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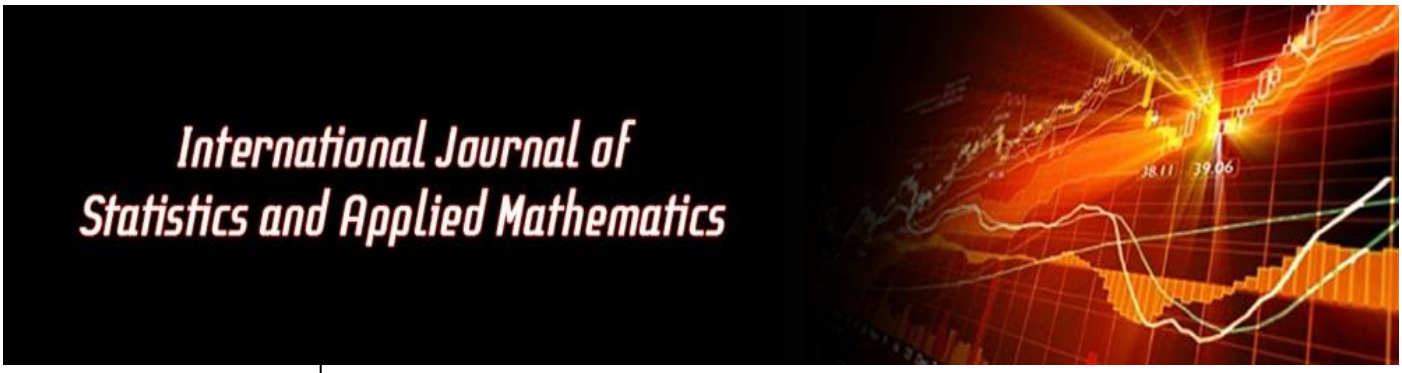

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Coupled fixed points for $(\psi-\alpha-\beta)-$ general contractions on partially ordered partial metric spaces

KPR Sastry, KKM Sarma and P Krishna Kumari

Abstract
The notion of $(\psi-\alpha-\beta)-$general contraction F on a partially ordered partial metric space with respect to another map g is introduced. We obtain conditions for the existence of coupled coincidence points and common coupled fixed points of F and g.

Keywords: Partially ordered Partial Metric Space, $(\psi-\alpha-\beta)-$contractions, $(\psi-\alpha-\beta)-$ general contractions, coupled coincidence point, common coupled fixed point
Subject Classification: MSC: 54H25, 47H10

Introduction
In 1994 Matthews. SG, [14] introduced the notion of a Partial Metric Space (PMS) as a part of the study of denotational Semantics of data flow networks. He also showed that the Banach Contraction Principle can be generalized to the PMSs. Many researchers have proved fixed point theorems in PMSs [2, 3, 5, 7, 8, 10, 11, 12, 17, 18].
Not only in Mathematics, PMSs have vast applications in Computer Science, Information Science and Biological Sciences.
Bhaskar. TG, Lakshmikantham. V [9] introduced the notion of a coupled fixed point and proved some fixed point theorems in partially ordered Metric Spaces. Later some researchers proved coupled fixed and coupled common fixed point theorems [1, 4, 11, 16, 19, 20, 21, 22].
Matthews. SG, [13, 14], Oltra. S and Valera. O [15] and Altun *et al*. [6] proved some fixed point theorems in PMSs for a single map, taking generalized contractions in PMSs.
Rao *et al*. [21] introduced the notion of $(\psi-\alpha-\beta)-$ contraction maps with respect to another map on a Partial metric space.
In this paper we introduce $(\psi-\alpha-\beta)-$ general contraction map F with respect to another map g on a partial metric space and obtain conditions for the existence of coupled coincidence points and common coupled fixed points of F and g. Supporting examples and applications are provided.

2. Preliminaries
In this section, we give the necessary definitions and results which are needed in section 3.

2.1 Definition (Matthews [13, 14])
A partial metric on non-empty set X is a function $p: X\times X\to R^{+}$ such that for all$x,y,z\in X$,
(2.1.1)$p(x,x)=p(x,y)=p(y,y)$ if and only if $x=y$
(2.1.2)$p(x,x)\le p(x,y), p(y,y)\le p(x,y)$
(2.1.3)$p(x,y)=p(y,x)$
(2.1.4)$p(x,y)\le p(x,z)+p(z,y)-p(z,z)$
The pair (X,p) is called a Partial Metric Space (PMS).

2.2 Example (Rao KPR *et al.* [19])
Let $X=\{[a,b]: a,b\in R, a\le b\}$ and define$p\{[a,b],[c,d]\}=max\{b,d\}-min\{a,c\}$. Then (X,p) is a PMS.

Each partial metric p on X generates a Topology T_p on X which has a base of the family of open p-balls $\{B_p(x,\epsilon), x \in X, \epsilon > 0\}$
Where $B_p(x,\epsilon) = \{y \in X : p(x,y) < p(x,x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.
We now state some basic Topological notions such as convergence, Cauchy sequence and completeness.

2.3 Definition (Rao KPR *et al.* [21])
A sequence $\{x_n\}$ in the PMS (X,p) converges to a point $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$. We may write this as $x_n \to x$.

2.4 Definition [21]
A sequence $\{x_n\}$ in the PMS (X,p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.

2.5 Definition [21]
A PMS (X,p) is called complete if every Cauchy Sequence $\{x_n\}$ in X converges to a point $x \in X$ so that$p(x,x) = \lim_{n,m\to\infty} p(x_n,x_m)$.

2.6 Lemma [21]
Assume $x_n \to z$ as $n \to \infty$ in a PMS (X,p), such that $p(z,z) = 0$,
Then
1. $\lim_{n\to\infty} p(x_n,y) = p(z,y)$ for every $y \in X$,
2. $x_n \to x \Longrightarrow x = z$,
3. $\{x_n\}$ is Cauchy.

2.7 Lemma [21]
Let (X,p) be a PMS. Then
(2.7.1) If $p(x,y) = 0$, then $x = y$.
(2.7.2) If $x \neq y$, then $p(x,y) > 0$.

2.8 Remark [21]
If $x = y$, $p(x,y)$ may not be zero.

2.9 Definition (Bhaskar. TG, Lakshmikantham. V [9])
Let $(X,\leq)$ be a partially ordered set and $F: X \times X \to X$. Then the map F is said to have mixed monotone property if $F(x,y)$ is monotone non-decreasing in x and monotone non-increasing in y, i.e., for any $x,y \in X$,
$x_1 \leq x_2 \Longrightarrow F(x_1,y) \leq F(x_2,y)$ for all $y \in X$
$y_1 \leq y_2 \Longrightarrow F(x,y_2) \leq F(x,y_1)$ for all $x \in X$

2.10 Definition [9]
An element $(x,y) \in X \times X$ is called a coupled fixed point of a mapping $F: X \times X \to X$ if $F(x,y) = x$ and $F(y,x) = y$.

2.11 Definition (Abbas. M *et al.* [1])
An element $(x,y) \in X \times X$ is called
1. A coupled coincidence point of mappings $F: X \times X \to X$ and $f: X \to X$ if $fx = F(x,y)$ and $fy = F(y,x)$.
2. A common coupled fixed point of mappings $F: X \times X \to X$ and $f: X \to X$ if $x = fx = F(x,y)$ and $y = fy = F(y,x)$.

2.12 Definition [1]
The mappings $F: X \times X \to X$ and $f: X \to X$ are called w-compatible if $f(F(x,y)) = F(fx,fy)$ and $f(F(y,x)) = F(fy,fx)$ whenever $fx = F(x,y)$ and $fy = F(y,x)$.

2.13 Definition (Lakshmikantham. V, Ciric. L [12])
Let $(X,\leq)$ be a partially ordered set, $F: X \times X \to X$ and $g: X \to X$ be mappings. Then the map F is said to have a mixed g-monotone property if $F(x,y)$ is monotone g-non-decreasing in x as well as monotone g-non-increasing in y, i.e., for any $x,y \in X$,
$gx_1 \leq gx_2 \Longrightarrow F(x_1,y) \leq F(x_2,y)$ for all $y \in X$ and
$gy_1 \leq gy_2 \Longrightarrow F(x,y_2) \leq F(x,y_1)$ for all $x \in X$.
Rao *et al.* [21] introduced the notion of $(\psi - \alpha - \beta)$- contraction map F with respect to another map g as follows.

2.14 Definition [21]
Let (X,p) be a PMS and $(X,\leq)$ be a partially ordered set.
Let $F: X \times X \to X$ and $g: X \to X$ be two mappings. We say that F satisfies a $(\psi - \alpha - \beta)$ − contraction with respect to g if there exist
$\psi, \alpha, \beta: [0,\infty) \to [0,\infty)$ Satisfying the following
(2.14.1) ψ is continuous and monotonically non-decreasing, α is continuous and β is lower semi continuous.
(2.14.2) $\psi(t) = 0$ if and only if $t = 0$, $\alpha(0) = \beta(0) = 0$
(2.14.3) $\psi(t) - \alpha(t) + \beta(t) > 0$ for $t > 0$,

(2.14.4) $\psi\left(p\big(F(x,y),F(u,v)\big)\right) \leq \alpha\big(m(x,y,u,v)\big) - \beta\big(m(x,y,u,v)\big)$ for all $x,y,u,v \in X$, with $gx \leq gu$, $gy \geq gv$ where

$$m(x,y,u,v) = max\left\{\begin{matrix} p(gx,gu), p(gy,gv), p\big(gx,F(x,y)\big), p\big(gy,F(y,x)\big), p\big(gu,F(u,v)\big), p\big(gv,F(v,u)\big), \\ \frac{p(gx,F(x,y))p(gy,F(y,x))}{1+p(gx,gu)+p(gy,gv)+p\big(F(x,y),F(u,v)\big)}, \frac{p(gu,F(u,v))p(gv,F(v,u))}{1+p(gx,gu)+p(gy,gv)+p\big(F(x,y),F(u,v)\big)} \end{matrix}\right\}.$$

Using the above definition Rao *et al*. [21] proved the following coupled fixed point result in partially ordered partial metric space.

2.15 Theorem [21]: Let $(X,\leq)$ be a partially ordered set and p be a partial metric so that $(X,p,\leq)$ is a partially ordered partial metric space. Let $F: X \times X \to X$ and $g: X \to X$ be such that
(2.15.1) F is a $(\psi - \alpha - \beta) -$ contraction with respect to g,
(2.15.2) $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X
(2.15.3) F has mixed g monotone property
(2.15.4) (a) If a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x \; \forall \; n$
(b) If a non-increasing sequence $\{y_n\} \to y$, then $y \leq y_n \; \forall \; n$

If there exist $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0,y_0)$ and $gy_0 \geq F(y_0,x_0)$, then F and g have a coupled coincidence point in $X \times X$.

2. Main Results

In this section, we introduce $(\psi - \alpha - \beta) -$ general contraction map F with respect to another map g on a partially ordered partial metric space and obtain conditions for the existence of coupled coincidence points and common coupled fixed points of F and g.

3.1 Definition

Let (X,p) be a PMS and $(X,\leq)$ be a partially ordered set. Let $F: X \times X \to X$ and $g: X \to X$ be two mappings. We say that F is a $(\psi - \alpha - \beta) -$ general contraction with respect to g if there exist $\psi, \alpha, \beta: [0,\infty) \to [0,\infty)$ satisfying the following
(3.1.1) ψ is continuous and monotonically non-decreasing, α is continuous and β is continuous.
(3.1.2) $\psi(t) = 0$ if and only if $t = 0$ and $\alpha(0) = \beta(0) = 0$
(3.1.3) $\psi(t) - \alpha(t) + \beta(t) > 0$ for $t > 0$,
(3.1.4) $\psi\left(p\big(F(x,y),F(u,v)\big)\right) \leq \alpha\big(M(x,y,u,v)\big) - \beta\big(M(x,y,u,v)\big)$ for all $x,y,u,v \in X$, with $gx \leq gu$, $gy \geq gv$ where

$$M(x,y,u,v) = max\left\{\begin{matrix} p(gx,gu), p(gy,gv), p\big(gx,F(x,y)\big), p\big(gy,F(y,x)\big), p\big(gu,F(u,v)\big), p\big(gv,F(v,u)\big), \\ \frac{p(gx,F(x,y))p(gy,F(y,x))}{p(gx,gu)+p(gy,gv)+p\big(F(x,y),F(u,v)\big)}, \frac{p(gu,F(u,v))p(gv,F(v,u))}{p(gx,gu)+p(gy,gv)+p\big(F(x,y),F(u,v)\big)} \end{matrix}\right\}$$

Whenever the denominator is non-zero.
Now we state and prove our first main theorem.

3.2 Theorem: Let $(X,\leq)$ be a partially ordered set and p be a partial metric on X, so that $(X,p,\leq)$ is a partially ordered partial metric space. Let $F: X \times X \to X$ and $g: X \to X$ be such that
(3.2.1) F is a $(\psi - \alpha - \beta) -$ general contraction with respect to g, where ψ, α, β satisfy
(3.1.1)-(3.1.4) of definition 3.1
(3.2.2) $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X
(3.2.3) F has mixed g monotone property
(3.2.4) (a) If a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x \; \forall \; n$
(b) If a non-increasing sequence $\{y_n\} \to y$, then $y_n \geq y \; \forall \; n$
Suppose there exist $x_0, y_0 \in X$ such that $gx_0 \leq F(x_0,y_0)$ and $gy_0 \geq F(y_0,x_0)$. Then F and g have a coupled coincidence point in$X \times X$.
Proof: Let $x_0, y_0 \in X$ be such that $gx_0 \leq F(x_0,y_0)$ and $gy_0 \geq F(y_0,x_0)$
Since $F(X \times X) \subseteq g(X)$,
We choose $x_1, y_1 \in X$ such that $gx_0 \leq F(x_0,y_0) = gx_1$ and $gy_0 \geq F(y_0,x_0) = gy_1$ and choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1,y_1)$ and $gy_2 = F(y_1,x_1)$.
Since F has mixed g monotone property, we obtain $gx_0 \leq gx_1 \leq gx_2$ and $gy_0 \geq gy_1 \geq gy_2$.
Continuing this process, we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $gx_{n+1} = F(x_n,y_n)$ and $gy_{n+1} = F(y_n,x_n), n = 0,1,2, \ldots$
So that $gx_0 \leq gx_1 \leq gx_2 \leq \cdots$ and $gy_0 \geq gy_1 \geq gy_2 \geq \cdots$
Case (i) Suppose $gx_m = gx_{m+1}$ and $gy_{m+1} = gy_m$ for some m. Then (x_m, y_m) is a coupled coincidence point of F and g in $X \times X$.
Case (ii) Assume $gx_n \neq gx_{n+1}$ or $gy_n \neq gy_{n+1}$ for all n
Since $gx_n \leq gx_{n+1}$ and $gy_n \geq gy_{n+1}$, from (3.2.1), we obtain $\psi\big(p(gx_n,gx_{n+1})\big) = \psi\left(p\big(F(x_{n-1},y_{n-1}),F(x_n,y_n)\big)\right)$
$\leq \alpha\big(M(x_{n-1},y_{n-1},x_n,y_n)\big) - \beta\big(M(x_{n-1},y_{n-1},x_n,y_n)\big)$

where $M(x_{n-1}, y_{n-1}, x_n, y_n) = max\left\{\begin{array}{c} p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n), p\big(gx_{n-1}, F(x_{n-1}, y_{n-1})\big), \\ p\big(gy_{n-1}, F(y_{n-1}, x_{n-1})\big), p\big(gx_n, F(x_n, y_n)\big), p\big(gy_n, F(y_n, x_n)\big), \\ \frac{p(gx_{n-1}, F(x_{n-1}, y_{n-1}))p(gy_{n-1}, F(y_{n-1}, x_{n-1}))}{p(gx_{n-1}, gx_n)+p(gy_{n-1}, gy_n)+p\left(F(x_{n-1}, y_{n-1}), F(x_n, y_n)\right)}, \\ \frac{p(gx_n, F(x_n, y_n))p(gy_n, F(y_n, x_n))}{p(gx_{n-1}, gx_n)+p(gy_{n-1}, gy_n)+p\left(F(x_{n-1}, y_{n-1}), F(x_n, y_n)\right)} \end{array}\right\}$

$$= max\left\{\begin{array}{c} p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n), p(gx_{n-1}, gx_n), \\ p(gy_{n-1}, gy_n), p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}), \\ \frac{p(gx_{n-1}, gx_n)p(gy_{n-1}, gy_n)}{p(gx_{n-1}, gx_n) + p(gy_{n-1}, gy_n) + p(gx_n, gx_{n+1})}, \\ \frac{p(gx_n, gx_{n+1})p(gy_n, gy_{n+1})}{p(gx_{n-1}, gx_n) + p(gy_{n-1}, gy_n) + p(gx_n, gx_{n+1})} \end{array}\right\}$$

[If $p(gx_{n-1}, gx_n) + p(gy_{n-1}, gy_n) + p(gx_n, gx_{n+1}) = 0$ then $p(gx_{n-1}, gx_n) = 0 \Longrightarrow gx_n = gx_{n-1}$, $p(gy_{n-1}, gy_n) = 0 \Longrightarrow gy_{n-1} = gy_n$ which is a contradiction to case (ii).

$\therefore p(gx_{n-1}, gx_n) + p(gy_{n-1}, gy_n) + p(gx_n, gx_{n+1}) \neq 0$]

But $\frac{p(gx_{n-1}, gx_n)p(gy_{n-1}, gy_n)}{p(gx_{n-1}, gx_n)+p(gy_{n-1}, gy_n)+p(gx_n, gx_{n+1})} \leq \max\{p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1})\}$ and $\frac{p(gx_n, gx_{n+1})p(gy_n, gy_{n+1})}{p(gx_{n-1}, gx_n)+p(gy_{n-1}, gy_n)+p(gx_n, gx_{n+1})} \leq p(gy_n, gy_{n+1})$

Therefore $M(x_{n-1}, y_{n-1}, x_n, y_n) = max\{p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n), p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1})\}$
Hence $\psi\big(p(gx_n, gx_{n+1})\big) \leq$
$\alpha(max\{p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n), p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1})\}) -$
$\beta(max\{p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n), p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1})\})$ (3.2.5)

Similarly
$\psi\big(p(gy_n, gy_{n+1})\big) \leq$
$\alpha(max\{p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n), p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1})\}) -$
$\beta(max\{p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n), p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1})\})$ (3.2.6)

Put $R_n = max\{p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1})\}$ then $R_n \neq 0$ for all $n \geq 1$ (by case (ii))
Suppose that $R_n > R_{n-1}$. Then from 3.2.5 $\psi(R_n) \leq \alpha(\ max\{R_n, R_{n-1}\}) - \beta(\max\{R_n, R_{n-1}\})$
$= \alpha(R_n) - \beta(R_n)$
$\therefore \psi(R_n) \leq \ \alpha(R_n) - \beta(R_n)$
i.e. $\psi(R_n) - \ \alpha(R_n) + \beta(R_n) \leq 0$
$\therefore R_n = 0$, which is a contradiction
$\therefore R_n \leq R_{n-1}$ (3.2.7)

Thus $\{R_n\}$ is a non-increasing sequence of non-negative real numbers and hence converges to some $r \geq 0$.
Also $\psi(R_n) \leq \ \alpha(R_{n-1}) - \beta(R_{n-1})$
Letting $n \to \infty$, we get $\psi(r) \leq \ \alpha(r) - \beta(r)$
From (3.1.2) and (3.1.3) we get $r = 0$
Thus $\lim_{n\to\infty} \max\{p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1})\} = 0$,
Consequently $\lim_{n\to\infty} p(gx_n, gx_{n+1}) = 0 = \lim_{n\to\infty} p(gy_n, gy_{n+1})$ (3.2.8)

Hence from (2.1.2), we have
$\lim_{n\to\infty} p(gx_n, gx_n) = 0 = \lim_{n\to\infty} p(gy_n, gy_n)$ (3.2.9)

Now we prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences
To the contrary, suppose that either $\{gx_n\}$ or $\{gy_n\}$ is not Cauchy
So that either $p(gx_m, gx_n) \nrightarrow 0$ or $p(gy_m, gy_n) \nrightarrow 0$ as $n, m \to \infty$
Consequently $\max\{p(gx_m, gx_n), p(gy_m, gy_n)\} \nrightarrow 0$ as $n, m \to \infty$

Then there exist an $\epsilon > 0$ and monotone increasing sequences of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k > k$ such that
$\max\{p(gx_{m_k}, gx_{n_k}), p(gy_{m_k}, gy_{n_k})\} \geq \epsilon$ (3.2.10)
And $\max\{p(gx_{m_k}, gx_{n_k-1}), p(gy_{m_k}, gy_{n_k-1})\} < \epsilon$ (3.2.11)

From (3.2.10) and (3.2.11) we have
$\epsilon \leq \max\{p(gx_{m_k}, gx_{n_k}), p(gy_{m_k}, gy_{n_k})\}$

$\leq \max\{p(gx_{m_k}, gx_{n_k-1}), p(gy_{m_k}, gy_{n_k-1})\} + \max\{p(gx_{n_k-1}, gx_{n_k}), p(gy_{n_k-1}, gy_{n_k})\}$
$< \epsilon + \max\{p(gx_{n_k-1}, gx_{n_k}), p(gy_{n_k-1}, gy_{n_k})\}$

Letting $k \to \infty$ and using (3.2.8), we get

$\lim_{k\to\infty} \max\{p(gx_{m_k}, gx_{n_k}), p(gy_{m_k}, gy_{n_k})\} = \epsilon$ (3.2.12)

And $\lim_{k\to\infty} \max\{p(gx_{m_k}, gx_{n_{k-1}}), p(gy_{m_k}, gy_{n_k-1})\} + \max\{p(gx_{n_k-1}, gx_{n_k}), p(gy_{n_k-1}, gy_{n_k})\} = \epsilon$ (3.2.13)

$\therefore \lim_{k\to\infty} \max\{p(gx_{m_k}, gx_{n_{k-1}}), p(gy_{m_k}, gy_{n_k-1})\} = \epsilon$ (3.2.14)

Consider $p(gx_{m_k}, gx_{n_k}) \leq p(gx_{m_k}, gx_{m_k-1}) + p(gx_{m_k-1}, gx_{n_k}) - p(gx_{m_k-1}, gx_{m_k-1})$
$\leq p(gx_{m_k}, gx_{m_k-1}) + p(gx_{m_k-1}, gx_{n_k-1}) + p(gx_{n_k-1}, gx_{n_k}) - p(gx_{n_k-1}, gx_{n_k-1}) - p(gx_{m_k-1}, gx_{m_k-1})$
$\leq \max\{p(gx_{m_k}, gx_{m_k-1}), p(gy_{m_k}, gy_{m_k-1})\}$
$+ \max\{p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1})\} + \max\{p(gx_{n_k-1}, gx_{n_k}), p(gy_{n_k-1}, gy_{n_k})\}$

Similarly
$p(gy_{m_k}, gy_{n_k}) \leq \max\{p(gy_{m_k}, gy_{m_k-1}), p(gx_{m_k}, gx_{m_k-1})\} + \max\{p(gy_{m_k-1}, gy_{n_k-1}), p(gx_{m_k-1}, gx_{n_k-1})\} + \max\{p(gx_{y-1}, gy_{n_k}), p(gx_{n_k-1}, gx_{n_k})\}$

$\therefore \max\{p(gx_{m_k}, gx_{n_k}), p(gy_{m_k}, gy_{n_k})\} \leq \max\{p(gx_{m_k}, gx_{m_k-1}), p(gy_{m_k}, gy_{m_k-1})\}$
$+ \max\{p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1})\}$
$+ \max\{p(gx_{n_k-1}, gx_{n_k}), p(gy_{n_k-1}, gy_{n_k})\}$
Now $p(gx_{m_k-1}, gx_{n_k-1}) \leq p(gx_{m_k-1}, gx_{m_k}) + p(gx_{m_k}, gx_{n_k-1}) - p(gx_{m_k}, gx_{m_k})$
$\leq p(gx_{m_k-1}, gx_{m_k}) + p(gx_{m_k}, gx_{n_k}) + p(gx_{n_k}, gx_{n_k-1}) - p(gx_{n_k}, gx_{n_k}) - p(gx_{m_k}, gx_{m_k})$
$\leq \max\{p(gx_{m_k-1}, gx_{m_k}), p(gy_{m_k-1}, gy_{m_k})\} + \max\{p(gx_{m_k}, gx_{n_k}), p(gy_{m_k}, gy_{n_k})\}$
$+ \max\{p(gx_{n_k}, gx_{n_k-1}), p(gy_{m_k}, gy_{n_k-1})\}$

Similarly
$p(gy_{m_k-1}, gy_{n_k-1}) \leq \max\{p(gx_{m_k-1}, gx_{m_k}), p(gy_{m_k-1}, gy_{m_k})\} + \max\{p(gx_{m_k}, gx_{n_k}), p(gy_{m_k}, gy_{n_k})\}$
$+ \max\{p(gx_{n_k}, gx_{n_k-1}), p(gy_{n_k}, gy_{n_k-1})\}$
$\therefore \max\{p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1})\} \leq \max\{p(gx_{m_k-1}, gx_{m_k}), p(gy_{m_k-1}, gy_{m_k})\}$
$+ \max\{p(gx_{m_k}, gx_{n_k}), p(gy_{m_k}, gy_{n_k})\} + \max\{p(gx_{m_k}, gx_{n_k-1}), p(gy_{n_k}, gy_{n_k-1})\}$ (3.2.15)

We have
$\max\{p(gx_{m_k}, gx_{n_k}), p(gy_{m_k}, gy_{n_k})\} \leq \max\{p(gx_{m_k}, gx_{m_k-1}), p(gy_{m_k}, gy_{m_k-1})\}$
$+ \max\{p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1})\} + \max\{p(, gx_{n_k-1}, gx_{n_k}), p(gy_{n_k-1}, gy_{n_k})\}$
Letting $k \to \infty$ in (3.2.15)
$\lim_{k\to\infty} \max\{p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1})\} \leq 0 + \epsilon + 0$
$\therefore \epsilon \leq \lim_{k\to\infty} \inf(\max\{p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1})\})$
$\leq \lim_{k\to\infty} \sup(\max\{p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1})\})$
$\leq \epsilon$
$\therefore \lim_{k\to\infty} \inf(\max\{p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1})\})$
$= \lim_{k\to\infty} \sup(\max\{p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1})\}) = \epsilon$
$\therefore \lim_{k\to\infty} \max\{p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1})\} = \epsilon$
Consider $\psi\left(p(gx_{m_k}, gx_{n_k})\right) = \psi\left(p\left(F(x_{m_k-1}, y_{m_k-1}), F(x_{n_k-1}, y_{n_k-1})\right)\right)$
$\leq \alpha\left(M(x_{m_k-1}, y_{m_k-1}, x_{n_k-1}, y_{n_k-1})\right) - \beta\left(M(x_{m_k-1}, y_{m_k-1}, x_{n_k-1}, y_{n_k-1})\right)$ (3.2.16)

Where

$$M(x_{m_k-1}, y_{m_k-1}, x_{n_k-1}, y_{n_k-1}) = \max\left\{\begin{array}{c} p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1}), p(gx_{m_k-1}, gx_{m_k}), \\ p(gy_{m_k-1}, gy_{m_k}), p(gx_{n_k-1}, gx_{n_k}), p(gy_{n_k-1}, gy_{n_k}), \\ \dfrac{p(gx_{m_k-1}, gx_{m_k})p(gy_{m_k-1}, gy_{m_k})}{p(gx_{m_k-1}, gx_{n_k-1}) + p(gy_{m_k-1}, gy_{n_k-1}) + p(gx_{m_k}, gx_{n_k})}, \\ \dfrac{p(gx_{n_k-1}, gx_{n_k})p(gy_{n_k-1}, gy_{n_k})}{p(gx_{m_k-1}, gx_{n_k-1}) + p(gy_{m_k-1}, gy_{n_k-1}) + p(gx_{m_k}, gx_{n_k})} \end{array}\right\}$$

$= \max\{p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1})\}$ For large k

(since $\max\left\{\begin{matrix} p(gx_{m_k-1}, gx_{m_k}), p(gy_{m_k-1}, gy_{m_k}), p(gx_{n_k-1}, gx_{n_k}), p(gy_{n_k-1}, gy_{n_k}) \\ \frac{p(gx_{m_k-1}, gx_{m_k})p(gy_{m_k-1}, gy_{m_k})}{p(gx_{m_k-1}, gx_{n_k-1})+p(gy_{m_k-1}, gy_{n_k-1})+p(gx_{m_k}, gx_{n_k})}, \frac{p(gx_{n_k-1}, gx_{n_k})p(gy_{n_k-1}, gy_{n_k})}{p(gx_{m_k-1}, gx_{n_k-1})+p(gy_{m_k-1}, gy_{n_k-1})+p(gx_{m_k}, gx_{n_k})} \end{matrix}\right\}$

$\to 0$ as $k \to \infty$)

From (3.2.16), for large k,

$\psi\left(p(gx_{m_k}, gx_{n_k})\right) \leq \alpha\left(M(x_{m_k-1}, y_{m_k-1}, x_{n_k-1}, y_{n_k-1})\right) - \beta\left(M(x_{m_k-1}, y_{m_k-1}, x_{n_k-1}, y_{n_k-1})\right)$

$= \alpha(\max\{p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1})\})$

$-\beta(\max\{p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1})\})$

Similarly, for large k

$\psi\left(p(gy_{m_k}, gy_{n_k})\right) \leq \alpha(\max\{p(gy_{m_k-1}, gy_{n_k-1}), p(gx_{m_k-1}, gx_{n_k-1})\})$

$-\beta\left(p(gy_{m_k-1}, gy_{n_k-1}), p(gx_{m_k-1}, gx_{n_k-1})\right)$

$\therefore \psi(\max\{p(gx_{m_k}, gx_{n_k}), p(gy_{m_k}, gy_{n_k})\}) = \max\left\{\psi\left(p(gx_{m_k}, gx_{n_k})\right), \psi\left(p(gy_{m_k}, gy_{n_k})\right)\right\}$

$\leq \alpha(\max\{p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1})\})$

$-\beta(\max\{p(gx_{m_k-1}, gx_{n_k-1}), p(gy_{m_k-1}, gy_{n_k-1})\})$

For large , since ψ is increasing.

So letting $k \to \infty$,

$\psi(\epsilon) \leq \alpha(\epsilon) - \beta(\epsilon)$, a contradiction, since $\epsilon > 0$.

Hence $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in the partial metric space (X, p).

From (3.2.9), we have

$$\lim_{n\to\infty} p(gx_n, gx_m) = 0 = \lim_{n\to\infty} p(gy_n, gy_m) \quad (3.2.17)$$

Suppose $g(X)$ is a complete subspace of X.

Since $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in the complete partial metric space $(g(X), p)$, then $\{gx_n\}$ and $\{gy_n\}$ converge to u and v in $g(X)$ respectively, say

i.e. $gx_n \to u$, $gy_n \to v$

Since$u, v \in g(X)$, there exist $x, y \in X$ such that $u = gx$ and $v = gy$

From (3.2.1)

$$p(u, u) = \lim_{n\to\infty} p(gx_n, u) = 0, p(v, v) = \lim_{n\to\infty} p(gy_n, v) = 0 \quad (3.2.18)$$

Now we prove that $\lim_{n\to\infty} p(F(x, y), gx_n) = p(F(x, y), u)$,

And $\lim_{n\to\infty} p(F(y, x), gy_n) = p(F(y, x), v)$

From (2.1.4), we have

$p(u, F(x, y)) \leq p(u, gx_{n+1}) + p(gx_{n+1}, F(x, y)) - p(gx_{n+1}, gx_{n+1})$

$\leq p(u, gx_{n+1}) + p(gx_{n+1}, F(x, y))$

Letting $n \to \infty$

$p(u, F(x, y)) \leq 0 + \lim_{n\to\infty} p(F(x_n, y_n), F(x, y))$

Also from (3.2.4), we get $gx_n \leq gx$ and $gy_n \geq gy$

Since ψ is continuous and non-decreasing, we get

$\psi\left(p(u, F(x, y))\right) = \psi\left(p(F(x_n, y_n), F(x, y))\right)$

$\leq \alpha(M(x_n, y_n, x, y)) - \beta(M(x_n, y_n, x, y))$

Where

$$M(x_n, y_n, x, y) = max\left\{\begin{matrix} p(gx_n, gx), p(gy_n, gy), p(gx_n, F(x_n, y_n)), \\ p(gy_n, F(y_n, x_n)), p(gx, F(x, y)), p(gy, F(y, x)), \\ \frac{p(gx_n, gx_{n+1})p(gy_n, gy_{n+1})}{p(gx_n, gx)+p(gy_n, gy)+p(gx_{n+1}, F(x, y))}, \\ \frac{p(gx, F(x, y))p(gy, F(y, x))}{p(gx_n, gx)+p(gy_n, gy)+p(gx_{n+1}, F(x, y))} \end{matrix}\right\}$$

$= max\{p(u, F(x, y)), p(v, F(y, x))\}$ For large n

(Since $\max\left\{\begin{matrix} p(gx_n, F(x_n,y_n)), p(gy_n, F(y_n,x_n)), \\ p(gx, F(x,y)), p(gy, F(y,x)), \\ \frac{p(gx_n,gx_{n+1})p(gy_n,gy_{n+1})}{p(gx_n,gx)+p(gy_n,gy)+p(gx_{n+1},F(x,y))}, \\ \frac{p(gx,F(x,y))p(gy,F(y,x))}{p(gx_n,gx)+p(gy_n,gy)+p(gx_{n+1},F(x,y))} \end{matrix}\right\} \to 0$ as $n \to \infty$)

$\therefore \psi\left(p\left(u, F(x,y)\right)\right) \leq \lim_{n\to\infty} \psi\left(p\left(F(x_n,y_n), F(x,y)\right)\right)$

$\leq \lim_{n\to\infty} \left(\alpha\left(M(x_n,y_n,x,y)\right) - \beta\left(M(x_n,y_n,x,y)\right)\right)$

$= \lim_{n\to\infty} (\alpha \max\{p(gx_n,gx), p(gy_n,gy)\} - \beta \max\{p(gx_n,gx), p(gy_n,gy)\})$

$= \lim_{n\to\infty} \alpha \max\{p(gx_n,gx), p(gy_n,gy)\} - \lim_{n\to\infty} \beta \max\{p(gx_n,gx), p(gy_n,gy)\}$

$= \alpha(\max\{p(u,gx), p(v,gy)\}) - \beta(\max\{p(u,gx), p(v,gy)\})$

$= \alpha(\max\{p(u,u), p(v,v)\}) - \beta(\max\{p(u,u), p(v,v)\})$

$= \alpha(0) - \beta(0)$

$= 0$

$\Longrightarrow p\left(u, F(x,y)\right) = 0$

$\Longrightarrow u = F(x,y)$

$\Longrightarrow gx = F(x,y)$

Similarly $p\left(v, F(y,x)\right) = 0$

$\Longrightarrow v = F(y,x)$

$\Longrightarrow gy = F(y,x)$.

Hence F and g have a coupled coincidence point in $X \times X$.

Note: Theorem 3.2 holds even when β is lower semi continuous.

3.3 Corollary: (Theorem 2.15)

Proof: Since m(x,y,u,v) ≤ M (x,y,u,v), the result follows in view of the note above.

Now we state and prove our second main theorem.

3.4 Theorem

In addition to the hypothesis of Theorem 3.2, we suppose that (x,y) and (x',y') are coupled fixed points of F and g, such that (gx,gy) and (gx',gy') are comparable. Then

$F(x,y) = gx = gx' = F(x',y')$ And $F(y,x) = gy = gy' = F(y',x')$ (3.3.1)

Moreover if (F,g) is w −compatible, then F and g have a unique common coupled fixed point in $X \times X$, in the sense that any two comparable common coupled fixed points coincide.

Proof: We may suppose that $gx \leq gx'$ And$gy \geq gy'$. Then from theorem (3.2) we have

$\psi\left(p\left(F(x,y), F(x',y')\right)\right) \leq \alpha\left(M(x,y,x',y')\right) - \beta\left(M(x,y,x',y')\right)$

Where $M(x,y,x',y') = \max\left\{\begin{matrix} p(gx,gx'), p(gy,gy'), p(gx,gx), \\ p(gy,gy), p(gx',gx'), p(gy',gy'), \\ \frac{p(gx,gx).p(gy,gy)}{p(gx,gx')+p(gy,gy')+p(gx,gx')}, \\ \frac{p(gx',gx').p(gy',gy')}{p(gx,gx')+p(gy,gy')+p(gx,gx')} \end{matrix}\right\}$

$= \max\left\{\begin{matrix} p(gx,gx'), p(gy,gy'), \\ \frac{p(gx,gx).p(gy,gy)}{2p(gx,gx')+p(gy,gy')}, \\ \frac{p(gx',gx').p(gy',gy')}{2p(gx,gx')+p(gy,gy')} \end{matrix}\right\}$ (By 2.1.2)

Now $\max\{p(gx,gx'), p(gy,gy')\} \geq \left\{\frac{p(gx,gx).p(gy,gy)}{2p(gx,gx')+p(gy,gy')}\right\}$

$\therefore \frac{p(gx,gx).p(gy,gy)}{2p(gx,gx')+p(gy,gy')} \leq \frac{p(gx,gx').p(gy,gy')}{2p(gx,gx')+p(gy,gy')}$

$\leq \frac{p(gx,gx').p(gy,gy')}{p(gy,gy')}$

$= p(gx,gx')$

$\leq \max\{p(gx,gx'), p(gy,gy')\}$

And

$\frac{p(gx',gx').p(gy',gy')}{2p(gx,gx')+p(gy,gy')} \leq \frac{p(gx,gx').p(gy,gy')}{2p(gx,gx')+p(gy,gy')}$

$\leq \frac{p(gx,gx').p(gy,gy')}{p(gy,gy')}$

$\leq p(gx, gx')$
$\leq \max\{p(gx, gx'), p(gy, gy')\}$
$\therefore \psi\left(p\left(F(x,y), F(x',y')\right)\right) = \psi\left(p(gx, gx')\right)$
$\leq \alpha(\max\{p(gx, gx'), p(gy, gy')\}) - \beta(\max\{p(gx, gx'), p(gy, gy')\})$

Similarly
$\psi\left(p(gy, gy')\right) \leq \alpha(\max\{p(gx, gx'), p(gy, gy')\}) - \beta(\max\{p(gx, gx'), p(gy, gy')\})$

$\therefore \max\{\psi\left(p(gx, gx')\right), \psi\left(p(gy, gy')\right)\}$
$\leq \alpha(\max\{p(gx, gx'), p(gy, gy')\}) - \beta(\max\{p(gx, gx'), p(gy, gy')\})$

$\therefore \psi(\max\{p(gx, gx'), p(gy, gy')\})$
$=\max\left\{\psi\left(p(gx, gx'), \psi\left(p(gy, gy')\right)\right)\right\}$
$\leq \alpha(\max\{p(gx, gx'), p(gy, gy')\}) - \beta(\max\{p(gx, gx'), p(gy, gy')\})$
$\Longrightarrow \psi(t) - \alpha(t) + \beta(t) \leq 0$ Where $t = \max\{p(gx, gx'), p(gy, gy')\}$

$\therefore t = 0$
$\Longrightarrow \max\{p(gx, gx'), p(gy, gy')\} = 0$
$\Longrightarrow p(gx, gx') = 0, p(gy, gy') = 0$
$\Longrightarrow gx = gx'$ and $gy = gy'$.

Suppose (x, y) and (x', y') are comparable coupled fixed points of F and g. We may suppose that $(x, y) \leq (x', y')$, that is $x \leq x'$ and $y \geq y'$.
Then $x = gx = F(x, y)$ and $y = gy = F(y, x)$
$x' = gx' = F(x', y')$ and $y' = gy' = F(y', x')$

Hence
By (3.3.1), $x = F(x, y) = gx = gx' = F(x', y') = x'$
And $y = F(y, x) = gy = gy' = F(y', x') = y'$
Hence $x = x'$ and $y = y'$
$\therefore (x, y) = (x', y')$

Hence any two comparable common coupled fixed points coincide.

3.5 Corollary
Let $(X, \preceq)$ be a partially ordered set and p be a partial metric such that (X, p) is a complete partial metric space. Let $F: X \times X \to X$ be such that
(3.4.1) $\psi\left(p\left(F(x,y), F(u,v)\right)\right) \leq \alpha(\max\{p(x,u), p(y,v)\}) - \beta(\max\{p(x,u), p(y,v)\})\ \forall\ x, y, u, v \in X$,
$x \preceq u$ and $y \succeq v$, where ψ,α and β are as defined in def. 3.1 and
(3.4.2) (a) If a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x\ \forall\ n$ and
(b) If a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n\ \forall\ n$.

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F\left(y_0, x_0\right)$, then F has a unique coupled fixed point in $X \times X$.
Proof:
In theorem 3.2, if we take g as the identity map $I: X \to X$, the result follows.
The following example supports our results.

3.6 Example
Let $X = [0, 1]$. Let $\preceq$ be the partially order on X defined by $\preceq y \Leftrightarrow x \geq y$.
The mapping $F: X \times X \to X$ is defined by $F(x, y) = \frac{x^2+y^2}{8(x+y+1)}$ and g is the identity map.
Define $p: X \times X \to [0, \infty)$ by $p(x, y) = \max\{x, y\}$ then $p$is a complete partial metric on X.
Define $\psi, \alpha, \beta: [0, \infty) \to [0, \infty)$ by $\psi(t) = t$, $\alpha(t) = \frac{t}{2}$ and $\beta(t) = \frac{t}{4}$.
Take $x_0 = y_0 = 1$. Then $gx_0 = g1 = 1 \geq \frac{{x_0}^2+{y_0}^2}{8(x_0+y_0+1)} = \frac{1}{12}$
So that$gx_0 \preceq F(x_0, y_0)$. Similarly$F(y_0, x_0) \preceq gy_0$
Let $p(x, u) = r, p(y, v) = \delta, x^2 \leq r^2, y^2 \leq \delta^2, u^2 \leq r^2, v^2 \leq \delta^2$

$\therefore x^2 + y^2 \leq r^2 + \delta^2$ and $u^2 + v^2 \leq r^2 + \delta^2$

$\therefore \frac{x^2+y^2}{x+y+1} \leq \frac{r^2+\delta^2}{x+y+1}, \frac{u^2+v^2}{u+v+1} \leq \frac{r^2+\delta^2}{u+v+1}, \frac{x^2}{x+y+1} \leq \frac{x^2}{x+1} \leq x, \frac{y^2}{x+y+1} \leq y$

$\therefore \frac{x^2+y^2}{x+y+1} \leq x+y \leq r+\delta\,, \frac{u^2+v^2}{u+v+1} \leq u+v \leq r+\delta$

$$\therefore \max\left\{\frac{x^2+y^2}{x+y+1}, \frac{u^2+v^2}{u+v+1}\right\} \leq r+\delta$$

$$\frac{1}{8}\max\left\{\frac{x^2+y^2}{x+y+1}, \frac{u^2+v^2}{u+v+1}\right\} \leq \frac{1}{8}(r+\delta) \leq \frac{1}{4}(\max\{r,\delta\})$$
$$= \tfrac{1}{2}(\max\{r,\delta\}) - \tfrac{1}{4}(\max\{r,\delta\})$$
$$= \alpha(\max\{p(x,u),p(y,v)\}) - \beta(\max\{p(x,u),p(y,v)\})$$

$$\therefore \psi\Big(p\big(F(x,y),F(u,v)\big)\Big) \leq \alpha(\max\{p(x,u),p(y,v)\}) - \beta(\max\{p(x,u),p(y,v)\})$$

Hence all conditions of theorem 3.4 hold and
$(0,0)$ Is the unique coupled fixed point of F In $X \times X$.

4. Application

In this section we give an application of our main results.

4.1 Theorem

Let (X,p) be a complete partial metric space. U be an open subset of X and $\overline{U}$ be a closed subset of X such that $U \subseteq \overline{U}$, Suppose $H\colon \overline{U} \times \overline{U} \times [0,1] \to X$ is an operator such that the following conditions are satisfied

1. $x \neq H(x,y,\lambda)$ and $y \neq H(y,x,\lambda)$ for each $x,y \in \partial U$ and $\lambda \in [0,1]$ (here ∂U denotes the boundary of U in X)
2. $\psi\Big(p\big(H(x,y,\lambda),H(u,v,\lambda)\big)\Big) \leq \alpha(\max\{p(x,u),p(y,v)\}) - \beta(\max\{p(x,u),p(y,v)\})$ for all $x,y \in \overline{U}$ and $\lambda \in [0,1]$, where $\psi\colon [0,\infty) \to [0,\infty)$ is continuous and non-decreasing and $\alpha,\beta\colon [0,\infty) \to [0,\infty)$ are continuous with $\psi(t) - \alpha(t) + \beta(t) > 0$ for $t > 0$
3. There exists $M \geq 0$ such that $p\big(H(x,y,\lambda),H(x,y,\mu)\big) \leq M|\lambda-\mu|$ for all $x \in \overline{U}$ and $\lambda,\mu \in [0,1]$.

Then $H(.,\lambda_0)$ has a coupled fixed point for some $\lambda_0 \in [0,1]$ if and only if $H(.,\mu)$ has a coupled fixed point for all $\mu \in [0,1]$

Proof: Consider the set $A = \{\lambda \in [0,1]\colon (x,y) = \big(H(x,y,\lambda),H(y,x,\lambda)\big)$ for some $x,y \in U\}$
Since $H(.,\lambda_0)$ has a coupled fixed point in U, we have $\lambda_0 \in A$, so that A is a non-empty set
We will show that A is both open and closed in [0, 1], so by connectedness we have $A = [0,1]$
As a result $H(.,\mu)$ has a coupled fixed point in U for all $\mu \in [0,1]$.
First we show that A is closed in [0, 1].
Suppose $\{\mu_n\}_{n=1}^{\infty} \subseteq A$ with $\mu_n \to \mu$ as $n \to \infty$ where $\mu \in [0,1]$
We show that $\mu \in A$.
Since $\mu_n \in A$ for $n = 1,2,3,\ldots$, there exist $x_n, y_n \in U$ with $(x_n,y_n) = \big(H(x_n,y_n,\mu_n),H(y_n,x_n,\mu_n)\big)$
Consider $p(x_n,x_{n+1}) = p\big(H(x_n,y_n,\mu_n),H(x_{n+1},y_{n+1},\mu_{n+1})\big)$

$$\leq p\big(H(x_n,y_n,\mu_n),H(x_{n+1},y_{n+1},\mu_n)\big) + p\big(H(x_{n+1},y_{n+1},\mu_n),H(x_{n+1},y_{n+1},\mu_{n+1})\big) - p\big(H(x_{n+1},y_{n+1},\mu_n),H(x_{n+1},y_{n+1},\mu_n)\big)$$
$$\leq p\big(H(x_n,y_n,\mu_n),H(x_{n+1},y_{n+1},\mu_n)\big) + \mathrm{M}|\mu_n - \mu_{n+1}|$$
$$\therefore p(x_n,x_{n+1}) \leq p\big(H(x_n,y_n,\mu_n),H(x_{n+1},y_{n+1},\mu_n)\big) + \mathrm{M}|\mu_n - \mu_{n+1}|$$

Letting $n \to \infty$
$$\lim_{n\to\infty} p(x_n,x_{n+1}) \leq \lim_{n\to\infty} p\big(H(x_n,y_n,\mu_n),H(x_{n+1},y_{n+1},\mu_n)\big)$$

Since ψ is continuous and non-decreasing, we obtain
$$\lim_{n\to\infty} \psi\big(p(x_n,x_{n+1})\big) \leq \lim_{n\to\infty} \alpha(\max\{p(x_n,x_{n+1}),p(y_n,y_{n+1})\}) - \beta(\max\{p(x_n,x_{n+1}),p(y_n,y_{n+1})\})$$

Similarly
$$\lim_{n\to\infty} \psi\big(p(y_n,y_{n+1})\big) \leq \lim_{n\to\infty} \alpha(\max\{p(x_n,x_{n+1}),p(y_n,y_{n+1})\}) - \beta(\max\{p(x_n,x_{n+1}),p(y_n,y_{n+1})\})$$

$$\therefore \lim_{n\to\infty}\big(\psi(\max\{p(x_n,x_{n+1}),p(y_n,y_{n+1})\})\big)$$
$$= \max\Big\{\lim_{n\to\infty} \psi\big(p(x_n,x_{n+1})\big), \lim_{n\to\infty} \psi\big(p(y_n,y_{n+1})\big)\Big\}$$
$$\leq \lim_{n\to\infty} \alpha(\max\{p(x_n,x_{n+1}),p(y_n,y_{n+1})\}) - \beta(\max\{p(x_n,x_{n+1}),p(y_n,y_{n+1})\})$$
$$\Longrightarrow \lim_{n\to\infty} \psi(\max\{p(x_n,x_{n+1}),p(y_n,y_{n+1})\})$$
$$- \lim_{n\to\infty} \alpha(\max\{p(x_n,x_{n+1}),p(y_n,y_{n+1})\})$$

$+\lim_{n\to\infty}\beta(\max\{p(x_n,x_{n+1}),p(y_n,y_{n+1})\})=0$
$\Longrightarrow \max\{p(x_n,x_{n+1}),p(y_n,y_{n+1})\}=0.$ (4.1.1)

Now we prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X,p)
On the contrary, let us suppose that $\{x_n\}$ or $\{y_n\}$ is not Cauchy
There exist an $\epsilon>0$ and monotone increasing sequences of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k>m_k$ such that
$\max\{p(x_{m_k},x_{n_k}),p(y_{m_k},y_{n_k})\}\geq\epsilon$ (4.1.2)
And $\max\{p(x_{m_k},x_{n_k-1}),p(y_{m_k},y_{n_k-1})\}<\epsilon$ (4.1.3)

From (4.1.2) and (4.1.3)

$$\begin{aligned}\epsilon &\leq \max\{p(x_{m_k},x_{n_k}),p(y_{m_k},y_{n_k})\}\\ &\leq \max\left\{\left(p(x_{m_k},x_{n_k-1})+p(x_{n_k-1},x_{n_k})\right),p(y_{m_k},y_{n_k-1})+p(y_{n_k-1},y_{n_k})\right\}\\ &\leq \max\left\{max\left(p(x_{m_k},x_{n_k-1}),p(y_{m_k},y_{n_k-1})\right)+max\left(p(x_{n_k-1},x_{n_k}),p(y_{n_k-1},y_{n_k})\right)\right\}\\ &\leq \max\{\epsilon+\max\{p(x_{n_k},x_{n_k-1}),p(y_{n_k},y_{n_k-1})\}\}\\ &\to \epsilon \text{ as } k\to\infty\end{aligned}$$

$\therefore \max\{p(x_{m_k},x_{n_k}),p(y_{m_k},y_{n_k})\}\to\epsilon$ as $k\to\infty$
and $\max\{p(x_{m_k},x_{n_k-1}),p(y_{m_k},y_{n_k-1})\}\to\epsilon$ as $k\to\infty$ (4.1.4)

$$\begin{aligned}p(x_{m_k},x_{n_k}) &\leq p(x_{m_k},x_{n_k+1})+p(x_{n_k+1},x_{n_k})\\ &\leq p(x_{m_k},x_{n_k})+2p(x_{n_k},x_{n_k+1})\end{aligned}$$

Similarly

$$\begin{aligned}p(y_{m_k},y_{n_k}) &\leq p(y_{m_k},y_{n_k+1})+p(y_{n_k+1},y_{n_k})\\ &\leq p(y_{m_k},y_{n_k})+2p(y_{n_k},y_{n_k+1})\end{aligned}$$

$\therefore \max\{p(x_{m_k},x_{n_k}),p(y_{m_k},y_{n_k})\}\leq\max\{p(x_{m_k},x_{n_k+1}),p(y_{m_k},y_{n_k+1})\}+\max\{p(x_{n_k+1},x_{n_k}),p(y_{n_k+1},y_{n_k})\}$

Letting $k\to\infty$, we get
$\lim_{k\to\infty}\max\{p(x_{m_k},x_{n_k+1}),p(y_{m_k},y_{n_k+1})\}=\epsilon$ (4.1.5)
Consider $p(x_{m_k},x_{n_k+1})=p\left(H(x_{m_k},y_{m_k},\mu_{m_k}),H(x_{n_k+1},y_{n_k+1},\mu_{n_k+1})\right)$

$$\leq\begin{pmatrix}p\left(H(x_{m_k},y_{m_k},\mu_{m_k}),H(x_{m_k+1},y_{m_k+1},\mu_{n_k+1})\right)\\ +p\left(H(x_{m_k},y_{m_k},\mu_{n_k+1}),H(x_{n_k+1},y_{n_k+1},\mu_{n_k+1})\right)\\ -p\left(H(x_{m_k},y_{m_k},\mu_{n_k+1}),H(x_{m_k},y_{m_k},\mu_{n_k+1})\right)\end{pmatrix}$$

$$\leq M|\mu_{m_k}-\mu_{n_k+1}|+p\left(H(x_{m_k},y_{m_k},\mu_{n_k+1}),H(x_{n_k+1},y_{n_k+1},\mu_{n_k+1})\right)$$ (4.1.6)

Consider $p(y_{m_k},y_{n_k+1})=p\left(H(y_{m_k},x_{m_k},\mu_{m_k}),H(y_{n_k+1},x_{n_k+1},\mu_{n_k+1})\right)$

$$\leq\begin{pmatrix}p\left(H(y_{m_k},x_{m_k},\mu_{m_k}),H(y_{m_k},x_{m_k},\mu_{n_k+1})\right)\\ +p\left(H(y_{m_k},x_{m_k},\mu_{n_k+1}),H(y_{n_k+1},x_{n_k+1},\mu_{n_k+1})\right)\\ -p\left(H(y_{m_k},x_{m_k},\mu_{n_k+1}),H(y_{m_k},x_{m_k},\mu_{n_k+1})\right)\end{pmatrix}$$

$$\leq M|\mu_{m_k}-\mu_{n_k+1}|+p\left(H(y_{m_k},x_{m_k},\mu_{n_k+1}),H(y_{n_k+1},x_{n_k+1},\mu_{n_k+1})\right)$$ (4.1.7)

Consider
$\max\{p(x_{m_k},x_{n_k+1}),p(y_{m_k},y_{n_k+1})\}$

$$\leq\max\begin{Bmatrix}M|\mu_{m_k}-\mu_{n_k+1}|+p\left(H(x_{m_k},y_{m_k},\mu_{n_k+1}),H(x_{n_k+1},y_{n_k+1},\mu_{n_k+1})\right),\\ M|\mu_{m_k}-\mu_{n_k+1}|+p\left(H(y_{m_k},x_{m_k},\mu_{n_k+1}),H(y_{n_k+1},x_{n_k+1},\mu_{n_k+1})\right)\end{Bmatrix}$$

$$=M|\mu_{n_k}-\mu_{n_k+1}|+max\begin{Bmatrix}p\left(H(x_{m_k},y_{m_k},\mu_{n_k+1}),H(x_{n_k+1},y_{n_k+1},\mu_{n_k+1})\right),\\ p\left(H(y_{m_k},x_{m_k},\mu_{n_k+1}),H(y_{n_k+1},x_{n_k+1},\mu_{n_k+1})\right)\end{Bmatrix}$$

$\therefore \max\{p(x_{m_k}, x_{n_k+1}), p(y_{m_k}, y_{n_k+1})\} - M|\mu_{n_k} - \mu_{n_k+1}| \leq$
$\max\left\{p\left(H(x_{m_k}, y_{m_k}, \mu_{n_k+1}), H(x_{n_k+1}, y_{n_k+1}, \mu_{n_k+1})\right), p\left(H(y_{m_k}, x_{m_k}, \mu_{n_k+1}), H(y_{n_k+1}, x_{n_k+1}, \mu_{n_k+1})\right)\right\}$

$$\therefore \psi(\max\{p(x_{m_k}, x_{n_k+1}), p(y_{m_k}, y_{n_k+1})\} - M|\mu_{n_k} - \mu_{n_k+1}|) \leq \psi\left(\max\begin{Bmatrix} p\left(H(x_{m_k}, y_{m_k}, \mu_{n_k+1}), H(x_{n_k+1}, y_{n_k+1}, \mu_{n_k+1})\right), \\ p\left(H(y_{m_k}, x_{m_k}, \mu_{n_k+1}), H(y_{n_k+1}, x_{n_k+1}, \mu_{n_k+1})\right) \end{Bmatrix}\right)$$

$$= \max\begin{Bmatrix} \psi\left(p\left(H(x_{m_k}, y_{m_k}, \mu_{n_k+1}), H(x_{n_k+1}, y_{n_k+1}, \mu_{n_k+1})\right)\right), \\ \psi\left(p\left(H(y_{m_k}, x_{m_k}, \mu_{n_k+1}), H(y_{n_k+1}, x_{n_k+1}, \mu_{n_k+1})\right)\right) \end{Bmatrix}$$

(Since ψ is continuous and non-decreasing)

$$\leq \max\begin{Bmatrix} \alpha(\max\{p(x_{m_k}, x_{n_k+1}), p(y_{m_k}, y_{n_k+1})\}) - \beta(\max\{p(x_{m_k}, x_{n_k+1}), p(y_{m_k}, y_{n_k+1})\}), \\ \alpha(\max\{p(y_{m_k}, y_{n_k+1}), p(x_{m_k}, x_{n_k+1})\}) - \beta(\max\{p(y_{m_k}, y_{n_k+1}), p(x_{m_k}, x_{n_k+1})\}) \end{Bmatrix}$$

$$= \alpha(\max\{p(x_{m_k}, x_{n_k+1}), p(y_{m_k}, y_{n_k+1})\}) - \beta(\max\{p(x_{m_k}, x_{n_k+1}), p(y_{m_k}, y_{n_k+1})\})$$

Letting $\to \infty$, from (4.1.5)
$\psi(\epsilon) \leq \alpha(\epsilon) - \beta(\epsilon)$
$\Longrightarrow \psi(\epsilon) - \alpha(\epsilon) + \beta(\epsilon) \leq 0$
$\therefore \epsilon \leq 0$, which is a contradiction
$\therefore$ $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in (X, p)
Since (X, p) is complete, by def. 2.4 there exiss $u, v \in U$ with$x_n \to u$, $y_n \to v$
So that $p(u, u) = 0 = p(v, v)$
From lemma 2.6
$\lim_{n\to\infty} p(x_n, , H(u, v, \mu)) = p(u, H(u, v, \mu))$
Consider $p(x_n, , H(u, v, \mu)) = p(H(x_n, y_n, \mu_n), H(u, v, \mu))$

$$\leq \begin{pmatrix} p(H(x_n, y_n, \mu_n), H(x_n, y_n, \mu)) + \\ p(H(x_n, y_n, \mu), H(u, v, \mu)) - \\ p(H(x_n, y_n, \mu), H(x_n, y_n, \mu)) \end{pmatrix}$$

$$\leq M|\mu_n - \mu| + p(H(x_n, y_n, \mu), H(u, v, \mu))$$

$\leq M|\mu_n - \mu| + \alpha(\max\{p(x_n, u), p(y_n, v)\}) - \beta(\max\{p(x_n, u), p(y_n, v)\})$
Letting $n \to \infty$
$\lim_{n\to\infty} p(x_n, H(u, v, \mu)) = p(u, H(u, v, \mu)) \leq 0$
$\therefore p(u, H(u, v, \mu)) = 0$
Similarly $p(v, H(v, u, \mu)) = 0$
$\therefore u = H(u, v, \mu)$ And $v = H(v, u, \mu)$
$\therefore \mu \in A$
Hence A is closed in [0, 1].
Let $\mu_0 \in A$, then there exist $x_0, y_0 \in U$ such that $(x_0, y_0) = (H(x_0, y_0, \mu_0), H(y_0, x_0, \mu_0))$ where $p(H(x, y, \mu), x_0) = p(H(x, y, \mu), H(x_0, y_0, \mu_0))$

$$\leq \begin{pmatrix} p(H(x, y, \mu), H(x, y, \mu_0)) + \\ p(H(x, y, \mu_0), H(x_0, y_0, \mu_0)) - \\ p(H(x, y, \mu_0), H(x, y, \mu_0)) \end{pmatrix}$$

$$\leq M|\mu - \mu_0| + p(H(x, y, \mu_0), H(x_0, y_0, \mu_0))$$

Suppose $|\mu - \mu_0| < \frac{1}{M^n}$ for large n
$\leq \frac{1}{M^{n-1}} + p(H(x, y, \mu_0), H(x_0, y_0, \mu_0))$ (4.8)
Similarly $p(H(y, x, \mu), y_0) \leq \frac{1}{M^{n-1}} + p(H(y, x, \mu_0), H(y_0, x_0, \mu_0))$ (4.9)
Now consider max $\{p(H(x, y, \mu), x_0), p(H(y, x, \mu), y_0)\} \leq \frac{1}{M^{n-1}} \max\begin{Bmatrix} p(H(x, y, \mu_0), H(x_0, y_0, \mu_0)), \\ p(H(y, x, \mu_0), H(y_0, x_0, \mu_0)) \end{Bmatrix}$

$$\therefore \psi\left(\max\{p(H(x,y,\mu),x_0),p(H(y,x,\mu),y_0)\}-\frac{1}{M^{n-1}}\right) \leq \max\begin{Bmatrix}\psi\left(p\big(H(x,y,\mu_0),H(x_0,y_0,\mu_0)\big)\right),\\ \psi\left(p\big(H(y,x,\mu_0),H(y_0,x_0,\mu_0)\big)\right)\end{Bmatrix}$$
$$\leq \max\begin{Bmatrix}\alpha(\max\{p(x,x_0),p(y,y_0)\})-\beta(\max\{p(x,x_0),p(y,y_0)\}),\\ \alpha(\max\{p(x,x_0),p(y,y_0)\})-\beta(\max\{p(x,x_0),p(y,y_0)\})\end{Bmatrix}$$
$$=\alpha(\max\{p(x,x_0),p(y,y_0)\})-\beta(\max\{p(x,x_0),p(y,y_0)\})$$
$$\leq \alpha(\max\{p(x,x_0),p(y,y_0)\})$$

Since ψ is non decreasing, α is continuous and β is continuous,
$\max\{p(H(x,y,\mu),x_0),p(H(y,x,\mu),y_0)\} \leq \max\{p(x,x_0),p(y,y_0)\}$
Thus for each $\mu \in (\mu_0-\epsilon,\mu_0+\epsilon)$
(ii)Holds and ψ is continuous and non-decreasing and α,β are continuous with $\psi(t)-\alpha(t)+\beta(t)>0$ for $t>0$. Thus all conditions of theorem 3.4 are satisfied
Hence we deduce that $H(.,\mu)$ has a coupled fixed point in $\bar{U}$. But this coupled fixed point is in U, since (i) holds.
Thus $\mu \in A$ for any $\mu \in (\mu_0-\epsilon,\mu_0+\epsilon)$
Hence $(\mu_0-\epsilon,\mu_0+\epsilon) \subseteq A$ and therefore A is open in [0,1]
Consequently A is a non empty, clopen set in [0,1] which is a connected set.
Hence $A=[0,1]$.
Converse part also follows in the same way.

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