International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2018; 3(5): 128-133

www.mathsjournal.com
Received: 15-07-2018
Accepted: 16-08-2018

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# Skew normal operators acting on a Hilbert space

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**Abstract**
The motive of the present study is to analyze the basic properties of skew normal operators acting on a Hilbert space H. New theorems and examples are given on some properties in the Hilbert space using skew normal operators.

**Keywords:** Normal operator, Polar decomposition, Self-adjoint operator, Skewnormal operator

## 1. Introduction
Let H be a Hilbert space and L (H) be the algebra of all bounded linear operators acting on H. The basic properties of various operators have been generalized by different authors. The classes of normal and skew normal operators have received greater importance in the study of operator theory. Several interesting aspects regarding these operators are investigated by many outstanding researchers [1, 3, 6]. A skew normal operator is a generalization of normal operator. In this paper such kind of operators are examined and results are derived.

## 2. Skew Normal Operator
An operator $T \in L(H)$ is called self-adjointif $T^* = T$, normal if $T^*T = TT^*$,i.e., T commutes with $T^*$ [2], quasinormal if $T\left(T^*T\right) = \left(T^*T\right)T$ [5]. The class of Quasinormal operators was first introduced and studied by A.Brown. Panayappan and Sivamani [2] discussed about n-binormal operators. Meenambika and Seshaiah [3, 4] have worked on Aluthge transformation of Semi-hyponormaloperators. An operator T is called isometry if $T^*T = I$, 2-isometry if $T^{*^2}T^2 - 2T^*T + I = 0$. An operator $T \in L(H)$ is skew normal if $\left(TT^*\right)T = T\left(T^*T\right)$ i.e., T commutes with normal operator and it is denoted by [sN]. The proposed method analyses some of the relationship between self-adjoint and skew-normal operators, also some results are proved based on skew-normal operator.

**Theorem: 2.1**

If $T \in [sN]$ then so are

i) kT for any real number k
ii) The restriction T/M of T to any closed subspace M of H that reduces T.

**Proof: i)**
The proof is obvious.
ii) If T is skew normal then $\left(TT^*\right)T = T\left(T^*T\right)$
Consider,

$$\left[(T/M)(T/M)^*\right](T/M) = \left[TT^*/M\right](T/M)$$
$$= TT^*T/M$$

$= T/M\ T^*/M\ T/M$

$= (T/M)[T^*/M\ T/M]$

$= (T/M)[(T/M)^*(T/M)]$

Hence $T/M \in [sN]$

**Theorem: 2.2**

**Let** $T \in L(H)$ be skew normal operators which is unitarily equivalent to an operator S if and only if TU=UT, $TU^* = U^*T$, $T^*U = UT^*$ .Then S is skew normal.

**Proof:**

Since T is unitarily equivalent to S,there is a unitary operator U such that $S = U^*TU$ which implies $S^* = U^*T^*U$ .We must show that $(SS^*)S = S(S^*S)$.Since T is skew normal then $(TT^*)T = T(T^*T)$

Now,

$(SS^*)S = S(S^*S)$

$(U^*TUU^*T^*U)U^*TU = U^*TU(U^*T^*UU^*TU)$

$(U^*TT^*U)U^*TU = U^*TU(U^*T^*TU)$

$(U^*TT^*U)U^*UT = U^*UT(U^*T^*TU) \quad (By\ UT = TU)$

$(U^*TT^*U)T = T(U^*T^*TU)$

$(TU^*UT^*)T = T(T^*U^*UT) \quad (By\ TU^* = U^*T, T^*U = UT)$

$(TT^*)T = T(T^*T)$

Therefore S is Skew Normal.

**Theorem: 2.3**

If $T \in L(H)$ is normal then,it is skew-normal.

**Proof:**

An operator T is normal if $T^*T = TT^*$

When T commutes with normal operator, it becomes,

$(TT^*)T = T(T^*T)$

Hence T is skew normal.

The following example shows that the converse need not be true.

**Example: 2.4**

Let $T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ is skew normal but not normal operator.

**Theorem: 2.5**

Let $T \in [sN]\ and\ S \in [sN]$ . If T and S are doubly commuting then TS is skew-normal.

**Proof:**

Consider,

$$[(TS)(TS)^*](TS) = [TSS^*T^*]TS$$
$$= [STT^*S^*]TS$$
$$= STT^*S^*TS$$
$$= STT^*TS^*S \quad (\because S^*T = TS^*)$$
$$= S(TT^*)TS^*S$$
$$= ST(T^*T)S^*S \quad (\because T\ is\ Skew\ Normal)$$

$= STT^*TS^*S$

$= TST^*S^*TS \quad (\because ST = TS, TS^* = S^*T)$

$= TSS^*T^*TS \quad (\because S^*T^* = T^*S^*)$

$= TSS^*T^*ST \quad (\because ST = TS)$

$= TSS^*ST^*T \quad (\because ST^* = T^*S)$

$= T(SS^*)ST^*T$

$= TS(S^*S)T^*T \quad (\because S \text{ is Skew Normal})$

$= TSS^*ST^*T$

$= (TS)[S^*T^*ST] \quad (\because ST^* = T^*S)$

$= (TS)[(TS)^*(TS)]$

Hence, TS is skew-normal.

**Example: 2.6**

Let $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ be not commuting skew normal operators. Then ST is not skew normal.

The following example shows that the sum and difference of two commuting skew normal operators need not be skew normal.

**Example: 2.7**

Let $S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$ on $R^2$. Then S and T are commuting skew normal operators but $S + T = \begin{pmatrix} 12 & 31 \\ 17 & 27 \end{pmatrix}$ is not skew normal and $S - T = \begin{pmatrix} 8 & 13 \\ -5 & -8 \end{pmatrix}$ is not skew normal.

**Theorem: 2.8**

Let $T = U|T|$ be the polar decomposition of an operator T. Then $T = U|T|$ is skew normal if $U|T| = |T|U$

**Proof:**

Assume $U|T| = |T|U$, then

$(TT^*)T - T(T^*T) = (U|T||T|U^*)U|T| - U|T|((|T|U^*U|T|)$

$= (U|T|^2U^*)U|T| - U|T|((|T|^2)$

$= (|T|^2UU^*)U|T| - U|T|((|T|^2)$

$= (|T|^2)U|T| - U|T|((|T|^2)$

$= 0$

So T is skew normal.

**Theorem: 2.9**

Let $T \in L(H)$ with the Cartesian decomposition T=A+iB, then T is skew-normal if and only if $i)\ AB^2 = B^2A$, $ii)\ A^2B = BA^2$

**Proof:**

Since T is skew-normal,

$(TT^*)T = T(T^*T)$

Consider,

$$\left(TT^{*}\right)T = \left[(A+iB)(A-iB)\right](A+iB)$$
$$= \left[A^{2}-(iB)^{2}\right](A+iB)$$
$$= \left[A^{2}+B^{2}\right](A+iB)$$
$$= A^{2}A + B^{2}A + iA^{2}B + iB^{3}$$
$$= A^{3} + AB^{2} + iBA^{2} + iB^{3} \text{.......... .......... ...}(1)$$

$$T\left(TT^{*}\right) = (A+iB)\left[(A+iB)(A-iB)\right]$$
$$= (A+iB)\left[A^{2}-(iB)^{2}\right]$$
$$= (A+iB)\left[A^{2}+B^{2}\right]$$
$$= AA^{2} + AB^{2} + iBA^{2} + iBB^{2}$$
$$= A^{3} + AB^{2} + iBA^{2} + iB^{3} \text{.......... .......... ...}(2)$$

From (1) and (2), T is Skew Normal operator.
The following example shows that skew normal operator need not be isometric operator.

**Example: 2.10**

Consider the operator $T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ which is a skew-normal but not isometric operator.

### 3. Self-adjoint and Skew-normal operators

**Theorem: 3.1**

If T is skew normal and $\lambda$ is any scalar which is real then $\lambda T$ is also skew-normal operator.

**Proof:**
Since T is skew normal operator,we have,

$$\left(TT^{*}\right)T = T\left(T^{*}T\right)$$

If $\lambda$ is any scalar which is real,then $(\lambda T)^{*} = \lambda T^{*}$
Consider,

$$\left[(\lambda T)(\lambda T)^{*}\right]\lambda T = \lambda^{3}\left(TT^{*}\right)T \text{..................}(3)$$

$$\lambda T\left[(\lambda T)^{*}(\lambda T)\right] = \lambda^{3}T\left(T^{*}T\right)$$
$$= \lambda^{3}\left(TT^{*}\right)T \text{................}(4) \qquad (Since\ T\ is\ skew-normal)$$

From (3) and (4), $\lambda$ T is skew –normal.

**Theorem: 3.2**

If T is skew-normal operator which is a self-adjoint operator, then $T^{*}$ is also skew-normal operator.

**Proof:**
Since T is skew normal operator, we have,

$$\left(TT^{*}\right)T = T\left(T^{*}T\right) \text{.......................................} (5)$$

Since T is self-adjoint we have $T^{*} = T$
Replace $T^{*}$ *by* $T$ in (5),
Consider,

$$\left(T^{*}T^{*}\right)T = T^{*}T^{*}T$$
$$= TTT^{*} \text{.......... .......... .......}(6)$$
$$T\left(T^{*}T^{*}\right) = TT^{*}T^{*}$$
$$= T^{*}TT$$
$$= TTT^{*} \text{.......... .......... .......}(7)$$

From (6) and (7), $T^{*}$ self adjoint operator.

**Theorem: 3.3**
If T is self adjointoperator, then T is skew-normal operator.

**Proof:**
Since T is self adjoint operator, $T^{*} = T$
Now,
$$\left(TT^{*}\right)T = \left(TT\right)T = T^{3} \text{.......... .......}(8)$$
$$T\left(T^{*}T\right) = T\left(TT\right) = T^{3} \text{.......... .......}(9)$$

**From (8) and (9),T is skew normal operator.**
**Theorem: 3.4**
Let T be any operator on a Hilbert space H.Then
i) $\left(T + T^{*}\right)$ is skew normal.
ii) $TT^{*}$ is skew normal.
iii) $T^{*}T$ is skew normal.
iv) $I + T^{*}T$, $I + TT^{*}$ are skew normal.

**Proof:**

i. Let $N = T + T^{*}$
$$N^{*} = \left(T + T^{*}\right)^{*} = T^{*} + T = N$$
Hence, N is a self adjoint operator.
We know that every self adjoint operator is skew-normal.
Therefore, $N = T + T^{*}$ is skew –normal operator.

ii. $\left(TT^{*}\right)^{*} = T^{**}T^{*} = TT^{*}$
   a. Hence, $TT^{*}$ is self-adjoint operator, so $TT^{*}$ is skew-normal operator.

iii. $\left(T^{*}T\right)^{*} = T^{*}T^{**} = T^{*}T$
   a. Hence, $T^{*}T$ is self-adjoint operator, so $T^{*}T$ is skew-normal operator.

iv. $\left(I + T^{*}T\right)^{*} = \left(I^{*} + T^{*}T^{**}\right) = \left(I + T^{*}T\right)$
$$\left(I + TT^{*}\right)^{*} = \left(I^{*} + T^{**}T^{*}\right) = \left(I + TT^{*}\right)$$

Hence, $I + T^{*}T$, $I + TT^{*}$ are self-adjoint operator,so $I + T^{*}T$, $I + TT^{*}$ areskew-normal operators.

**Theorem: 3.5**
If T is a self -adjoint operator, then $T^{-1}$ is also a skew normal operator.

**Proof:**
Since T is self adjoint operator, $T^{*} = T$

To prove $T^{-1}$ is a skew normal operator, it is enough to check if it is a self -adjoint operator.

Now,
$$\left(T^{-1}\right)^{*} = \left(T^{*}\right)^{-1} = T^{-1}$$

$\Rightarrow T^{-1}$ Self-adjoint operator.

Hence, $T^{-1}$ is skew-normal operator.

**Proof verification**

Assume that $T^{-1}$ is self-adjoint operator.

$\therefore \left(T^{-1}\right)^{*} = T^{-1}$

To prove that $T^{-1}$ is skew-normal operator,
Consider,

$$\left[T^{-1}\left(T^{-1}\right)^{*}\right]T^{-1} = \left[\left(T^{-1}\right)T^{-1}\right]T^{-1}$$

$$= (T^{-1})^{3}$$

$$= T^{-3} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

$$T^{-1}\left[\left(T^{-1}\right)^{*}T^{-1}\right] = T^{-1}\left[\left(T^{-1}\right)T^{-1}\right]$$

$$= (T^{-1})^{3}$$

$$= T^{-3} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(11)$$

From (10) and (11),

$T^{-1}$ is skew-normal operator.

**Theorem: 3.6**

Let T be a self-adjoint operator on a Hilbert space H,then $S^{*}TS$ is skew normal.

**Proof:**

Since T is self adjoint operator, $T^{*} = T$
Consider,

$$\left(S^{*}TS\right)^{*} = S^{*}T^{*}S^{**} = S^{*}TS$$

$\Rightarrow S^{*}TS$ is self adjoint.

By theorem (2.3), we have the result that if $S^{*}TS$ is self-adjoint, then it is skew-normal.

**Proof verification**

Assume $S^{*}TS$ is self adjoint,
Consider,

$$\left[\left(S^{*}TS\right)\left(S^{*}TS\right)^{*}\right]\left(S^{*}TS\right) = \left[\left(S^{*}TS\right)\left(S^{*}TS\right)\right]\left(S^{*}TS\right)$$

$$= \left(S^{*}TS\right)^{3} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

$$\left(S^{*}TS\right)\left[\left(S^{*}TS\right)^{*}\left(S^{*}TS\right)\right] = \left(S^{*}TS\right)\left[\left(S^{*}TS\right)\left(S^{*}TS\right)\right]$$

$$= \left(S^{*}TS\right)^{3} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13)$$

From (12) and (13), $S^{*}TS$ is skew normal operator.

## 4. Conclusion

This paper has presented a new class of operators called Skew normal operators. Some of the characters of skew normal operators were studied. The described work is focused on relationship between self adjoint and skew normal operators. As for future work is concerned, this class of operators is extended to higher class of operators.

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