t-\sigma) \geq u_*^+\left(t + (\tau - \sigma)\right)$, $t \in \left(A - (\tau - \sigma)\right)$. From condition (2.2), we have

$$\infty > \int_E f\left(t, y(t-\sigma)\right)dt + \sum_{E \le t_k} f_k\left(t_k, y(t_k-\sigma)\right)$$

$$\ge \int_E f\left(t, u_*^+\left(t+(\tau-\sigma)\right)\right)dt + \sum_{E \le t_k} f_k\left(t_k, u_*^+\left(t_k+(\tau-\sigma)\right)\right)$$

Which contradicts assumption (2.1). This completes the proof of Theorem 3.1.

We see this analogy in the following illustration:

Example 2.1: Consider the equation

$$\begin{cases} \left[y(t)+y(t-\pi)\right]'' + t^2 y(t-2\pi) = \sin t, \ t \notin S \\ \Delta\left[y(t_k)+y(t_k-\pi)\right]' + t^2 y(t_k-2\pi) = \sin t_k, \ \forall\, t_k \in S \end{cases} \tag{2.3}$$

We can see that condition (2.1) holds. Therefore, every solution of equation (2.3) is oscillatory. In fact, $y(t)=\sin t$ is such a solution.

We now consider equation (1.1) in the more general form as

$$\begin{cases} \left[y(t)+p(t)y(t-\tau)\right]'' + f\left(t, y\left(g(t)\right), y'\left(\sigma(t)\right)\right) = Q(t), t \notin S \\ \Delta\left[y(t_k)+p_k y(t_k-\tau)\right]' + f_k\left(t_k, y\left(g(t_k)\right), \Delta y\left(\sigma(t_k)\right)\right) = Q(t_k), \ \forall\, t_k \in S \end{cases} \tag{2.4}$$

Theorem 2.2: Assume that
1. $p(t) \in PC\left([t_0,\infty), R_+\right)$ and $p_k \ge 0, \ t_k \ge t_0$;
2. $g(t), \sigma(t) \in PC\left([t_0,\infty), R\right)$ And $Q(t) \in PC\left([t_0,\infty), R\right), \ t \ge t_0$.
3. $g(t)$ Is non-decreasing and $\lim_{t\to\infty} g(t) = \infty$.
4. $f \in C\left([t_0,\infty) x R^2, R\right)$ and $f(t,u,v)u > 0, \ u \ne 0$;
5. For any $T \ge t_0$.

$$\begin{cases} \liminf_{t\to\infty}\left[\int_T^t Q(s)ds + \sum_{T \le t_k < t} Q_k\right] = -\infty, \quad \limsup_{t\to\infty}\left[\int_T^t Q(s)ds + \sum_{T \le t_k < t} Q_k\right] = \infty, \\ \liminf_{t\to\infty}\left[\int_T^t \int_T^t Q(u)duds + \sum_{T \le t_k < t}\ \sum_{T \le t_k < s} Q_k\right] = -\infty, \quad \limsup_{t\to\infty}\left[\int_T^t \int_T^s Q(u)duds + \sum_{T \le t_k < t}\ \sum_{T \le t_k < s} Q_k\right] = \infty. \end{cases} \tag{2.5}$$

Then every solution of equation (2.4) is oscillatory.

Proof: Without loss of generality, let us assume by contradiction that $y(t)$ is a finally positive solution of equation (2.4). Set $z(t) = y(t) + p(t)y(t-\tau)$. Then $z(t) > 0, \ t \ge T \ge t_0$. From equation (2.4), we discover that $z''(t) < Q(t)$. Thus,

$$z'(t) - z'(T) < \int_T^t Q(s)ds + \sum_{T \le t_k < t} Q_k \ . \tag{2.6}$$

By condition (v), there exists a sufficiently large $T^* \ge t_0$ such that $z'\left(T^*\right) < 0$. Replacing T by T^* in inequality (2.6), we obtain

$$z'(t) < \int_{T^*}^t Q(s)ds + \sum_{T^* \le t_k < t} Q_k$$

And

$$z(t)-z(T^*)<\int_{T^*}^{t}\int_{T^*}^{s}Q(u)\,duds+\sum_{T^*\le t_k<t}\ \sum_{T^*\le t_k<s}Q_k\ .$$

Therefore

$$\liminf_{t\to\infty}\ z(t)=-\infty\,,$$

Which contradicts the positivity of $z(t)$. This completes the proof of Theorem 2.2.

This is seen in the following illustration:

Example 2.2: Consider the equation

$$\begin{cases}\left[y(t)+y(t-\pi)\right]''+ty(t-2\pi)=t\sin t,\ t\notin S\\ \Delta\left[y(t_k)+y(t_k-\pi)\right]'+t_k y(t_k-2\pi)=t_k\sin t_k,\ \forall\ t_k\in S.\end{cases}\tag{2.7}$$

It is easy to see that all assumptions of Theorem 2.2 are satisfied. Therefore, every solution of equation (2.7) is oscillatory In fact, $y(t)=\sin t$ is such a solution.

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