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Inequalities for square functions induced by operators on a Hilbert space

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Abstract
Let $U : H \to H$ be a unitary operatör on a Hilbert space H and

$$A_n f = \frac{1}{n}\sum_{i=1}^{n} U^i f$$

for all $f \in H$. Suppsoe that (n_k) is a lacunary sequence with no non-trivial common divisor, then there exists a positive constant C such that

$$\|f\|_H \leq C\left(\sum_{k=1}^{\infty}\left\|A_{n_{k+1}} f - A_{n_k} f\right\|_H^2\right)^{1/2}$$

for all $f \in H$ with $\int f = 0$.

Let T be a contraction on a Hilbert space H and let

$$A_n(T) f = \frac{1}{n}\sum_{i=1}^{n} T^i f$$

for all $f \in H$. Suppose that (n_k) is a lacunary sequence with no non-trivial common divisor, then there exist a Hilbert space K containing H as a closed subspace, and an ortogonal projection $P : K \to H$ such that

$$\|P\|\cdot\|f\|_H \leq C\left(\sum_{k=1}^{\infty}\left\|A_{n_{k+1}}(T) f - A_{n_k}(T) f\right\|_H^2\right)^{1/2}$$

for all $f \in H$ with $\int f = 0$, where C is a positive constant.

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Introduction

Let (n_k) be an increasing sequence of positive integers we say that (n_k) is lacunary if there exists a constant $\beta > 1$ such that

$$\frac{n_{k+1}}{n_k} \geq \beta$$

for all $k \geq 1$.

Our first result is the following:

Theorem 1. Let $U : H \to H$ be a unitary operatör on a Hilbert space H and

$$A_n f = \frac{1}{n}\sum_{i=1}^{n} U^i f$$

for all $f \in H$. Suppsoe that (n_k) is a lacunary sequence with no non-trivial common divisor, then there exists a positive constant C such that

$$\| f \|_H \le C\left(\sum_{k=1}^{\infty} \left\| A_{n_{k+1}} f - A_{n_k} f \right\|_H^2 \right)^{1/2}$$

for all $f \in H$ with $\int f = 0$.

Proof. Let

$$a_n(\alpha) = \frac{1}{n}\sum_{k=1}^{n} \alpha^k$$

for $\alpha \in \mathrm{T}$. By the spectral theorem for unitray operators, it is sufficient to show that there exists a constnt c such that for all α, $|\alpha| = 1,\ \alpha \ne 1$,

$$\sum_{k=1}^{\infty} \left| a_{n_{k+1}}(\alpha) - a_{n_k}(\alpha) \right|^2 \ge c$$

The fact that the sequence (n_k) has no non-trivial common divisor guarantees that for sufficiently large N

$$\sum_{k=1}^{\infty} \left| a_{n_{k+1}}(\alpha) - a_{n_k}(\alpha) \right|^2$$

is never zero for $|\alpha| = 1$ except for $\alpha = 1$. Thus we only need to consider α near 1. Let $\alpha = e^{it}$ with $\pi \le t \le \pi$. Suppose that $\frac{1}{n_1} \ge |t| > 0$. Then for some k we have $\frac{1}{n_k} \ge |t| \ge \frac{1}{n_{k+1}}$. On the other hand, it is easy to verify that

$$\left| a_{n_{k+1}}(\alpha) - a_{n_k}(\alpha) \right| = \left| \frac{\alpha^{n_{k+1}} - 1}{n_{k+1}(\alpha - 1)} - \frac{\alpha^{n_k} - 1}{n_k(\alpha - 1)} \right|$$

$$\ge \left| \frac{\sin(n_{k+1}t)}{n_{k+1}t} - \frac{\sin(n_k t)}{n_k t} \right|$$

$$\ge \gamma\left(1 - \frac{n_k}{n_{k+1}}\right)$$

$$\ge \gamma\left(1 - \frac{1}{\beta}\right)$$

for some constant γ where β is the lacunarity constant for the sequence (n_k). Thus we have

$$\sum_{k=1}^{\infty} \left| a_{n_{k+1}}(\alpha) - a_{n_k}(\alpha) \right|^2 \ge \gamma^2 \left(1 - \frac{1}{\beta}\right)^2$$

and this completes the proof.

Corollary 2. Let X be a measure space, $U : L^2(X) \to L^2(X)$ be a unitary operator and

$$A_n f = \frac{1}{n}\sum_{i=1}^{n} U^i f$$

for all $f \in L^2(X)$. Suppsoe that (n_k) is a lacunary sequence with no non-trivial common divisor, then there exists a positive constant C such that

$$\| f\|_2 \le C\left\|\left(\sum_{k=1}^{\infty}\left|A_{n_{k+1}} f - A_{n_f} f\right|^2\right)^{1/2}\right\|_2$$

for all $f \in L^2(X)$ with $\int f = 0$.

Proof. When $H = L^2(X)$ we clearly have

$$\left(\sum_{k=1}^{\infty}\left\|A_{n_{k+1}} f - A_{n_k} f\right\|_H^2\right)^{1/2} = \left\|\left(\sum_{k=1}^{\infty}\left|A_{n_{k+1}} f - A_{n_k} f\right|^2\right)^{1/2}\right\|_2$$

and the Corollary follows from Theorem 1.
Let T be an operator on a Hilbert space H and define

$$Sf = \left(\sum_{k=1}^{\infty}\left\|A_{n_{k+1}}(T) f - A_{n_k}(T) f\right\|_H^2\right)^{1/2}.$$

Then we have the following result:

Theorem 3. Let T be a contraction on a Hilbert space H and let

$$A_n(T) f = \frac{1}{n}\sum_{i=1}^{n} T^i f$$

for all $f \in H$. Suppose that (n_k) is a lacunary sequence with no non-trivial common divisor, then there exist a Hilbert space K containing H as a closed subspace, and an ortogonal projection $P : K \to H$ such that

$$\|P\| \cdot \| f\|_H \le C\left(\sum_{k=1}^{\infty}\left\|A_{n_{k+1}}(T) f - A_{n_k}(T) f\right\|_H^2\right)^{1/2}$$

for all $f \in H$ with $\int f = 0$, where C is a positive constant.

Proof. By the dilation theorem (see Sz-Nagy and Foias [1]) there exists a Hilbert space K containing H as a closed subspace, an orthogonal projection $P : K \to H$, and a unitary operator $U : K \to K$ with $PU^i f = T^i f$ for all $i \ge 0$ and $f \in H$. Let now $f \in H$. Then by Theorem 1 we have

$$\sum_{k=1}^{N}\left\|A_{n_{k+1}}(T) f - A_{n_k}(T) f\right\|_H^2 = \sum_{k=1}^{N}\left\|P(A_{n_{k+1}}(U) f - A_{n_k}(U) f\right\|_H^2$$

$$= \|P\|^2 \sum_{k=1}^{N}\left\|A_{n_{k+1}}(U) f - A_{n_k}(U) f\right\|_H^2$$

$$\geq C\|f\|_H^2\,\|P\|^2$$

for some positive constant C.

Corollary 4. Let X be a measure space, T be a contraction on $L^2(X)$ and define

$$A_n(T)f = \frac{1}{n}\sum_{i=1}^{n} T^i f$$

for all $f \in L^2(X)$. Define

$$Sf(x) = \left(\sum_{k=1}^{\infty}\left|A_{n_{k+1}}(T)f(x) - A_{n_k}(T)f(x)\right|^2\right)^{1/2}.$$

Suppose that (n_k) is a lacunary sequence with no non-trivial common divisor, then there exists a Hilbert space K containing $L^2(X)$ as a closed subspace, and an orthogonal projection $P: K \to L^2(X)$ such that

$$\|P\|\cdot\|f\|_2 \leq C\|Sf\|_2$$

for all $f \in L^2(X)$ with $\int f = 0$, where C is a positive constant.

Proof. When $H = L^2(X)$ we have

$$\left(\sum_{k=1}^{\infty}\left\|A_{n_{k+1}}(T)f - A_{n_k}(T)f\right\|_H^2\right)^{1/2} = \left\|\left(\sum_{k=1}^{\infty}\left|A_{n_{k+1}}(T)f - A_{n_k}(T)f\right|^2\right)^{1/2}\right\|_2.$$

Thus the Corollary follows from Theorem 3.

References

1. Sz-Nagy B, C Foias. Analyse harmonique des operateures de l'espace de Hilbert, Akad. Kiado, Budapest. Mason, paris, 1967.