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Modeling the survival and existence of a resource dependent population: Effects of toxicants emitted from external sources on the resource biomass

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Abstract
In this paper, a non-linear mathematical model is proposed and analyzed to study the existence and survival of a resource dependent population when the resource is affected by a toxicant (pollutant), emitted into the environment from external sources. Although the population density is also affected, here in the model we have only considered the adverse effects on resource biomass taking up the consideration of the uptake of toxicants by population.

Keywords: Population, resource

Introduction

Due to fast development in the field of industrialization, human population and its activities has been key factor in the disturbance of nature and its resources. From the industries various kinds of toxicants (pollutants) are discharged into both aquatic and terrestrial environments affecting living beings as well as their resources on which they depend for their survival. These toxicants are emitted into the environment from different sources (e.g. chimneys, vehicular traffic, etc).

Several investigators have studied effects of toxicants on biological species using mathematical models. In particular, Hallam and collaborators have proposed and analyzed mathematical models to study effects of toxicants on the biological species when these are emitted into the environment from external sources.

It may be noted here that in above studies toxicants are emitted from external sources into the environment with constant rates. It may also be noted here that, in these studies, effect of toxicant has been studied by considering that the intrinsic growth rate of species is affected by uptake concentration only but it may happen that environmental concentration of toxicant, apart from affecting carrying capacity, may also directly affect the growth rate of biological species. These considerations have got to be used in the modelling process while studying effects of toxicants on biological species. Here, in the model the uptake of resource as well as population both has been considered.

In making of the model the following assumptions are made.

1) The densities of biological population as well as its resource are assumed to be governed by generalized logistic equations with respective intrinsic growth rates and carrying capacities.
2) The toxicant is emitted into the environment at a constant rate (cumulative rate due to emissions from different sources such as industrial chimneys, vehicles, etc.) its concentration is augmented due to various activities of population.
3) The concentration of the toxicant in the environment decreases due to its simultaneous assimilation, (absorption, deposition, uptake), etc. by species population and its resource biomass, the magnitudes being proportional to environmental concentration of toxicant as well as densities of population and its resource. These assimilated amounts affect the densities of population and its resource biomass in the following two ways.

i. A fraction of the corresponding assimilated amount becomes part of the uptake phase and this uptaken toxicant interacts with the population and the resource biomass through biophysical processes leading to decrease in the intrinsic growth rate (growth rate per capita of biomass) of it resource biomass density.

ii. The remaining fraction of this assimilated amount of toxicant in each case decreases the respective growth rate of population (resource biomass density) directly.

4) The environmental concentration of the toxicant decreases the carrying capacities of population as well as its resource biomass.
5) The concentrations of toxicant in the environment as well as in its uptake phase decrease due to natural factors by an amount which is proportional to respective concentrations in the two phases. Also due to recycling there may exist a transformation of toxicant from uptake phase to environmental phase.

In the following we propose a general model governing the above mentioned problem and analyze it by using stability theory of differential equations [10].

Mathematical model

We model here the survival of a resource dependent population (such as human beings) when both population and its resource are affected simultaneously by a toxicant (pollutant) emitted from external sources as well as formed by precursors of population.

Keeping in view the above considerations and assumptions, a six dimensional model governing the problem is proposed as follows,

$$\begin{aligned}
\frac{dN}{dt} &= r(U)N - \frac{r_0 N^2}{K} + \beta_1 NB \\
\frac{dB}{dt} &= s(U)B - \frac{s_0 B^2}{L(T)} - \beta_2 NB - k\alpha BT \\
\frac{dT}{dt} &= Q - \delta T - \alpha BT + \pi\nu UB \\
\frac{dU}{dt} &= (1-k)\alpha BT - \phi U - \phi_1 UN - \nu UB
\end{aligned} \qquad (2.1)$$

$$N(0) = N_0 \geq 0,\ B(0) = B_0 \geq 0,\ T(0) = T_0,\ U(0) = U_0,\ 0 \leq \pi \leq 1$$

In model (2.1), $N(t)$ is the density of biological species, $B(t)$ is the density of resource biomass, $T(t)$ is the concentrations of toxicant (pollutant) in the environment, $U(t)$ is uptake concentration of the resource biomass respectively. The constant Q is the cumulative rate of emissions of the same toxicants (pollutants) into the environment from various external sources, β_1 is the growth rate coefficient of N and β_2 is the depletion rate coefficient of B. The constants δ, ϕ and ϕ_1 are the depletion rate coefficients of the toxicant in the environment, resource biomass and biological species respectively, The constant α is the rates of depletion of pollutant in the environment due to uptake of pollutant by resource biomass respectively. Some of part of the may die out at a rate ν due to pollutant and a fractions π of this may again re-enter in the environment. Here k is a fraction of αBT directly effecting population and remaining $(1-k)$ of it is up taken by the species, which decreases the intrinsic growth rate.

In model (2.1) the function $s(U)$ represents the intrinsic growth rate of resource biomass which decrease with U of the toxicant. Hence we assume,

$$s(0) = s_0 > 0,\ \frac{ds(U)}{dU} \leq 0 \text{ for } U \geq 0. \qquad (2.1a)$$

Where s_0 is the maximum of $s(U)$.

The function $L(T)$ denote the maximum densities of resource biomass which the environment can support. It is assumed that these are decreasing functions of T and so we write

$$L(0) = L_0 > 0,\ \frac{dL(T)}{dT} \leq 0 \text{ for } T \geq 0 \qquad (2.1b)$$

Where L_0 is the maximum of $L(T)$.

The function $r(U)$ represents the intrinsic growth rate of resource biomass which decrease with U of the toxicant. Hence we assume,

$$r(0) = r_0 > 0, \ \frac{dr(U)}{dU} \leq 0 \text{ for } U \geq 0. \tag{2.1c}$$

Where r_0 is the maximum of $r(U)$.
In our model, the constants K denotes the carrying capacity of population.
It is noted that $N(t)$ is partially dependent on the resource density $B(t)$.

Equilibrium analysis

The given model (2.1) has four non-negative equilibria in $N-B-T-U$ space, namely $E_0 = (0,0,T,0), E_1(\hat{N},0,\hat{T},0)$, $E_2(0,\tilde{B},\tilde{T},\tilde{U})$and $E^*(N^*,B^*,T^*,U^*)$. Existence of E_0, E_1 and E_2 can be shown easily. We show the existence of E^*as follows. Here N^*,B^*,T^* and $U^*$are the positive solutions of the algebraic equations.
In this case N^*,B^*,T^*,U^* are the positive solutions given by the following equations,

$$N = \frac{(r(U) + \beta_1 B)K}{r_0} = f(B), \tag{2.2}$$

$$B = \frac{(s(U) - k\alpha T - \beta_2 N)L(T)}{s_0}, \tag{2.3}$$

$$T = \frac{Q(\phi + \phi_1 N + \nu B)}{f_1(B)} = g(B) \tag{2.4}$$

$$U = \frac{Q(1-k)\alpha B}{f_1(B)} = h(B) \tag{2.5}$$

Where,

$$f_1(B) = \delta\phi + \delta\nu B + \phi\nu B + \phi_1 \delta N + (1-(1-k)\pi)\nu\alpha B^2$$

It is noted here that in the second equation of the model, which can be written as $\frac{dB}{dt} = (s(U) - \beta_2 N - k\alpha T)B - \frac{s_0 B^2}{L(T)}$

Where $(s(U) - \beta_2 N - k\alpha T)$ denotes the intrinsic growth rate of $\frac{dB}{dt}$ and it increases only if $(s(U) - \beta_2 N - k\alpha T) > 0$ $N \geq 0, B \geq 0, T \geq 0$ and $U \geq 0$.

To show the existence of E^*, let us take

$$F(B) = s_0 B - s(h(B))L(g(B)) + \beta_2 f(B)L(g(B)) + k\alpha g(B)L(g(B)) \tag{2.6}$$

From the (2.6) we note the following:

$$F(0) = -L\left(\frac{Q}{\delta}\right)\left[s_0 - \left(\frac{\beta_2 rK}{r_0} + k\alpha\frac{Q}{\delta}\right)\right] < 0,$$

Further,

$$F(L_0) = s_0 L_0 - s(h(L_0))L(g(L_0)) + \beta_2 f(L_0)L(g(L_0)) + k\alpha g(L_0)L(g(L_0)) > 0$$

Hence their exists a root B^* in the interval $0 < B^* < L_0$.

$$F\left(B^*\right)=0$$

For B^* to be unique we must have $F'(B)$ positive in $0<B^*<L_0$, which can be found from (2.6) as

$$F'(B)=s_0-s_0\frac{B}{L(g)}\frac{dL}{dg}\frac{dg}{dB}-L(g)\frac{ds}{dh}\frac{dh}{dg}+k\alpha g\frac{dg}{dB}+\beta_2 K\frac{\beta_1}{r_0}L(g)$$

It is noted that $B=B^*,\ F'(B)>0$

Thus the condition for unique and positive B^* is $F'(B)>0$, in $0<B^*<L_0$. Once B^* is determined N^*, T^* and U^* can be found from above equations.

Further, from the 2nd equation of the model at equilibrium we get,

$$r_0 B=s(U)L(g)-\beta_2\left(rk+\beta_1 KB\right)L(g)-k\alpha g L(g)$$

On differentiation, we have,

$$s_0\frac{dB}{dQ}=s(U)\frac{dL}{dg}\frac{dg}{dQ}-\beta_2\left(rK+\beta_1 KB\right)\frac{dL}{dg}\frac{dg}{dQ}-k\alpha g\frac{dL}{dg}\frac{dg}{dQ}+L(g)\frac{ds}{dU}\frac{dU}{dQ}$$

$$-\beta_2 L(g)\beta_1 K\frac{dB}{dQ}-k\alpha L(g)\frac{dg}{dQ}$$

Using the formulas $\frac{dg}{dQ}=\frac{\partial g}{\partial B}\frac{dB}{dQ}+\frac{\partial g}{\partial Q}$ and similarly for other variables, we get

$$\frac{dB}{dQ}<0$$

These show that as emission rate of toxicant increases the resource biomass density decreases.

Stability analysis

Local stability

To study the local stability behavior, we compute the varitional matrices corresponding to each equilibrium, from which the following is noted.

i. E_0 is a saddle point with unstable manifold in $N-B$ direction and stable manifold in $T-U$ direction.

ii. E_1 is a saddle point with unstable manifold in $N-B$ direction if $\left(r>\frac{2r_0\hat{N}}{K}\right)$ and $s_0>\left(\beta_2\hat{N}+k\alpha\hat{T}\right)$ then stable manifold in $T-U$ direction.

iii. E_2 is a saddle point with unstable manifold in N direction and stable manifold in $B-T-U$ direction.

The stability behavior of E^* is not obvious from the corresponding Jacobian matrix, therefore, by using Liapunov's method, in the following theorem we have found sufficient conditions for E^* to be locally asymptotically stable.

Theorem 1. *Let the following inequalities hold:*

$$\left[\frac{\beta_1}{\beta_2}\left(\frac{s_0}{\left[L\left(T^*\right)\right]^2}L'\left(T^*\right)-k\alpha\right)-C_3\left(\alpha T^*+\pi\nu U^*\right)\right]^2<4\frac{\beta_1 s_0 C_3}{\beta_2 L\left(T^*\right)}\left(\delta+\alpha B^*\right)$$

$$\left[r_1'\left(U^*\right)-C_4\phi U^*\right]^2<4\frac{C_4 r_0}{K}\left(\phi+\phi_1 N^*+\nu B^*\right)$$

where,

$$C_3 = \frac{C_4(1-k)\alpha}{\pi\nu},\; C_4 = \frac{-\beta_1 s'(U^*)}{\beta_2\{(1-k)\alpha T^* - \nu U^*\}}$$

Then E^* is locally asymptotically stable.
Proof: - Using the following Liapunov's function for the linearlized system (2.1).

$$V = \frac{1}{2}\frac{C_1 n^2}{N^*} + \frac{1}{2}\frac{C_2 b^2}{B^*} + \frac{1}{2}C_3\tau^2 + \frac{1}{2}C_4 u^2$$

which on differentiation gives

$$\dot{V} = \frac{C_1 n\dot{n}}{N^*} + \frac{C_2 b\dot{b}}{B^*} + C_3\tau\dot{\tau} + C_4 u\dot{u}$$

On substituting values of $\dot{n}, \dot{b}, \dot{\tau}$ and $\dot{u}$

$$\dot{V} = -\frac{C_1 r_0}{K}n^2 - \frac{C_2 s_0}{L(T^*)}b^2 - C_3(\delta + \alpha B^*)\tau^2 - C_4(\phi + \phi_1 N^* + \nu B^*)u^2 + (C_1\beta_1 - C_2\beta_2)nb$$

$$[C_1 r(U^*) - C_4\phi U^*]nu + \left[C_2\left(\frac{s_0}{[L(T^*)]^2}L'(T^*) - k\alpha\right) - C_3(\alpha T^* + \pi\nu U^*)\right]b\tau +$$

$$[C_2 s'(U^*) + C_4((1-k)\alpha T^* - \nu U^*)]bu + [C_3\pi\nu B^* + C_4(1-k)\alpha B^*]\tau u$$

On choosing the values of constant as follows (since $((1-k)\alpha T^* - \nu U^*) > 0$)

$$C_1 = 1,\; C_2 = \frac{\beta_1}{\beta_2},\; C_4 = \frac{-\beta_1 s'(U^*)}{\beta_2((1-k)\alpha T^* - \nu U^*)}$$

The equation reduces to the following form

$$\dot{V} = -\frac{r_0}{K}n^2 - \frac{\beta_1 s_0}{\beta_2 L(T^*)}b^2 - C_3(\delta + \alpha B^*)\tau^2 - C_4(\phi + \phi_1 N^* + \nu B^*)u^2 + [r(U^*) - C_4\phi U^*]nu +$$

$$\left[\frac{\beta_1}{\beta_2}\left(\frac{s_0}{[L(T^*)]^2}L'(T^*) - k\alpha\right) - C_3(\alpha T^* + \pi\nu U^*)\right]b\tau + [C_3\pi\nu B^* + C_4(1-k)\alpha B^*]\tau u$$

For $\dot{V}$ to be negative definite the following conditions must hold.

$$\left[\frac{\beta_1}{\beta_2}\left(\frac{s_0}{[L(T^*)]^2}L'(T^*) - k\alpha\right) - C_3(\alpha T^* + \pi\nu U^*)\right]^2 < 4\frac{\beta_1 s_0 C_3}{\beta_2 L(T^*)}(\delta + \alpha B^*)$$

$$[C_3\pi\nu B^* + C_4(1-k)\alpha B^*]^2 < 4C_3C_4(\delta + \alpha B^*)(\phi + \phi_1 N^* + \nu B^*)$$

$$[r_1'(U^*) - C_4\phi U^*]^2 < 4\frac{C_4 r_0}{K}(\phi + \phi_1 N^* + \nu B^*)$$

Rewriting the above inequality as follows

$$\left[C_3\pi\nu B^* - C_4(1-k)\alpha B^*\right]^2 + 4C_3C_4\pi\nu(1-k)\alpha B^{*2} < 4C_3C_4\left(\delta+\alpha B^*\right)\left(\phi+\phi_1 N^* + \nu B^*\right)$$

This equation can be reduced on choosing $C_3 = \dfrac{C_4(1-k)\alpha}{\pi\nu}$, where C_4 is defined above

Keeping the values of the constant we get the same inequalites as mentioned above

$$\left[\frac{\beta_1}{\beta_2}\left(\frac{s_0}{\left[L(T^*)\right]^2}L'(T^*) - k\alpha\right) - C_3\left(\alpha T^* + \pi\nu U^*\right)\right]^2 < 4\frac{\beta_1 s_0 C_3}{\beta_2 L(T^*)}\left(\delta+\alpha B^*\right)$$

$$\left[r_1'(U^*) - C_4\phi U^*\right]^2 < 4\frac{C_4 r_0}{K}\left(\phi+\phi_1 N^* + \nu B^*\right)$$

It can be checked that the derivative of V with respect to t is negative definite under the above-mentioned conditions.

The following theorem characterizes the non-linear stability behavior of the equilibrium point E^*. The following lemma, which establishes a of attraction for all solutions of the system initiating in the interior of the positive orthant.

Lemma: - The set

$$A = \left\{(N,B,T,U): 0 \le N \le \frac{K}{r_0}\left(r(U)+\beta_1 L_0\right), 0 \le B \le L_0, 0 \le T+U \le \frac{Q}{\phi_1}\right\}$$

where $\phi_1 = \min(\delta,\phi)$ attracts all solutions initiating in the interior of the positive orthant.

Proof: From the second equation of the model, we have

$$\frac{dB}{dt} = s(U)B - \frac{s_0 B^2}{L(T)} - \beta_2 NB - k\alpha BT$$

$$\le s_0 B - \frac{s_0 B^2}{L_0}$$

$$\Rightarrow 0 \le B \le L_0$$

From the first equation of the model, we see

$$\frac{dN}{dt} \le r(U)N - \frac{r_0 N^2}{K} + \beta_1 N L_0$$

$$\le \left(r(U)+\beta_1 L_0\right)N - \frac{r_0 N^2}{K}$$

$$\Rightarrow 0 \le N \le \frac{K}{r_0}\left(r(U)+\beta_1 L_0\right)$$

Further, from the model

$$\frac{dT}{dt} + \frac{dU}{dt} \le Q - \phi_1\left(T+U\right)$$

$$\Rightarrow 0 \le T+U \le \frac{Q}{\phi_1} \text{ where, } \phi_1 = \min(\delta,\phi)$$

Hence the lemma.

Theorem 2. In addition to the assumptions (2.1a) and (2.1b), let the functions $r(U)$, $s(U)$ and $L(T)$ satisfy the conditions

$$L_m \le L(T) \le L_0\,,\ 0 \le -s'(U) \le q\,, 0 \le -r'(U) \le a\,,\ 0 \le -L'(T) \le p$$

for some positive constants L_m $, m, p$ and q in A.
If the following inequalities holds

$$\left[\frac{\beta_1}{\beta_2}\left(s_0 B^* \frac{p}{L_m^2} + k\alpha\right) + C_3\left(\alpha T^* - \pi\nu U^*\right)\right]^2 < 4\frac{\beta_1 s_0 C_3}{\beta_2 L\left(T^*\right)}\left(\delta + \alpha B^*\right)$$

$$\left[a - C_4\phi U^*\right]^2 < 4\frac{C_4 r_0}{K}\left(\phi + \phi_1 N^* + \nu B^*\right)$$

Where $C_3 = \dfrac{(1-k)\alpha}{\pi\nu}$

Then E^* *is non-linearly asymptotically stable in* A.

Proof: - Consider the following positive definite function around E^*

$$W(N, B.T, U) = C_1\left(N - N^* - N^* \ln\frac{N}{N^*}\right) + C_2\left(B - B^* - B^* \ln\frac{B}{B^*}\right) + \frac{1}{2}C_3\left(T - T^*\right)^2 + \frac{1}{2}C_4\left(U - U^*\right)^2$$ which on differentiation gives

$$\dot{W} = C_1\left(\frac{N - N^*}{N}\right)\frac{dN}{dt} + C_2\left(\frac{B - B^*}{B}\right)\frac{dB}{dt} + C_3\left(T - T^*\right)\frac{dT}{dt} + C_4\left(U - U^*\right)\frac{dU}{dt}$$

On substituting values of $\dot{N}, \dot{B}, \dot{T}$ and $\dot{U}$

$$\dot{W} = -\frac{C_1 r_0}{K}\left(N - N^*\right)^2 - \frac{C_2 s_0}{L\left(T^*\right)}\left(B - B^*\right)^2 - C_3\left(\delta + \alpha B^*\right)\left(T - T^*\right)^2 - C_4\left(\phi + \phi_1 N^* + \nu B^*\right)\left(U - U^*\right)^2 +$$
$$\left(C_1\beta_1 - C_2\beta_2\right)\left(N - N^*\right)\left(B - B^*\right) + \left[C_1\kappa(U) - C_4\phi U^*\right]\left(N - N^*\right)\left(U - U^*\right) +$$
$$\left[C_2\left(s_0\xi(T)B^* - k\alpha\right) - C_3\left(\alpha T^* + \pi\nu U^*\right)\right]\left(B - B^*\right)\left(T - T^*\right) +$$
$$\left[C_2\eta(U) + C_4\left((1-k)\alpha T^* - \nu U^*\right)\right]\left(B - B^*\right)\left(U - U^*\right) + \left[C_3\pi\nu B^* + C_4(1-k)\alpha B^*\right]\left(T - T^*\right)\left(U - U^*\right)$$

Where,

$$\xi(T) = \begin{cases} \dfrac{\dfrac{1}{L(T)} - \dfrac{1}{L\left(T^*\right)}}{\left(T - T^*\right)}; T \ne T^* \\[2ex] -\dfrac{L'\left(T^*\right)}{\left[L\left(T^*\right)\right]^2}; T = T^* \end{cases}, \quad \eta(U) = \begin{cases} \dfrac{s(U) - s\left(U^*\right)}{\left(U - U^*\right)}; U \ne U^* \\[2ex] s'\left(U^*\right); U = U^* \end{cases}$$

And $\kappa(U)=\begin{cases}\dfrac{r(U)-r(U^*)}{(U-U^*)};U\neq U^*\\ s'(U^*);U=U^*\end{cases}$

Let the functions $r(U), s(U)$ and $L(T)$ satisfy the following conditions

$$L_m \le L(T) \le L_0,\ 0 \le -s'(U) \le q,\ 0 \le -r'(U) \le a,\ 0 \le -L'(T) \le p$$

Then by using the above equations and the mean value theorem, we get $|\xi(T)| \le \frac{p}{L_m^2}, |\kappa(U)| \le a$ and $|\eta(U)| \le q$

On choosing the value of the constant as

$$C_1 = 1,\ C_2 = \frac{\beta_1}{\beta_2}$$

The above equation reduces to the following form

$$\dot{W} = -\frac{r_0}{K}(N-N^*)^2 - \frac{\beta_1 s_0}{\beta_2 L(T^*)}(B-B^*)^2 - C_3(\delta+\alpha B^*)(T-T^*)^2 - C_4(\phi+\phi_1 N^* + \nu B^*)(U-U^*)^2 +$$

$$[\kappa(U) - C_4\phi U^*](N-N^*)(U-U^*) + \left[\frac{\beta_1}{\beta_2}(s_0\xi(T)B^* - k\alpha) - C_3(\alpha T^* + \pi\nu U^*)\right](B-B^*)(T-T^*) +$$

$$\left[\frac{\beta_1}{\beta_2}\eta(U) + C_4((1-k)\alpha T^* - \nu U^*)\right](B-B^*)(U-U^*) + [C_3\pi\nu B^* + C_4(1-k)\alpha B^*](T-T^*)(U-U^*)$$

For $\dot{W}$ to be negative definite the following conditions must hold.

$$\left[\frac{\beta_1}{\beta_2}(s_0 B^*\xi(T) + k\alpha) + C_3(\alpha T^* - \pi\nu U^*)\right]^2 < 4\frac{\beta_1 s_0 C_3}{\beta_2 L(T^*)}(\delta + \alpha B^*)$$

$$[C_3\pi\nu B^* + C_4(1-k)\alpha B^*]^2 < 4C_3C_4(\delta+\alpha B^*)(\phi + \phi_1 N^* + \nu B^*)$$

$$[\kappa(U) - C_4\phi U^*]^2 < 4\frac{C_4 r_0}{K}(\phi + \phi_1 N^* + \nu B^*)$$

Rewriting the above inequality by using the maximum values of $\xi(T), \eta(U)$ and $\kappa(U)$

$$\left[\frac{\beta_1}{\beta_2}\left(s_0 B^* \frac{p}{L_m^2} + k\alpha\right) + C_3(\alpha T^* - \pi\nu U^*)\right]^2 < 4\frac{\beta_1 s_0 C_3}{\beta_2 L(T^*)}(\delta + \alpha B^*)$$

$$[C_3\pi\nu B^* + C_4(1-k)\alpha B^*]^2 < 4C_3C_4(\delta+\alpha B^*)(\phi + \phi_1 N^* + \nu B^*)$$

$$[a - C_4\phi U^*]^2 < 4\frac{C_4 r_0}{K}(\phi + \phi_1 N^* + \nu B^*)$$

This equation can be reduced on choosing $C_3 = \frac{C_4(1-k)\alpha}{\pi\nu}$, where C_4 is defined above

Keeping the values of the constant we get the same inequalites as mentioned above

$$\left[\frac{\beta_1}{\beta_2}\left(s_0 B^* \frac{p}{L_m^2} + k\alpha\right) + C_3\left(\alpha T^* - \pi\nu U^*\right)\right]^2 < 4\frac{\beta_1 s_0 C_3}{\beta_2 L(T^*)}\left(\delta + \alpha B^*\right)$$

$$\left[a - C_4 \phi U^*\right]^2 < 4\frac{C_4 r_0}{K}\left(\phi + \phi_1 N^* + \nu B^*\right)$$

The above theorems imply that under some conditions the equilibrium level of resource biomass density decreases as the emission rate of the toxicant into the environment increases leading to decrease in the density of resource dependent population. Further it is pointed out that if the amount of toxicant in the environment goes on increasing, the survival of species may be threatened.

Conclusion

In this paper, a mathematical model have been proposed and analyzed to study the survival and existence of a resource dependent population, considering the uptake phase by population and resource biomass. The cumulative rate of the emission of the toxicant into the environment is assumed to be constant. The existence of non-trivial equilibrium has been proved and its stability behavior is analyzed. It has been shown that with the increase in the emission rate of toxicant the equilibrium level of biological population decreases leading to the decrease in the equilibrium level of the resource dependent population. It has been also noted that for large rates of its emission and transformation, the resource biomass may become extinct and the population depending on it may be threatened to extinction.

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