International Journal of Statistics and Applied Mathematics 2018; 3(5): 149-155

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2018; 3(5): 149-155

www.mathsjournal.com
Received: 19-07-2018
Accepted: 20-08-2018

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On two multiple integral formulas involving humbert functions of two variables Φ_2 and Ψ_2

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Abstract
The purpose of this paper is to apply the generalized Kummer's summation theorem due to Lavoie *et al.* [3] to establish two general multiple integrals involving Humbert hypergeometric functions of two variables Φ_2 and Ψ_2. Some interesting applications of our main results are also presented.
MSC: 33C05, 33C65, 33C70.

Keywords: Humbert functions, multiple integral fourmulas, Kummer's theorem

1. Introduction

In the usual notation, let ${}_pF_q$ denote generalized hypergeometric function of one variable with p numerator parameters and q denominator parameters, defined by [5]

$$ {}_pF_q\left[\begin{matrix} a_1,\dots,a_p\,; \\ b_1,\dots,b_q\,; \end{matrix} x\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n\dots(a_p)_n}{(b_1)_n\dots(b_q)_n}\frac{x^n}{n!}\,, \tag{1.1} $$

where $(a)_n$ is the Pochhammer's symbol defined by

$$ (a)_n=\begin{cases} 1 & ,\ if\ \ n=0 \\ a(a+1)(a+2)\dots(a+n-1) & ,\ if\ \ n=1,2,3,\dots\dots \end{cases} \tag{1.2} $$

The special case of (1.1) when $p=2,\ q=1$ is usually called Gauss's hypergeometric function ${}_2F_1(a,b;c;\,x)$.

The confluent hypergeometric functions of Humbert Φ_2 and Ψ_2 are defined as follows [5]:

$$ \Phi_2\left[a,a';b;x,y\right]=\sum_{m,n=0}^{\infty}\frac{(a)_m\,(a')_n}{(b)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}\,, \tag{1.3} $$

$$ \Psi_2\left[a;b,b';x,y\right]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}}{(b)_m(b')_n}\frac{x^m}{m!}\frac{y^n}{n!}\,, \tag{1.4} $$

Exton [1, 2] gave the definitions and the Laplace integral representations of the quadruple hypergeometric functions K_{12} and K_{20} as follows:

$$ K_{12}(a,a,a,a;b_1,b_2,b_3,b_4;c_1,c_1,c_2,c_2;x,y,z,t) $$

$$ =\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(a)_{p+q+r+s}\ (b_1)_p\ (b_2)_q\ (b_3)_r\ (b_4)_s\ x^p\ y^q\ z^r\ t^s}{(c_1)_{p+q}(c_2)_{r+s}\ \ p!q!r!s!} $$

$$=\frac{1}{\Gamma(b_1)}\frac{1}{\Gamma(b_2)}\frac{1}{\Gamma(b_3)}\frac{1}{\Gamma(b_4)}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-s_1-s_2-s_3-s_4} s_1^{b_1-1} s_2^{b_2-1} s_3^{b_3-1} s_4^{b_4-1}$$

$$\times \Psi_2(a; c_1, c_2;\ xs_1+ys_2, zs_3+ts_4)\ ds_1\, ds_2\, ds_3\, ds_4 \tag{1.5}$$

and

$$K_{20}(a_1,a_1,b_3,b_4;b_1,b_2,a_2,a_2;c,c,c,c;x,y,z,t)$$

$$=\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(a_1)_{p+q}(b_3)_r(b_4)_s(b_1)_p(b_2)_q(a_2)_{r+s}\ x^p\, y^q\, z^r\ t^s}{(c)_{p+q+r+s}\ \ p!\,q!\,r!\,s!}$$

$$=\frac{1}{\Gamma(b_1)}\frac{1}{\Gamma(b_2)}\frac{1}{\Gamma(b_3)}\frac{1}{\Gamma(b_4)}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-s_1-s_2-s_3-s_4} s_1^{b_1-1} s_2^{b_2-1} s_3^{b_3-1} s_4^{b_4-1}$$

$$\times \Phi_2(a_1, a_2; c;\ xs_1+ys_2, zs_3+ts_4)\ ds_1\, ds_2\, ds_3\, ds_4\,. \tag{1.6}$$

The Kampé de Fériet function of two variables $F^{p:q;k}_{l:m;n}[x,y]$ is defined and represented as follows [5]:

$$F\begin{matrix} p:q;k \\ l:m;n\end{matrix}\left[\begin{matrix}(a_p):(b_q);(c_k); \\ (\alpha_l):(\beta_m);(\gamma_n);\end{matrix} x,y\right]=\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s}\ \frac{x^r}{r!}\ \frac{y^s}{s!}\,. \tag{1.7}$$

In the present investigation, we shall require the following generalization of the classical Kummer's theorem for the series ${}_2F_1(-1)$ [3]:

$${}_2F_1\left[\begin{matrix}a,b & ; \\ 1+a-b+i & ;\end{matrix} -1\right]=\frac{\Gamma(\frac{1}{2})\Gamma(1+a-b+i)\Gamma(1-b)}{2^a\Gamma(1-b+\frac{1}{2}(i+|i|))}$$

$$\times\left\{\frac{A_i}{\Gamma(\frac{1}{2}a+\frac{1}{2}i+\frac{1}{2}-[\frac{1+i}{2}])\Gamma(1+\frac{1}{2}a-b+\frac{1}{2}i)}\right.$$

$$\left.+\frac{B_i}{\Gamma(\frac{1}{2}a+\frac{1}{2}i-[\frac{i}{2}])\Gamma(\frac{1}{2}+\frac{1}{2}a-b+\frac{1}{2}i)}\right\} \tag{1.8}$$

For $(i=0,\pm1,\pm2,\pm3,\pm4,\pm5)$,

where $[x]$ denotes the greatest integer less than or equal to x and $|x|$ denotes the usual absolute value of x. The coefficients A_i and B_i are given respectively in [3]. When $i=0$, (1.8) reduces immediately to the classical Kummer's theorem [4]

$${}_2F_1\left[\begin{matrix}a,b & ; \\ 1+a-b & ;\end{matrix} -1\right]=\frac{\Gamma(1+a-b)\Gamma(\frac{1}{2})}{2^a\Gamma(1+\frac{1}{2}a-b)\Gamma(\frac{1}{2}a+\frac{1}{2})}\,. \tag{1.9}$$

The following results will be required also [5]:

$$(\alpha)_{m+n}=(\alpha)_m(\alpha+m)_n\,, \tag{1.10}$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}A(n,m)=\sum_{m=0}^{\infty}\sum_{n=0}^{m}A(n,m-n)\,, \tag{1.11}$$

$$(\alpha)_{m-n} = \frac{(-1)^n (\alpha)_m}{(1-\alpha-m)_n}, 0 \le n \le m \tag{1.12}$$

and

$$(m-n)! = \frac{(-1)^n\, m!}{(-m)_n},\ 0 \le n \le m. \tag{1.13}$$

2. Main Integral Formulas

Theorem 1. The following general integrals involving Ψ_2 holds true.

$$\frac{1}{\Gamma(b_1-i)}\frac{1}{\Gamma(b_1)}\frac{1}{\Gamma(b_2-i)}\frac{1}{\Gamma(b_2)}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-s_1-s_2-s_3-s_4} s_1^{b_1-i-1} s_2^{b_1-1} s_3^{b_2-i-1} s_4^{b_2-1}$$

$$\times \Psi_2(a; c_1, c_2; x(s_1-s_2), y(s_3-s_4))\ ds_1 ds_2 ds_3 ds_4$$

$$= \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \frac{(a)_{2m_1+2m_2}(b_1-i)_{2m_1}(b_2-i)_{2m_2}\, x^{2m_1} y^{2m_2}}{(c_1)_{2m_1}(c_2)_{2m_2}(2m_1)!(2m_2)!}$$

$$\times (A_i^{'}E_1 + B_i^{'}F_1)(A_i^{''}E_2 + B_i^{''}F_2)$$

$$+ \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \frac{(a)_{2m_1+2m_2+1}(b_1-i)_{2m_1+1}(b_2-i)_{2m_2}\, x^{2m_1+1} y^{2m_2}}{(c_1)_{2m_1+1}(c_2)_{2m_2}(2m_1+1)!(2m_2)!}$$

$$\times (C_i^{'}G_1 + D_i^{'}H_1)(A_i^{''}E_2 + B_i^{''}F_2)$$

$$+ \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \frac{(a)_{2m_1+2m_2+1}(b_1-i)_{2m_1}(b_2-i)_{2m_2+1}\, x^{2m_1} y^{2m_2+1}}{(c_1)_{2m_1}(c_2)_{2m_2+1}(2m_1)!(2m_2+1)!}$$

$$\times (A_i^{'}E_1 + B_i^{'}F_1)(C_i^{''}G_2 + D_i^{''}H_2)$$

$$+ \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \frac{(a)_{2m_1+2m_2+2}(b_1-i)_{2m_1+1}(b_2-i)_{2m_2+1}\, x^{2m_1+1} y^{2m_2+1}}{(c_1)_{2m_1+1}(c_2)_{2m_2+1}(2m_1+1)!(2m_2+1)!}$$

$$\times (C_i^{'}G_1 + D_i^{'}H_1)(C_i^{''}G_2 + D_i^{''}H_2). \tag{2.1}$$

Theorem 2. The following general integrals involving Φ_2 holds true.

$$\frac{1}{\Gamma(b_1-i)}\frac{1}{\Gamma(b_1)}\frac{1}{\Gamma(b_2-i)}\frac{1}{\Gamma(b_2)}\times\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-s_1-s_2-s_3-s_4} s_1^{b_1-i-1} s_2^{b_1-1} s_3^{b_2-i-1} s_4^{b_2-1}$$

$$\times \Phi_2[a, c; d; x(s_1-s_2), y(s_3-s_4)]\ ds_1 ds_2 ds_3 ds_4$$

$$= \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} \frac{(a)_{2m_1}(b_1-i)_{2m_1}(c)_{2m_2}(b_2-i)_{2m_2}\, x^{2m_1} y^{2m_2}}{(d)_{2m_1+2m_2}(2m_1)!(2m_2)!}$$

$$\times (A_i^{'}E_1 + B_i^{'}F_1)(A_i^{''}E_2 + B_i^{''}F_2)$$

$$+\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(a)_{2m_1+1}(b_1-i)_{2m_1+1}(c)_{2m_2}(b_2-i)_{2m_2}\,x^{2m_1+1}y^{2m_2}}{(d)_{2m_1+2m_2+1}(2m_1+1)!(2m_2)!}$$

$$\times(C_i^{'}G_1+D_i^{'}H_1)(A_i^{''}E_2+B_i^{''}F_2)$$

$$+\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(a)_{2m_1}(b_1-i)_{2m_1}(c)_{2m_2+1}(b_2-i)_{2m_2+1}\,x^{2m_1}y^{2m_2+1}}{(d)_{2m_1+2m_2+1}(2m_1)!(2m_2+1)!}$$

$$\times(A_i^{'}E_1+B_i^{'}F_1)(C_i^{''}G_2+D_i^{''}H_2)$$

$$+\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(a)_{2m_1+1}(b_1-i)_{2m_1+1}(c)_{2m_2+1}(b_2-i)_{2m_2+1}\,x^{2m_1+1}y^{2m_2+1}}{(d)_{2m_1+2m_2+2}(2m_1+1)!(2m_2+1)!}$$

$$\times(C_i^{'}G_1+D_i^{'}H_1)(C_i^{''}G_2+D_i^{''}H_2)\,, \qquad (2.2)$$

where

$$E_r=\frac{2^{2m_r}\,\Gamma(\tfrac{1}{2})\Gamma(1-2m_r-b_r+i)\,\Gamma(1-b_r)}{\Gamma(1-b_r+\tfrac{1}{2}(i+|i|))\Gamma(-m_r+\tfrac{1}{2}i+\tfrac{1}{2}-[\tfrac{1+i}{2}])\Gamma(1-m_r-b_r+\tfrac{1}{2}i)}$$

$$F_r=\frac{2^{2m_r}\,\Gamma(\tfrac{1}{2})\Gamma(1-2m_r-b_r+i)\,\Gamma(1-b_r)}{\Gamma(1-b_r+\tfrac{1}{2}(i+|i|))\Gamma(-m_r+\tfrac{1}{2}i-[\tfrac{i}{2}])\Gamma(-m_r+\tfrac{1}{2}-b_r+\tfrac{1}{2}i)}$$

$$G_r=\frac{2^{2m_r+1}\,\Gamma(\tfrac{1}{2})\Gamma(-2m_r-b_r+i)\,\Gamma(1-b_r)}{\Gamma(1-b_r+\tfrac{1}{2}(i+|i|))\Gamma(-m_r+\tfrac{1}{2}i-[\tfrac{1+i}{2}])\Gamma(-m_r+\tfrac{1}{2}-b_r+\tfrac{1}{2}i)}$$

$$H_r=\frac{2^{2m_r+1}\,\Gamma(\tfrac{1}{2})\Gamma(-2m_r-b_r+i)\,\Gamma(1-b_r)}{\Gamma(1-b_r+\tfrac{1}{2}(i+|i|))\Gamma(-m_r-\tfrac{1}{2}+\tfrac{1}{2}i-[\tfrac{i}{2}])\Gamma(-m_r-b_r+\tfrac{1}{2}i)}$$

for $i=0,\pm1,\pm2,\pm3,\pm4,\pm5$; $r=1,2.$

The coefficients $A_i^{'}$ and $B_i^{'}$ can be obtained from the tables of A_i and B_i given in [3] by replacing a and b by $-2m_1$ and b_1, the coefficients $A_i^{''}$ and $B_i^{''}$ can be obtained from the tables of A_i and B_i by replacing a and b by $-2m_2$ and b_2, the coefficients $C_i^{'}$ and $D_i^{'}$ can be obtained from the tables of A_i and B_i by replacing a and b by $-2m_1-1$ and b_1, the coefficients $C_i^{''}$ and $D_i^{''}$ can be obtained from the tables of A_i and B_i by replacing a and b by $-2m_2-1$ and b_2 .

Proof of Theorem 1. Denoting the left hand side of (2.1) by S, then from the definition (1.5), we have

$$S=\sum_{m_1,p_1,m_2,p_2=0}^{\infty}\frac{(a)_{m_1+p_1+m_2+p_2}(b_1-i)_{m_1}(b_1)_{p_1}(b_2-i)_{m_2}(b_2)_{p_2}\,x^{m_1}(-x)^{p_1}\;y^{m_2}(-y)^{p_2}}{(c_1)_{m_1+p_1}(c_2)_{m_2+p_2}\;m_1!\,p_1!\,m_2!\,p_2!}$$

Using the results (1.10)-(1.13), then after a little simplification, we obtain

$$S=\sum_{m_1,m_2=0}^{\infty}\frac{(a)_{m_1+m_2}(b_1-i)_{m_1}(b_2-i)_{m_2}\,x^{m_1}y^{m_2}}{(c_1)_{m_1}(c_2)_{m_2}\;m_1!\,m_2!}$$

$$\times{}_2F_1\left[\begin{matrix}-m_1,b_1 & ; \\ 1-m_1-b_1+i & ;\end{matrix}-1\right]{}_2F_1\left[\begin{matrix}-m_2,b_2 & ; \\ 1-m_2-b_2+i & ;\end{matrix}-1\right]$$

Now, separating into odd and even powers of each of x^{m_1} and y^{m_2}, we have

$$S=\sum_{m_1,m_2=0}^{\infty}\frac{(a)_{2m_1+2m_2}\,(b_1-i)_{2m_1}\,(b_2-i)_{2m_2}\;x^{2m_1}\;y^{2m_2}}{(c_1)_{2m_1}\,(c_2)_{2m_2}\;(2m_1)!\,(2m_2)!}$$

$$\times{}_2F_1\left[\begin{matrix}-2m_1\,,b_1 & ; \\ 1-2m_1-b_1+i & ;\end{matrix}-1\right]{}_2F_1\left[\begin{matrix}-2m_2\,,b_2 & ; \\ 1-2m_2-b_2+i & ;\end{matrix}-1\right].$$

$$+\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(a)_{2m_1+2m_2+1}(b_1-i)_{2m_1+1}(b_2-i)_{2m_2}\,x^{2m_1+1}y^{2m_2}}{(c_1)_{2m_1+1}(c_2)_{2m_2}(2m_1+1)!(2m_2)!}$$

$$\times{}_2F_1\left[\begin{matrix}-2m_1-1,b_1 & ; \\ -2m_1-b_1+i & ;\end{matrix}-1\right]{}_2F_1\left[\begin{matrix}-2m_2\,,b_2 & ; \\ 1-2m_2-b_2+i & ;\end{matrix}-1\right]$$

$$+\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(a)_{2m_1+2m_2+1}(b_1-i)_{2m_1}(b_2-i)_{2m_2+1}\,x^{2m_1}y^{2m_2+1}}{(c_1)_{2m_1}(c_2)_{2m_2+1}(2m_1)!(2m_2+1)!}$$

$$\times{}_2F_1\left[\begin{matrix}-2m_1\,,b_1 & ; \\ 1-2m_1-b_1+i & ;\end{matrix}-1\right]{}_2F_1\left[\begin{matrix}-2m_2-1,b_2 & ; \\ -2m_2-b_2+i & ;\end{matrix}-1\right]$$

$$+\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{(a)_{2m_1+2m_2+2}(b_1-i)_{2m_1+1}(b_2-i)_{2m_2+1}\,x^{2m_1+1}y^{2m_2+1}}{(c_1)_{2m_1+1}(c_2)_{2m_2+1}(2m_1+1)!(2m_2+1)!}$$

$$\times{}_2F_1\left[\begin{matrix}-2m_1-1,b_1 & ; \\ -2m_1-b_1+i & ;\end{matrix}-1\right]{}_2F_1\left[\begin{matrix}-2m_2-1,b_2 & ; \\ -2m_2-b_2+i & ;\end{matrix}-1\right].$$

Finally, if we apply the generalized Kummer's theorem (1.8) for each ${}_2F_1(-1)$, then we arrive at the right hand side of (2.1).This completes the proof of (2.1)**.** The proof of (2.2) is similar to that of (2.1) and we use here the definition (1.6).

3. Applications

The following results will be required in this section [5]:

$$(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\quad\text{and}\quad\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}=\frac{(-1)^n}{(1-\alpha)_n}\tag{3.1}$$

$$\Gamma\left(\tfrac{1}{2}\right)\Gamma\left(1+\alpha\right)=2^{\alpha}\;\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}\alpha\right)\Gamma\left(1+\tfrac{1}{2}\alpha\right)\tag{3.2}$$

$$(\alpha)_{2n}=2^{2n}\left(\tfrac{1}{2}\alpha\right)_n\left(\tfrac{1}{2}\alpha+\tfrac{1}{2}\right)_n\tag{3.3}$$

$$(2n)!=2^{2n}\left(\tfrac{1}{2}\right)_n n!\;\text{ and }\;(2n+1)!=2^{2n}\left(\tfrac{3}{2}\right)_n n!\,.\tag{3.4}$$

(i) Taking $i=0$ in (2.1) and (2.2) and using the results (3.1)-(3.4), we get the following integral formulas :

$$\left\{\frac{1}{\Gamma(b_1)}\frac{1}{\Gamma(b_2)}\right\}^2\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty}e^{-s_1-s_2-s_3-s_4}s_1^{\,b_1-1}s_2^{\,b_1-1}s_3^{\,b_2-1}s_4^{\,b_2-1}$$

$$\times\Psi_2(a;c_1,c_2;x(s_1-s_2),y(s_3-s_4))\;ds_1ds_2ds_3ds_4$$

$$=F\,{}^{2:1;1}_{0:2;2}\left[\begin{matrix}\tfrac{1}{2}a,\tfrac{1}{2}a+\tfrac{1}{2} & : & b_1 & ; & b_2 & ; \\ - & : & \tfrac{1}{2}c_1,\tfrac{1}{2}c_1+\tfrac{1}{2} & ; & \tfrac{1}{2}c_2,\tfrac{1}{2}c_2+\tfrac{1}{2} & ;\end{matrix}x^2,y^2\right]\tag{3.5}$$

and

$$\left\{\frac{1}{\Gamma(b_1)}\frac{1}{\Gamma(b_2)}\right\}^2\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-s_1-s_2-s_3-s_4}s_1^{b_1-1}s_2^{b_1-1}s_3^{b_2-1}s_4^{b_2-1}$$

$$\times\Phi_2[a,c;d;x(s_1-s_2),y(s_3-s_4)]\,ds_1ds_2ds_3ds_4$$

$$=F^{0:3;3}_{2:0;0}\left[\begin{matrix} - & :\tfrac{1}{2}a,\tfrac{1}{2}a+\tfrac{1}{2},b_1;\tfrac{1}{2}c,\tfrac{1}{2}c+\tfrac{1}{2},b_2; \\ \tfrac{1}{2}d,\tfrac{1}{2}d+\tfrac{1}{2}: & - \quad ; \quad - \quad ; \end{matrix} x^2,y^2\right]. \tag{3.6}$$

Further, taking $c_1=2b_1, c_2=2b_2$ in (3.5), we get the following integral in terms of Appell's functions F_4 [5]:

$$\left\{\frac{1}{\Gamma(b_1)}\frac{1}{\Gamma(b_2)}\right\}^2\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-s_1-s_2-s_3-s_4}s_1^{b_1-1}s_2^{b_1-1}s_3^{b_2-1}s_4^{b_2-1}$$

$$\times\Psi_2(a;2b_1,2b_2;x(s_1-s_2),y(s_3-s_4))\,ds_1ds_2ds_3ds_4$$

$$=F_4[\tfrac{1}{2}a,\tfrac{1}{2}a+\tfrac{1}{2};b_1+\tfrac{1}{2},b_2+\tfrac{1}{2};x^2,y^2] \tag{3.7}$$

(ii) Taking $i=1$ in (2.1) and (2.2) and using the results (3.1)-(3.4), we get the following integral formulas:

$$\frac{1}{\Gamma(b_1-1)}\frac{1}{\Gamma(b_1)}\frac{1}{\Gamma(b_2-1)}\frac{1}{\Gamma(b_2)}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-s_1-s_2-s_3-s_4}s_1^{b_1-2}s_2^{b_1-1}s_3^{b_2-2}s_4^{b_2-1}$$

$$\times\Psi_2(a;c_1,c_2;x(s_1-s_2),y(s_3-s_4))\,ds_1ds_2ds_3ds_4$$

$$=F^{2:1;1}_{0:2;2}\left[\begin{matrix} \tfrac{1}{2}a,\tfrac{1}{2}a+\tfrac{1}{2} : & b_1 \quad ; & b_2 \quad ; \\ - & :\tfrac{1}{2}c_1,\tfrac{1}{2}c_1+\tfrac{1}{2};\tfrac{1}{2}c_2,\tfrac{1}{2}c_2+\tfrac{1}{2}; \end{matrix} x^2,y^2\right]$$

$$-\frac{ax}{c_1}F^{2:1;1}_{0:2;2}\left[\begin{matrix} \tfrac{1}{2}a+\tfrac{1}{2},\tfrac{1}{2}a+1: & b_1 \quad ; & b_2 \quad ; \\ - & :\tfrac{1}{2}c_1+\tfrac{1}{2},\tfrac{1}{2}c_1+1;\tfrac{1}{2}c_2,\tfrac{1}{2}c_2+\tfrac{1}{2}; \end{matrix} x^2,y^2\right]$$

$$-\frac{ay}{c_2}F^{2:1;1}_{0:2;2}\left[\begin{matrix} \tfrac{1}{2}a+\tfrac{1}{2},\tfrac{1}{2}a+1: & b_1 \quad ; & b_2 \quad ; \\ - & :\tfrac{1}{2}c_1,\tfrac{1}{2}c_1+\tfrac{1}{2};\tfrac{1}{2}c_2+\tfrac{1}{2},\tfrac{1}{2}c_2+1; \end{matrix} x^2,y^2\right]$$

$$+\frac{a(a+1)xy}{c_1c_2}F^{2:1;1}_{0:2;2}\left[\begin{matrix} \tfrac{1}{2}a+1,\tfrac{1}{2}a+\tfrac{3}{2}: & b_1 \quad ; & b_2 \quad ; \\ - & :\tfrac{1}{2}c_1+\tfrac{1}{2},\tfrac{1}{2}c_1+1;\tfrac{1}{2}c_2+\tfrac{1}{2},\tfrac{1}{2}c_2+1; \end{matrix} x^2,y^2\right] \tag{3.8}$$

and

$$\frac{1}{\Gamma(b_1-1)}\frac{1}{\Gamma(b_1)}\frac{1}{\Gamma(b_2-1)}\frac{1}{\Gamma(b_2)}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-s_1-s_2-s_3-s_4}s_1^{b_1-2}s_2^{b_1-1}s_3^{b_2-2}s_4^{b_2-1}$$

$$\times\Phi_2[a,c;d;x(s_1-s_2),y(s_3-s_4)]\,ds_1ds_2ds_3ds_4$$

$$=F^{0:3;3}_{2:0;0}\left[\begin{matrix} - & :\tfrac{1}{2}a,\tfrac{1}{2}a+\tfrac{1}{2},b_1;\tfrac{1}{2}c,\tfrac{1}{2}c+\tfrac{1}{2},b_2; \\ \tfrac{1}{2}d,\tfrac{1}{2}d+\tfrac{1}{2}: & - \quad ; \quad - \quad ; \end{matrix} x^2,y^2\right]$$

$$-\frac{ax}{d} F^{0:3;3}_{2:0;0}\left[\begin{matrix} - & : \tfrac{1}{2}a+\tfrac{1}{2}, \tfrac{1}{2}a+1, b_1 ; \tfrac{1}{2}c, \tfrac{1}{2}c+\tfrac{1}{2}, b_2 ; \\ \tfrac{1}{2}d+\tfrac{1}{2}, \tfrac{1}{2}d+1 : & - \quad ; \quad - \quad ; \end{matrix} x^2, y^2\right]$$

$$-\frac{cy}{d} F^{0:3;3}_{2:0;0}\left[\begin{matrix} - & : \tfrac{1}{2}a, \tfrac{1}{2}a+\tfrac{1}{2}, b_1 ; \tfrac{1}{2}c+\tfrac{1}{2}, \tfrac{1}{2}c+1, b_2 ; \\ \tfrac{1}{2}d+\tfrac{1}{2}, \tfrac{1}{2}d+1 : & - \quad ; \quad - \quad ; \end{matrix} x^2, y^2\right]$$

$$+\frac{acxy}{d(d+1)} F^{0:3;3}_{2:0;0}\left[\begin{matrix} - & : \tfrac{1}{2}a+\tfrac{1}{2}, \tfrac{1}{2}a+1, b_1 ; \tfrac{1}{2}c+\tfrac{1}{2}, \tfrac{1}{2}c+1, b_2 ; \\ \tfrac{1}{2}d+1, \tfrac{1}{2}d+\tfrac{3}{2} : & - \quad ; \quad - \quad ; \end{matrix} x^2, y^2\right]. \tag{3.9}$$

Further, taking $c_1 = 2b_1 - 1, c_2 = 2b_2 - 1$ in (3.8), we get

$$\frac{1}{\Gamma(b_1-1)}\frac{1}{\Gamma(b_1)}\frac{1}{\Gamma(b_2-1)}\frac{1}{\Gamma(b_2)}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-s_1-s_2-s_3-s_4} s_1^{b_1-2} s_2^{b_1-1} s_3^{b_2-2} s_4^{b_2-1}$$

$$\times\Psi_2(a; 2b_1-1, 2b_2-1; x(s_1-s_2), y(s_3-s_4))\, ds_1 ds_2 ds_3 ds_4$$

$$= F_4\left[\tfrac{1}{2}a, \tfrac{1}{2}a+\tfrac{1}{2}; b_1-\tfrac{1}{2}, b_2-\tfrac{1}{2}; x^2, y^2\right]$$

$$-\frac{ax}{2b_1-1} F_4\left[\tfrac{1}{2}a+\tfrac{1}{2}, \tfrac{1}{2}a+1; b_1+\tfrac{1}{2}, b_2-\tfrac{1}{2}; x^2, y^2\right]$$

$$-\frac{ay}{2b_2-1} F_4\left[\tfrac{1}{2}a+\tfrac{1}{2}, \tfrac{1}{2}a+1; b_1-\tfrac{1}{2}, b_2+\tfrac{1}{2}; x^2, y^2\right]$$

$$+\frac{a(a+1)xy}{(2b_1-1)(2b_2-1)} F_4\left[\tfrac{1}{2}a+1, \tfrac{1}{2}a+\tfrac{3}{2}; b_1+\tfrac{1}{2}, b_2+\tfrac{1}{2}; x^2, y^2\right]. \tag{3.10}$$

The other special cases of the integrals (2.1) and (2.2) can also be obtained in the similar manner.

4. References

1. Exton H. Certain hypergeometric functions of four variables. Bull. Soc. Math. Gréce, (N.S.). 1972; 13:104-113.
2. Exton H. Multiple Hypergeometric Functions and Applications. John Wiley and Sons,
3. Halsted Press, New York, 1976.
4. Lavoie JL, Grondin F, Rathie AK. Generalizations of Whipple's theorem on the sum of a ${}_3F_2$. J. Comput. Appl. Math. 1996; 72:293-300.
5. Rainville ED. Special Functions. The Macmillan Company, New York, 1960.
6. Reprinted by Chelsea Pub. Co., Bronx, New York, 1971.
7. Srivastava HM, Manocha HL. A Treatise on Generating Functions. Halsted Press, New York, 1984.