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Fixed point theorem in cone banach spaces

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Abstract
In this paper we prove some fixed point theorem in Cone Banach Space by the help of continuous and compatibility of mappings. This results is a generalization of many results in this literature.
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1. Introduction
In 2007, Huang and Zhang [4] introduced the notion of Cone Metric Space, replacing the set of real numbers by ordered Banach Space and proved some fixed point theorems for functions satisfying Contractive conditions in this space. In the last decade many researchers fined fixed point theorems in this space and generalized many results such as M.Abbas, G.Junck [1], R.K. Gujetiya *et al.* [3] etc. very recently some results on fixed point theorems have been extended to Cone Banach Space. E. Karapinar [5] in 2009 proved some fixed point theorems for self mappings satisfying some contractive condition on a Cone Banach Space.

1.1 Definition 1: [7] Let E be a real Banach spaces and K be a subset of E. K is called a cone if and only if

a) K is closed, nonempty and $K \neq \{0\}$,
b) ax+by in K for all x, y in K and a, b ≥ 0,
c) $x \in K$ and $-x \in K \Rightarrow x=0 \Leftrightarrow K \cap (-K) = \{0\}$.

Consider a cone $K \subset E$. We define a partial ordering " $\leq$ " with respect to K by $x \leq y$ if and only if $y-x \in K$, we write $x < y$ to indicate that $x \leq y$ but $x \neq y$ and $x << y$ to indicate that $y-x \in$ intK. The int K denotes the interior of K.
Let X be a non empty set and $K \subset E$ be a real Banach space. Suppose the metric mapping d: $X \times X \rightarrow E$ is satisfies the following conditions:

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ iff x=y, for all x, y in X,
2. $d(x, y) = d(y, x)$, for all x, y in X,
3. $d(x, z) \leq d(x, y) + d(y, z)$; for all x, y, z in X.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

1.2 Definition 2: [7] Let, X be a vector space over R. Suppose the mapping $\|.\|: X \rightarrow E$ satisfies

1. $\|x\| > 0$, for all $x \in X$.
2. $\|x\| = 0$ if and only if x=0.
3. $\|x + y\| \leq \|x\| + \|y\|$, for all $x,y \in X$.
4. $\|kx\| = |k|\|x\|$, for all $k \in R$.

Then $\|.\|$ is called a norm on x, and $(X, \|.\|)$ is called a cone normed space. Clearly each cone normed space is a cone metric space with defined by $d(x,y) = \|x - y\|$.

1.3 Definition 3: [7] Let, $(X, \|.\|)$ be a cone normed space, $x \in X$ and $\{x_n\}$ a sequence in X. then

1. $\{x_n\}$ converges to x if for every $c \in E$ with $0 << c$ there is a natural number N such that $\|x_n - x\| \leq c$ for all $n \geq N$. we shall denote it by $\lim_{n\rightarrow\infty} x_n = x$ or, $x_n \rightarrow x$.

2. $\{x_n\}$ is a Cauchy sequence, if for every $c \in E$ with $0<<c$ there is a natural number N such that $\|x_n - x_m\| \le c$ for all $n,m \ge N$.
3. $(X,\|.\|)$ is a complete cone normed space if every Cauchy sequence is convergent.

A complete cone normed space is called a Cone Banach space.

1.4 Definition 4: Two self maps A and B of a cone normed space $(X,\|.\|)$ are said to be compatible if $\lim_{n\to\infty}\|ABx_n - BAx_n\| = 0$ for all a in X, where $\{x_n\}$ is a sequence in X such that if $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x$ for some x in X.

1.5 Definition 5: Maps f and g are said to be commuting if fgx = gfx, for all $x \in X$.

1.6 Definition 6: Let, f and g be two self maps on a set X, if fx=gx for some x in X then x is called coincidence point of f and g.

2. Main Theorem

2.1 Theorem 1: Let, $(X,\|.\|)$ be a Cone Banach Space and f,g,h, l be mappings on X such that

1. $\|hx - ly\| \le k \max\{\|fx - hx\|,\|gy - ly\|,\|fx - gy\|\}$, where $0<k<1$ …(1)
2. f and g are onto mappings.
3. f is continuous.
4. (f, h) and (g, l) are commute

Then, f, g, h and l have a unique common fixed point.

Proof: Let, $x_0 \in X$ be arbitrary, then $l(x_0) \in X$, since f is onto there is an $x_1 \in X$ such that $f(x_1)= l(x_0)$. let, $y_0= f(x_1)= l(x_0)$. Again $x_1 \in X$ since, g is onto there is a $x_2 \in X$ such that $g(x_2) =h(x_1)$. Let, $y_2= g(x_2) =h(x_1)$. Continuing like this we get a sequence $\{y_n\}$ such that $y_{2n}= f(x_{2n+1})= l(x_{2n})$ and $y_{2n+1}= g(x_{2n+2}) =h(x_{2n+1})$. We have from (1)

$\|y_{2n-1} - y_{2n}\| = \|h(x_{2n-1}) - l(x_{2n})\|$

$\le k \max\{ \|f(x_{2n-1}) - h(x_{2n-1})\|, \|g(x_{2n}) - l(x_{2n})\|, \|f(x_{2n-1}) - g(x_{2n})\|\}$,

$= k \max\{ \|y_{2n-2} - y_{2n-1}\|, \|y_{2n-1} - y_{2n}\|, \|y_{2n-2} - y_{2n-1}\|\}$ …(2)

If $\|y_{2n-1} - y_{2n}\|>\|y_{2n-2} - y_{2n-1}\|$ then, from (2) we get,

$\|y_{2n-1} - y_{2n}\| \le k \|y_{2n-1} - y_{2n}\|<\|y_{2n-1} - y_{2n}\|$ (as $0<k<1$)

Which is a contradiction, so $\|y_{2n-1} - y_{2n}\|<\|y_{2n-2} - y_{2n-1}\|$ and from (2) we get,

$\|y_{2n-1} - y_{2n}\|\le k \|y_{2n-2} - y_{2n-1}\|$ ...(3)

And, $\|y_{2n} - y_{2n+1}\| = \|h(x_{2n}) - l(x_{2n+1})\|$

$\le k \max\{ \|f(x_{2n}) - h(x_{2n})\|, \|g(x_{2n+1}) - l(x_{2n+1})\|, \|f(x_{2n}) - g(x_{2n+1})\|\}$,

$= k \max\{ \|y_{2n-1} - y_{2n}\|, \|y_{2n} - y_{2n+1}\|, \|y_{2n-1} - y_{2n}\|\}$. …(4)

If $\|y_{2n} - y_{2n+1}\|>\|y_{2n-1} - y_{2n}\|$ then, from (2) we get,

$\|y_{2n} - y_{2n+1}\| \le k \|y_{2n} - y_{2n+1}\|<\|y_{2n} - y_{2n+1}\|$ (as $0<k<1$)

Which is a contradiction, so $\|y_{2n} - y_{2n+1}\|<\|y_{2n-1} - y_{2n}\|$ and from (4) we get,

$\|y_{2n} - y_{2n+1}\|\le k \|y_{2n-1} - y_{2n}\|$, ...(5)

From (4) and (5) we get, $\|y_n - y_{n+1}\| \le k^n\|y_0 - y_1\|$ …(6)

Now, let $m > n$, then,

$\|y_n - y_m\| \le \|y_n - y_{n+1}\| +\|y_{n+1} - y_{n+2}\| +\ldots+\|y_{m-1} - y_m\|$

$\le(k^n +k^{n+1}+\ldots+k^{m-1}) \|y_0 - y_1\|$ (by (3))

$=\frac{k^n(1-k^m)}{1-k}\|y_0 - y_1\|\le\frac{k^n}{1-k}\|y_0 - y_1\|$. ..(7)

Let, $c > 0$, then there is a $\delta >0$ such that $c+N_\delta(0) \subseteq H$ where $N_\delta(0) =\{y \in X: \|y\| \le\delta\}$. Since $k <1$ there exists a positive integer N such that $\frac{k^n}{1-k}\|y_0 - y_1\| \le \delta$ for every $n \ge N$. Hence $\frac{k^n}{1-k}\|y_0 - y_1\| \in N_\delta(0)$, which implies $-\frac{k^n}{1-k}\|y_0 - y_1\| \in N_\delta(0)$. Therefore, $c -\frac{k^n}{1-k}\|y_0 - y_1\| \in c+ N_\delta(0) \subseteq H$ implies $\frac{k^n}{1-k}\|y_0 - y_1\| \le c$ for $n \ge N$. So, by definition $\{y_n\}$ is a Cauchy sequence in X.

Since X is complete there exists a z in X such that $\lim_{n\to\infty} y_n=z$, and $\lim_{n\to\infty} lx_{2n}= \lim_{n\to\infty} fx_{2n+1} = \lim_{n\to\infty} hx_{2n+1}=\lim_{n\to\infty} gx_{2n+2}= z$. Continuity of f implies $f^2x_{2n+1}\to fz$. Since h and f commute $h fx_{2n+1}=fhx_{2n+1}\to fz$. …………..(8)

We have from (1),

$\|hf(x_{2n+1}) - l(x_{2n})\|$

$\le k \max\{ \|ff(x_{2n+1}) - hf(x_{2n+1})\|,\|g(x_{2n}) - l(x_{2n})\|, \|ff(x_{2n+1}) - g(x_{2n})\|\}$,

$= k \max \{ \|fz - fz\|, \|z - z\|, \|fz - z\| \}$ (by(8))
$= k \max \{0, 0, \|fz - z\|\}$.

Taking $\lim_{n\to\infty}$ in the above inequality we get,
$\|fz - z\| \le k \|fz - z\| < \|fz - z\|$ (as $0<k<1$)

Which gives $\|fz - z\|=0$ i.e., fz=z. …….(9)

Now put x=z and y=x_{2n} in (1) and get,
$\|hz - l(x_{2n})\| \le k \max\{ \|fz - hz\|, \|g(x_{2n}) - l(x_{2n})\|, \|fz - g(x_{2n})\|\}$,
$= k \max \{\|z - hz\|, \|z - z\|, \|z - z\|\}$,
$= k \max \{\|z - hz\|, 0, 0\}$.
Taking $\lim_{n\to\infty}$ in the above inequality we get,
$\|hz - z\| \le k \|hz - z\| < \|hz - z\|$ (as $0<k<1$)

Which gives $\|hz - z\|=0$ i.e., hz=z. ……(10)

Since, g is onto, so there exists a u in X such that gu=z. …………(11)

Put x= fx_{2n+1} and y= u in (1) and get,
$\|hf(x_{2n+1}) - lu\| \le k \max\{ \|ff(x_{2n+1}) - hf(x_{2n+1})\|, \|gu - lu\|, \|ff(x_{2n+1}) - gu\|\}$,
$= k \max \{ \|fz - fz\|, \|z - lu\|, \|fz - z\|\}$, (by(11))
$= k \max \{0, \|lu - z\|, \|z - z\|\}$, (by (9))
$= k \max \{0, \|lu - z\|, 0 \}$.

Taking $\lim_{n\to\infty}$ in the above inequality we get,
$\|z - lu\| \le k \|lu - z\| < \|lu - z\|$ (as $0<k<1$)

Which gives $\|lu - z\|=0$ i.e., lu=z. ……….(12)

From (11) & (12) we get, lu=gu=z. ...……..(13)

Since, g and l are comute. So, lgu=glu i.e., lz=gz. (by (13)) ………..(14)

Now put x= x_{2n+1} and y= z in (1) and get,
$\|hx_{2n+1} - lz\| \le k \max\{ \|fx_{2n+1} - hx_{2n+1}\|, \|gz - lz\|, \|fx_{2n+1} - gz\|\}$,

Taking $\lim_{n\to\infty}$ in the above inequality and using (13) we get,
$\|z - lz\| \le k \max\{ \|z - z\|, \|z - z\|, \|z - z\|\} = k \max \{ 0,0,0\} = 0$

So, lz=z. ...…..(15)

So from (14) and (15) we get lz=gz=z. …….(16)

From (9), (10) and (16) we get, hz=fz=gz=lz=z.
Hence, z is a fixed point of h,f,g and l.
To prove uniqueness of z, let, if possible there exists another fixed point w(≠z) in X.

So, hw=fw=gw=lw=w. ………….(17)

Put, x=z and y=w in (1) and get,
$\|hz - lw\| \le k \max\{\|fz - hz\|, \|gw - lw\|, \|fz - gw\|\}$,
Or, $\|z - w\| \le k \max\{\|z - z\|, \|w - w\|, \|z - w\|\}$ (by (16) and (17))
$= k \max \{0, 0, \|z - w\|\} = k\|z - w\| < \|z - w\|$. (as $0<k<1$)
So, $\|z - w\|=0$. i.e., z=w.
So, the fixed point z of h,f, g and l is unique.

2.2 Theorem 2: Let, $(X, \|.\|)$ be a Cone Banach Space and f and g be two self mappings on X such that
$\|fx - gy\| \le k \max\{\|x - fx\|, \|y - gy\|, \|x - y\|\}$, where $0<k<1$ …(18)
Then, f and g have a unique common fixed point.

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