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# The Gerber-Shiu discounted penalty function in the thinning risk model with refunding

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**Abstract**
In this paper, the thinning risk model with refunding is considered, where the counting process of the refund and the counting process of the claim are the thinning of the premium arrival counting process with a binomial process. The Gerber-Shiu discounted penalty function of this model is discussed, and the renewal equation and the asymptotic estimate satisfied by the Gerber-Shiu discounted penalty function are also derived. And according to the characteristic of the Gerber-Shiu discounted penalty function, we obtain the asymptotic estimate of the ruin probability.

**Keywords:** The Gerber-Shiu discounted penalty function, the renewal equation, generating function, the thinning process

## 1. Introduction

The thinning process has an important application in risk theory. For example, the thinning risk model described by

$$U^p(t) = u + N(t) - \sum_{i=1}^{N^P(t)} X_i$$

is considered (see literature [7]), where $u \geq 0$ is the initial capital of an insurance company. $\{N(t), t = 1,2,3,\cdots\}$ is a binomial process with parameter $\theta(\theta > 0)$ and $N(t)$ denotes the total number of the policies sold by insurance company in [0,t]. And the number of insurance policy charged each time is constant 1. $\{N^p(t), t = 1,2,3,\cdots\}$ is called the p-thinning process of $\{N(t), t = 1,2,3,\cdots\}$, that is, $\{N^p(t), t = 1,2,3,\cdots\}$ is binomial process with parameter $p\theta(0<p<1)$, $N^p(t)$ indicates the amount of the claim in [0,t]. $\{X_i, i = 1,2,3,\cdots\}$ is a sequence of independent and identically distributed nonnegative random variables with common distribution function $f(x)$( $f(0) = 0$), which are independent of { $N(t), t = 1,2,3,\cdots$}, $X_i$ is the size of the $i$ -th claim with the mean $\mu$ .

Now, we generalize the thinning risk model by adding the refund as follows

$$U^{p,q}(t) = u + N(t) - \sum_{i=1}^{N^q(t)} Y_i - \sum_{i=1}^{N^P(t)} X_i,$$

Here $\{N^q(t), t = 1,2,3,\cdots\}$ is the q-thinning process of $\{N(t), t = 1,2,3,\cdots\}$, i.e. $\{N^q(t), t = 1,2,3,\cdots\}$ is a binomial process with parameter $q\theta(0 < p + q < 1)$, $N^q(t)$ represents the number of the refund in [0,t]. $\{Y_i, i = 1,2,3,\cdots\}$ is a sequence of independent and identically distributed nonnegative random variables with common distribution function ( $g(y)$ $g(0) = 0$ and let $g(y) = f(x)$ ), which are independent of { $N(t), t = 1,2,3\cdots$ } $X_i$ is the size of the $i$ -th claim with the mean $\mu$ .

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Being same with the classical discrete risk model, we define the time of ruin as $T = inf\{t : U^{p,q}(t) < 0\}$ with $inf\{\varnothing\} = \infty$, the ruin probability as $\psi(u) = P(T < \infty | U^{p,q}(0) = u, u \ge 0)$ and the Gerber-Shiu discounted penalty function as $m(u) = E[v^t \omega(U(T-), |U(T)|) I(T < \infty) | U^{p,q}(0) = u]$, which $U(T-)$ shows the instantaneous surplus before ruin, $|U(T)|$ expresses the deficit at the time of ruin, $\omega(x_1, x_2)(x_1 \ge 0, x_2 \ge 0)$ is a nonnegative continuous bounded function, $v(v \in (0,1])$ denotes the discount factor and $I(A)$ represents the indicator function of a set A.

The Gerber-Shiu discounted penalty function was investigated by many authors. J. Pan and G.J. Wang (2009) [3] studied a thinning risk model and derived the integral equation, the integral differential equation and the recursive formula for the expected discounted penalty function. L.J. He, C.Y. Wang and K. Zhang (2016) [4] observed a compound Poisson-Geometric risk model with variable premium rate and obtained a defective renewal equation of Gerber-Shiu discounted penalty function by applying the differential argument method. G. Shija and M.J. Jacob (2016) [2] considered a Markov modulated delayed by-claim type risk model with random incomes and derived the system of integral equations satisfied by the Gerber-Shiu discounted penalty function.

In section 2, the relevant lemma is given and proved. In section 3, the defective renewal equation and the asymptotic estimate satisfied by the Gerber-Shiu discounted penalty function are derived, then the asymptotic estimate of the ruin probability is also obtained.

## 2. Preliminary

We represent the generating function of $f(x)$ on $N = 0,1,2,\cdots$ as $\tilde{f}(z)$ (i.e. $\tilde{f}(z) = \sum_{u=0}^{\infty} z^u f(u)$).

**Hypothesis 1** There exits $\tilde{f}(z) \to +\infty$ as $z \to z_\infty$ ( $z_\infty$ may be $+\infty$ ) when there is $z_\infty > 1$ for the generating function $\tilde{f}(z)$ of X.

**Lemma 1** If $v \in (0,1]$, then the equation

$$z - zv(1-\theta) - v\theta(1-q)(1-p) - v\theta(p+q-pq)\tilde{f}(z) = 0 \tag{1}$$

has an unique positive solution $\rho$ ( $\rho \in (\frac{v\theta(1-q)(1-p)}{1-v(1-\theta)}, 1]$ and $\rho = 1$ as $v = 1$ ). If Hypothesis 1 is true, then the equation (1) has an only positive solution $R$ called the adjustment coefficient ( $R \in (1, z_\infty)$ ).

**Proof** By setting $g_1(z) = z - zv(1-\theta) - v\theta(1-q)(1-p)$ and $g_2(z) = v\theta(p+q-pq)\tilde{f}(z)$, we know that

$$g_1(0) = -v\theta(1-q)(1-p) < 0 = g_2(0),\ g_1\left(\frac{v\theta(1-q)(1-p)}{1-v(1-\theta)}\right) = 0 < g_2\left(\frac{v\theta(1-q)(1-p)}{1-v(1-\theta)}\right),$$

$$g_1(1) = 1 - v(1-\theta) - v\theta(1-q)(1-p) \ge v\theta(p+q-pq) = g_2(1),\ g_2'(z) > 0,\ g_2''(z) > 0 \text{ and } \frac{v\theta(1-q)(1-p)}{1-v(1-\theta)} < 1,$$ So

$g_1(z) = g_2(z)$ has an only positive solution $\rho$ in the interval $(\frac{v\theta(1-q)(1-p)}{1-v(1-\theta)}, 1]$, and it is easy to see that $\rho = 1$ as $v = 1$.

And because $\lim_{z \to z_\infty} \frac{g_2(z)}{g_1(z)} = \lim_{z \to z_\infty} \frac{v\theta(p+q-pq)\tilde{f}(z)/z}{1-v(1-\theta)-v\theta(1-q)(1-p)/z} = \infty$, the equation $g_1(z) = g_2(z)$ has an unique positive solution $R$ in the interval $(1, z_\infty)$.

**Lemma 2** Let $\{a_k, k=1,2,3,\cdots\}$ satisfying $\sum_{k=0}^{+\infty} a_k = 1,\ \sum_{k=0}^{+\infty} k a_k < \infty$ and $\{b_k, k=1,2,3,\cdots\}$ satisfying $\sum_{k=0}^{+\infty} b_k < \infty$ be two Nonnegative sequences. There exits $\lim_{n\to\infty} u_n$ and $\lim_{n\to\infty} u_n = \frac{\sum_{k=0}^{+\infty} b_k}{\sum_{k=1}^{+\infty} k a_k}, n = 0,1,2,\cdots$ when there is a bounded sequence $\{a_n, n=1,2,3,\cdots\}$ satisfying $u_n = \sum_{k=0}^{n} u_k a_{n-k} + b_n, n = 0,1,2,\cdots$.

**3. The Gerber-Shiu discounted penalty function**

**Theorem 1** With relatively safe load condition, the defective renewal equation for the Gerber-Shiu discounted penalty function is followed by

$$m(u) = \frac{\nu\theta}{1-\nu(1-\theta)}[(p+q-pq)\sum_{i=0}^{u} m(u-i)\sum_{j=i+1}^{+\infty}\rho^{j-i-1} f(j) + p\sum_{i=u+1}^{+\infty}\rho^{i-u-1}\omega(i)], \tag{2}$$

Where $\omega(u) = \sum_{k=u+1}^{+\infty}\omega(u,k-u) f(k),$ and

$$m(0) = \frac{p(\tilde{\omega}(\rho)-\omega(0))}{(1-q)(1-p)}. \tag{3}$$

**Proof** In the time interval $[0,1]$, it is well known that the probability is $1-\theta$ if there is no premium, the probability is $1-q$ if a refund occurs, and the probability is $1-p$ if a claim happens. Thus, by considering whether a premium, refund and claim occurs in the time interval $[0,1]$, the Gerber-Shiu discounted penalty function yields that

$$\begin{aligned} m(u) &= \nu[(1-\theta)m(u) + \theta(1-q)(1-p)m(u+1) + \theta q(1-p)\sum_{k=1}^{u+1} m(u+1-k) f(k) \\ &\quad + \theta p\sum_{k=1}^{u+1} m(u+1-k) f(k) + \theta p\sum_{k=u+2}^{+\infty}\omega(u+1,k-u-1) f(k)]. \end{aligned} \tag{4}$$

Let $\tilde{m}(z)$ be the generating function of $m(u)$ with respect to $u$ on the entire real line, and $\tilde{\omega}(z)$ is the corresponding generating function of $\omega(u)$. Now, taking the generating function both sides of the equation (4) with respect to $u$, we get

$$\begin{aligned} &[z - z\nu(1-\theta) - \nu\theta(1-q)(1-p) - \nu\theta(p+q-pq)\tilde{f}(z)]\tilde{m}(z) \\ &= \nu\theta p(\tilde{\omega}(z)-\omega(0)) - \nu\theta(1-q)(1-p)m(0). \end{aligned} \tag{5}$$

From Lemma 1, it is followed that $\nu\theta p(\tilde{\omega}(\rho)-\omega(0)) - \nu\theta(1-q)(1-p)m(0) = 0$ as $z=\rho$, the equation (3) is resolved. Then we substitute the equation (3) into the equation (5) and derive that

$$[z - z\nu(1-\theta) - \nu\theta(1-q)(1-p) - \nu\theta(p+q-pq)\tilde{f}(z)]\tilde{m}(z) = \nu\theta p(\tilde{\omega}(z)-\tilde{\omega}(\rho)). \tag{6}$$

And it is obvious to know that

$$\rho - \rho\nu(1-\theta) - \nu\theta(1-q)(1-p) - \nu\theta(p+q-pq)\tilde{f}(\rho) = 0, \tag{7}$$

we substitute the equation (7) into the equation (6) and yield that

$$[(z-\rho) - (z-\rho)\nu(1-\theta) - \nu\theta(p+q-pq)(\tilde{f}(z) - \tilde{f}(\rho)]\tilde{m}(z) = \nu\theta p(\tilde{\omega}(z)-\tilde{\omega}(\rho)),$$

i.e.

$$\tilde{m}(z)=\frac{\nu\theta}{1-\nu(1-\theta)}[(p+q-pq)\frac{\tilde{f}(z)-\tilde{f}(\rho)}{z-\rho}\tilde{m}(z)+p\frac{\tilde{\omega}(z)-\tilde{\omega}(\rho)}{z-\rho}].$$

Therefore, the equation (2) can be further obtained. And because $\frac{\nu\theta(p+q-pq)}{1-\nu(1-\theta)}[\sum_{i=0}^{+\infty}\sum_{j=i+1}^{+\infty}\rho^{j-i-1}f(j)]=\frac{\nu\theta(p+q-pq)}{1-\nu(1-\theta)}\sum_{j=1}^{+\infty}f(j)\sum_{i=0}^{j-1}\rho^{j-i-1}<1,$ it is easily known that the equation (2) is the defective renewal equation.

**Theorem 2** For $0<\nu<1$, the asymptotic estimate for the Gerber-Shiu discounted penalty function is
$m(u)\to CR^{-u}\ (u\to\infty),$

Here $C=\frac{p}{p+q-pq}\frac{R-\rho}{R}\frac{\tilde{\omega}(R)-\tilde{\omega}(\rho)}{(R-\rho)\tilde{f}'(R)-(\tilde{f}(R)-\tilde{f}(\rho))}.$

**Proof** Let $Q_\rho(i)=\frac{\nu\theta(p+q-pq)}{1-\nu(1-\theta)}\sum_{j=i+1}^{+\infty}\rho^{j-i-1}f(j),\ H_\rho(u)=\frac{p\nu\theta}{1-\nu(1-\theta)}\sum_{i=u+1}^{+\infty}\rho^{i-u-1}\omega(i),\ M(u)=R^u m(u),$
$A(i)=R^iQ_\rho(i)$

And $B(u)=R^uH_\rho(u).$ first, we multiply by $R^u$ both sides of the equation (2) and obtain that

$$M(u)=\sum_{i=0}^{u}M(u-i)A(i)+B(u). \tag{8}$$

Because

$$\sum_{i=0}^{+\infty}A(i)=\sum_{i=0}^{+\infty}R^iQ_\rho(i)=\frac{\nu\theta(p+q-pq)}{1-\nu(1-\theta)}\sum_{i=0}^{+\infty}R^i\sum_{j=i+1}^{+\infty}\rho^{j-i-1}f(j)$$
$$=\frac{\nu\theta(p+q-pq)}{1-\nu(1-\theta)}\sum_{j=1}^{+\infty}\rho^j f(j)\sum_{i=0}^{j-1}(\frac{R}{\rho})^i\frac{1}{\rho}=\frac{\nu\theta(p+q-pq)}{1-\nu(1-\theta)}\frac{\tilde{f}(\rho)-\tilde{f}(R)}{\rho-R}=1,$$

The equation (8) is a holomorphic renewal equation. Next, the condition $\sum_{u=0}^{+\infty}B(u)<\infty$ is needed to be proved for applying Lemma 2.
In fact, if $\|\omega\|=sup\{\omega(x,y)\},$ then $\|\omega\|<\infty,$ we have

$$0\le\sum_{u=0}^{+\infty}B(u)=\frac{p\nu\theta}{1-\nu(1-\theta)}\sum_{u=0}^{+\infty}R^u\sum_{i=u+1}^{+\infty}\rho^{i-u-1}\omega(i)$$
$$\le\frac{p\nu\theta\|\omega\|}{1-\nu(1-\theta)}\sum_{u=0}^{+\infty}R^u\sum_{i=u+1}^{+\infty}f(i)=\frac{p\nu\theta\|\omega\|}{1-\nu(1-\theta)}\sum_{i=1}^{+\infty}f(i)\sum_{u=0}^{i-1}R^u$$
$$=\frac{p\nu\theta\|\omega\|}{1-\nu(1-\theta)}\frac{1}{1-R}\sum_{i=1}^{+\infty}(1-R^i)f(i)=\frac{p\nu\theta\|\omega\|}{1-\nu(1-\theta)}\frac{1-\tilde{f}(R)}{1-R}<\infty.$$

Thus

$$B=\sum_{u=0}^{+\infty}B(u)=\sum_{u=0}^{+\infty}R^{u}H_{\rho}(u)=\frac{pv\theta}{1-v(1-\theta)}\sum_{u=0}^{+\infty}R^{u}\sum_{i=u+1}^{+\infty}\rho^{i-u-1}\omega(i)$$
$$=\frac{pv\theta}{1-v(1-\theta)}\frac{\tilde{\omega}(\rho)-\tilde{\omega}(R)}{\rho-R}$$

And

$$A=\sum_{i=0}^{+\infty}iA(i)=\sum_{i=0}^{+\infty}iR^{i}Q_{\rho}(i)=\frac{v\theta(p+q-pq)}{1-v(1-\theta)}\sum_{i=0}^{+\infty}iR^{i}\sum_{j=i+1}^{+\infty}\rho^{j-i-1}f(j)$$
$$=\frac{v\theta(p+q-pq)}{1-v(1-\theta)}\{\frac{R\tilde{f}'(R)}{R-\rho}-\frac{R}{(R-\rho)^{2}}[\tilde{f}(R)-\tilde{f}(\rho)]\}.$$

Finally, from Lemma 2, it is followed that

$$\lim_{u\to\infty}M(u)=\frac{B}{A}=\frac{p}{p+q-pq}\frac{R-\rho}{R}\frac{\tilde{\omega}(R)-\tilde{\omega}(\rho)}{(R-\rho)\tilde{f}'(R)-(\tilde{f}(R)-\tilde{f}(\rho))}=C.$$

The proving is completed.

**Example** Based on the feature of the Gerber-Shiu discounted penalty function, we can obtain the asymptotic estimate of the ruin probability. It is easily known that $m(u)=E[I(T<\infty)\mid U^{p,q}(0)=u]=\psi(u)$ as $\omega(x_1,x_2)=1, v=1$. From Theorem 2, the asymptotic estimate of the ruin probability $\psi(u)$ is written by

$$\psi(u)=C_0R^{-u}(u\to\infty),$$

here $C_0=\dfrac{p}{p+q-pq}\dfrac{(\tilde{f}(R)-1)-(R-1)\mu}{R[(R-1)\tilde{f}'(R)-(\tilde{f}(R)-1)]}$.

Particularly, $\psi(0)=\dfrac{p}{(1-q)(1-p)}(\mu-1)$ can also be obtained.

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