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Common fixed point theorem in digital spaces using contraction mappings

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Abstract
In this paper, we prove common fixed point theorem in digital metric spaces using contraction mappings. We also give example to prove validity of proved results.

Keywords: Digital metric spaces, commuting mappings, contraction mappings

1. Introduction

Digital topology is a developing area based on general topology and functional analysis which studies features of 2D and 3D digital images. Rosenfield [9] was the first to consider digital topology as the tool to study digital images. The digital versions of the topological concepts were given by Boxer [1], who later studied digital continuous functions [2]. Later, he gave results of digital homology groups of 2D digital images in [3] and [4]. Ege and Karaca [5, 8] give various fixed point theorems in digital metric spaces. In this paper, we prove common fixed point theorem in digital metric spaces using contraction mappings. We also give example to prove validity of proved results.

2. Preliminaries

Firstly, we give some definitions used in our paper:

Definition 2.1: [5] Let l, n be positive integers, $1 \leq l \leq n$ and two distinct points p = $(p_1, p_2, \dots p_n)$, q = $(q_1, q_2, \dots q_n) \in \mathbb{Z}^n$, p and q are k_l-adjacent if there are at most l indices i such that $|p_i - q_i| = 1$ and for all other indices j such that $|p_j - q_j| \neq 1, p_j = q_j$.

Definition 2.2: [5] Two points p and q in $\mathbb{Z}$ are 2-adjacent if $|p - q| = 1$.

Definition 2.3: [5] Two points p and q in $\mathbb{Z}^2$ are 8-adjacent if they are distinct and differ by at most 1 in each coordinate.

Definition 2.4: [5] For some digital image $X \subset \mathbb{Z}^n$ and $p \in X$, we define the digital k-neighborhoods of p with some radius ϵ:
$N_k(p) = \{x \in X : l_k(p, x) \leq \epsilon\} \cup \{p\}$,
Where $l_k(p, x)$ is the length of a shortest simple curve from p to x, $\epsilon \in \mathbb{N}$.

Definition 2.5: [5] A digital image $X \subseteq \mathbb{Z}^n$ is k-connected if and only if for every pair of different point x, y ∈ X, there is a set $\{x_0, x_1, \dots x_r\}$ of points of digital image X such that x = x_0, y = x_r and x_i and x_{i+1} are k- neighbourhood where i = 0,1,2…r−1.

Definition 2.6: [5] Let (X, k_0) $\subseteq \mathbb{Z}^{n_0}$, (Y, k_1) $\subseteq \mathbb{Z}^{n_1}$be digital images and f: X → Y be a function.

i. If for every k_0-connected subset U of X, f (U) is a k_1-connected subset of Y, then f is said to be (k_0, k_1)-continuous.
ii. f is (k_0, k_1)-continuous if and only if for every k_0-adjacent points {x_0, x_1} of X, either f(x_0) = f(x_1) or f(x_0) and f(x_1) are k_1-adjacent in Y.

Definition 2.7. [5] A $(2, k)$-continuous function f: $[0, \mathrm{m}]_{\mathbb{Z}} \rightarrow$ X such that f(0) = x and f(m) = y is called a digital k -path from x to y in a digital image X.

Definition 2.8 [9]: Let (X, k) be a digital image and f: (X, k)→ (X, k) be any (k, k)-continuous function. We say the digital image (X, k) has the fixed point property [10] if for every (k, k)-continuous map f: X→ X there exists x ∈ X such that f (x) = x.

Definition 2.8 [9]: Let (X, d, k) denote the digital metric space with k -adjacency where d is usual Euclidean metric for $\mathbb{Z}^n$.
A sequence $\{x_n\}$ of points of a digital metric space (X, d, k) is a Cauchy sequence if for all ϵ > 0, there exists α∈ℕ such that for all n, m > α, then d(x_n, x_m) < ϵ.

Definition 2.9 [5]: Let (X, d, k) denote the digital metric space with k -adjacency where d is usual Euclidean metric for $\mathbb{Z}^n$. A sequence $\{x_n\}$ of points of a digital metric space (X, d, k) converges to a limit a ∈ X if for all ϵ > 0, there exists α ∈ ℕ such that for all n > α, then d(x_n, a) < ϵ.

Definition 2.10 [5]: Let (X, d, k) denote the digital metric space with k -adjacency where d is usual Euclidean metric for $\mathbb{Z}^n$. A digital metric space (X, d, k) is a complete digital metric space if any Cauchy sequence $\{x_n\}$ of points of (X, d, k) converges to a point a of (X, d, k).

Definition 2.11 [5]: Let (X, d, k) denote the digital metric space with k -adjacency where d is usual Euclidean metric for $\mathbb{Z}^n$. Let (X, k) be any digital image. A function f: (X, k)→(X, k) is called right-continuous if f (a) = $\lim_{x \to a^+} f(x)$where a ∈ X.

Definition 2.12 [5]: Let (X, d, k) denote the digital metric space with k -adjacency where d is usual Euclidean metric for $\mathbb{Z}^n$. Let (X, d, k) be any digital metric space and f: (X, d, k)→ (X, d, k) be a self –digital map. If there exists β ∈ (0, 1) such that for all x, y ∈ X, d(f (x), f (y)) ≤ βd(x, y), then f is called a digital contraction map.

Theorem 2.1. [7] (Banach contraction principle) Let (X, d, k) be a complete digital metric space which has a usual Euclidean metric in $\mathbb{Z}^n$. Let f: X→ X be a digital contraction map. Then f has a unique fixed point, i.e. there exists a unique c ∈ X such that f (c) = c.

Definition 2.13: Two mappings f and g are said to be commuting maps if fgx = gfx for all x ∈X.

3. Main Results

Theorem 3.1. Let (X, d, k) be a complete digital metric space with k -adjacency where d is usual Euclidean metric for $\mathbb{Z}^n$and let f and g be self-mappings on X satisfying the following conditions:
(3.1) f(X) ⊆ g(X);
(3.2) g is continuous;
(3.3) d(fx, fy) ≤ q d(gx, gy), for every x y ∈X and 0 < q < 1.
Then f and g have a unique common fixed point in X provided f and g commute.

Proof. Let x_0 be an arbitrary point in X. By (3.1), one can choose a point $x_1$in X such that fx_0 = gx_1, In general choose x_{n+1}such that y_n = fx_n= gx_{n+1}.Now, we prove $\{y_n\}$ is a Cauchy sequence in X.
From (3.3), take x = x_n, y = x_{n+1}, z = x_{n+1} we have
d(fx_n, fx_{n+1},) ≤ q d(gx_n, gx_{n+1}) = q d(fx_{n-1}, fx_n,)
Continuing in the same way, we have
d(fx_n, fx_{n+1}) ≤ q^nd(fx_0, fx_1) implies that d(y_n, y_{n+1}) ≤ q^nd(y_0, y_1).
For n > m, d(y_n, y_m) ≤ d(y_n, y_{n-1}) + d(y_{n-1}, y_{n-2})+ · · · +d(y_m, y_{m+1})
≤$(q^{n-1} + q^{n-2} + \cdots q^m)d(y_0, y_1)$
≤$\frac{q^m}{1-q}$ d(y_0, y_1). This implies d(y_n, y_m) →1(n, m → ∞).
Hence (y_n) is a Cauchy sequence. Since(X, d, k) is complete digital metric space, therefore, there exists a point z in X such that $\lim_{n\to\infty} y_n$= $\lim_{n\to\infty} gx_n$= $\lim_{n\to\infty} fx_n$= z. Since g is continuous. Therefore
$\lim_{n\to\infty} ggx_n$= $\lim_{n\to\infty} gfx_n$= gz. Further, we have since f and g are commuting maps, therefore by definition, we get $\lim_{n\to\infty} gfx_n$= $\lim_{n\to\infty} fgx_n$= $\lim_{n\to\infty} ggx_n$= $\lim_{n\to\infty} gfx_n$= gz. From (3.3), take x = gx_n, y = x_n, we have d(fgx_n, fx_n) ≤ q d(ggx_n, gx_n). Proceeding limit as n → ∞, we have z = gz.
We now prove that z = fz.
Again from (3.3), setting x = x_n, y = z, we have d(fx_n, fz) ≤ q d(gx_n, gz).
Taking limit as n → ∞, we have z = fz. Therefore, we have gz = fz = z. Thus z is a common fixed point of f and g.

Uniqueness. We assume that z_1 (≠z) be another common fixed point of f and g.
Then d(z, z_1) > 0 and d(z, z_1) = d(fz, fz_1) ≤ q d(gz, gz_1) = q d(z, z_1) < d(z, z_1), a contradiction, therefore z = z_1. Hence uniqueness follows.

Example 3.1: Let us consider the digital metric space (X, d, k), with the standard digital metric d. Consider the self mapping f, g:$\rightarrow X$ given by, fx =2$-x^2$ and gx =x^2. Clearly f and g satisfy all the hypotheses of the Theorem 3.1 and hence have a unique common fixed point x =1.

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