International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2018; 3(6): 14-18

www.mathsjournal.com
Received: 03-09-2018
Accepted: 10-10-2018

Ahmed Ali Atash
Department of Mathematics, Aden University, Aden, Yemen

Salem Saleh Barahmah
Department of Mathematics, Aden University, Aden, Yemen

Maisoon Ahmed Kulib
Department of Mathematics, Aden University, Aden, Yemen

Correspondence
Ahmed Ali Atash
Department of Mathematics, Aden University, Aden, Yemen

On a new extensions of extended gamma and beta functions

Ahmed Ali Atash, Salem Saleh Barahmah and Maisoon Ahmed Kulib

Abstract
The purpose of the present paper is to introduce a new extension of extended Gamma and Beta functions given recently by Pucheta [5] and Rahaman *et al.* [6] Further, we present certain results including summation formulas, integral representations and Mellin transform.

Keywords: Extended gamma function, extended beta function, Mittag-Leffler function, summation formulas, integral representations, Mellin transform.
2010 Mathematics Subject Classification: 33B20, 33C20, 33B15, 33C05

Introduction
The Gamma and Beta functions are defined respectively, as follows (see) [8]:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt, (\mathrm{Re}(x) > 0). \quad (1.1)$$

and

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt, (\mathrm{Re}(x) > 0, \mathrm{Re}(y) > 0\,). \quad (1.2)$$

For each $x, y \in (0, +\infty)$, the Beta and Gamma functions have the following relation

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (1.3)$$

In 1903, Gosta Mittag-Leffler [4] introduced the function $E_\alpha(z)$ defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \alpha > 0, z \in \mathbb{C}. \quad (1.4)$$

$$E_1(z) = e^z.$$

The generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ [9] is defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, (\alpha, \beta \in \mathbb{C}, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0). \quad (1.5)$$

$$E_{\alpha,1}(z) = E_\alpha(z).$$

Recently, many authors have introduced certain extensions of extended Gamma and Beta functions (1.1) and (1.2) (see) [1-7].
The following two extensions of Gamma function are introduced by Chaudhry and Zubair [1] and Pucheta [5] respectively:

$$\Gamma_p(x) = \int_0^\infty t^{x-1} \exp\left(-t - \frac{p}{t}\right) dt, (Re(p) > 0) (1.6)$$

And

$$\Gamma^\alpha(x) = \int_0^\infty t^{x-1} E_\alpha(-t)\, dt, Re(x) > 0, \alpha > 0. (1.7)$$

Chaudhry *et al*. [2] presented the following extended Beta function:

$$B(x, y; p) = \int_0^1 t^{x-1} (1-t)^{y-1} \exp\left(\frac{-p}{t(1-t)}\right) dt, (1.8)$$

$(Re(p) > 0, Re(x) > 0, Re(y) > 0)$.

In 2014, Choi *et al*. [3] introduced the following extended Beta function:

$$B(x, y; p; q) = \int_0^1 t^{x-1} (1-t)^{y-1} \exp\left(\frac{-p}{t} - \frac{q}{(1-t)}\right) dt, (1.9)$$

$(Re(p) > 0, Re(q) > 0, Re(x) > 0, Re(y) > 0)$.

The following extended Beta function is given by Rahman *et al*. [6]

$$B_{p,q}^\alpha(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} E_\alpha\left(-\frac{p}{t}\right) E_\alpha\left(-\frac{q}{(1-t)}\right) dt, (1.10)$$

$(Re(x) > 0, Re(y) > 0, \alpha > 0, p, q \geq 0)$.

In the present paper, we introduce a new extensions of Gamma and Beta functions in the following forms.

$$\Gamma_p^{(\alpha,\beta)}(x) = \int_0^\infty t^{x-1} E_{\alpha,\beta}\left(-t - \frac{p}{t}\right) dt, (1.11)$$

$Re(x) > 0, p \geq 0, Re(\alpha) > 0, Re(\beta) > 0$

and

$$B_{p,q}^{(\alpha,\beta)}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} E_{\alpha,\beta}\left(-\frac{p}{t}\right) E_{\alpha,\beta}\left(-\frac{q}{(1-t)}\right) dt, (1.12)$$

$(Re(x) > 0, Re(y) > 0, \ Re(\alpha) > 0, Re(\beta) > 0, p, q \geq 0)$,

Remark 1.
(i) For $p = 0$, equation (1.11) reduces to the following extended Gamma function:

$$\Gamma_0^{(\alpha,\beta)}(x) = \Gamma^{(\alpha,\beta)}(x) = \int_0^\infty t^{x-1} E_{\alpha,\beta}(-t)\, dt, (1.13)$$

$Re(x) > 0, Re(\alpha) > 0, Re(\beta) > 0$.

(i) For $\alpha = \beta = 1$, equation (1.11) reduces to extended Gamma function (1.6).
(ii) For $p = 0, \beta = 1$, equation (1.11) reduces to extended Gamma function (1.7).
(iii) For$p = 0, \alpha = \beta = 1$, equation (1.11) reduces to the classical Gamma function (1.1).

Remark 2.
(i) For$\alpha = \beta = 1$ and $p = q$, equation (1.12) reduces to extended Beta function (1.8).
(ii) For$\alpha = \beta = 1$, equation (1.12) reduces to extended Beta function (1.9).
(iii) For$\beta = 1$, equation (1.12) reduces to extended Beta function (1.10).
(iv) For$\alpha = \beta = 1$ and $p = q = 0$, equation(1.12) reduces to the classical Beta function (1.2).

Properties and formulas
In this section, we present certain properties of extensions of Gamma and Beta functions including summation formulas, integral representations and Mellin transform

Theorem 1. The following summation formulas holds true

(i) $B_{p,q}^{(\alpha,\beta)}(x, 1-y) = \sum_{n=0}^{\infty} \frac{(y)_n}{n!} B_{p,q}^{(\alpha,\beta)}(x+n, 1)$, (2.1)

(ii) $B_{p,q}^{(\alpha,\beta)}(x, y) = \sum_{n=0}^{\infty} B_{p,q}^{(\alpha,\beta)}(x+n, y+1)$ (2.2)

Proof. From (1.12), we have

$$B_{p,q}^{(\alpha,\beta)}(x, y-1) = \int_0^1 t^{x-1}(1-t)^{-y} E_{\alpha,\beta}\left(-\frac{p}{t}\right) E_{\alpha,\beta}\left(-\frac{q}{(1-t)}\right) dt.$$

Using the generalized binomial theorem

$$(1-t)^{-y} = \sum_{n=0}^{\infty} \frac{(y)_n}{n!} t^n, |t| < 1,$$

we obtain

$$B_{p,q}^{(\alpha,\beta)}(x, 1-y) = \int_0^1 \sum_{n=0}^{\infty} \frac{(y)_n}{n!} t^{x+n-1} E_{\alpha,\beta}\left(-\frac{p}{t}\right) E_{\alpha,\beta}\left(-\frac{q}{(1-t)}\right) dt.$$

Now, interchanging the order of summation and integration in the above equation and using (1.12), we obtain the required result (2.1). The proof of (2.2) is similar to that of (2.1) and we use here the generalized binomial theorem

$$(1-t)^{y-1} = (1-t)^y \sum_{n=0}^{\infty} t^n, \qquad |t| < 1.$$

Theorem 2. The following summation formula holds true

$$\sum_{k=0}^{n} \binom{n}{k} B_{p,q}^{(\alpha,\beta)}(x+k, y+n-k) = B_{p,q}^{(\alpha,\beta)}(x, y), n \in \mathbb{N}_0 \ (2.3)$$

Proof. For prove of (2.3) we use the mathematical induction on ($n \in \mathbb{N}_0$) as follows:
Clearly, For $n = 0$ the equation (2.3) holds.

For$n = 1$, we have

$$B_{p,q}^{(\alpha,\beta)}(x+1, y) + B_{p,q}^{(\alpha,\beta)}(x, y+1)$$

$$= \int_0^1 \{t^x(1-t)^{y-1} + t^{x-1}(1-t)^y\} E_{\alpha,\beta}\left(-\frac{p}{t}\right) E_{\alpha,\beta}\left(-\frac{q}{(1-t)}\right) dt,$$

$$= \int_0^1 t^{x-1}(1-t)^{y-1}\{t + (1-t)\} E_{\alpha,\beta}\left(-\frac{p}{t}\right) E_{\alpha,\beta}\left(-\frac{q}{(1-t)}\right) dt,$$

$$= \int_0^1 t^{x-1}(1-t)^{y-1} E_{\alpha,\beta}\left(-\frac{p}{t}\right) E_{\alpha,\beta}\left(-\frac{q}{(1-t)}\right) dt,$$

$$= B_{p,q}^{(\alpha,\beta)}(x, y). \ (2.4)$$

Therefore, the equation (2.3) holds for $n = 1$.
Continuing this process for all ($n \in \mathbb{N}_0$), we finally obtain the desired relation (2.3).

Remark 3.
(i) If$\alpha = \beta = 1$ and $p = q$, then (2.4) reduces toa known result of Chaudhry *et al*. [2, p.23 (3.1)].
(ii) If $\alpha = \beta = 1$, then (2.4) reduces toa known result of Choi *et al*. [3, p.362 (3.1)].
(iii) If $\beta = 1$, then (2.4)r educes to a known result of Rahman *et al*. [6, p.7(3.1)].
(iv) If $\alpha = \beta = 1$ and $p = q = 0$, then (2.4) reduces to a known result for the classical Beta function.

Theorem 3. For the product of two extended gamma function defined in (1.13) the following formula holds true

$$\Gamma^{(\alpha,\beta)}(u)\Gamma^{(\alpha,\beta)}(v)=\frac{1}{B(x+u,y+v)}\int_0^\infty\int_0^\infty p^{u-1}q^{v-1}\ B_{p,q}^{(\alpha,\beta)}(x,y)dpdq,\ (2.5)$$

$(Re(x+u)>0, Re(y+v)>0, Re(\alpha)>0, Re(\beta)>0, Re(p), Re(q)>0,$

$Re(u), Re(v)>0).$

Proof. From (1.13), we have

$$\Gamma^{(\alpha,\beta)}(u)\Gamma^{(\alpha,\beta)}(v)=\frac{B(x+u,y+v)}{B(x+u,y+v)}\int_0^\infty\int_0^\infty w^{u-1}z^{v-1}\ E_{\alpha,\beta}(-w)\ E_{\alpha,\beta}(-z)\,dwdz$$

$$=\frac{1}{B(x+u,y+v)}\int_0^1 t^{x+u-1}(1-t)^{y+v-1}$$
$$\times\left\{\int_0^\infty\int_0^\infty w^{u-1}z^{v-1}\ E_{\alpha,\beta}(-w)\ E_{\alpha,\beta}(-z)\,dwdz\right\}dt$$

Using the substituting $w=\frac{p}{t}, z=\frac{q}{(1-t)}$, we get

$$\Gamma^{(\alpha,\beta)}(u)\Gamma^{(\alpha,\beta)}(v)=\frac{1}{B(x+u,y+v)}\int_0^1 t^{x-1}(1-t)^{y-1}$$

$$\times\left\{\int_0^\infty\int_0^\infty p^{u-1}q^{v-1}\ E_{\alpha,\beta}\left(-\frac{p}{t}\right)\ E_{\alpha,\beta}\left(-\frac{q}{(1-t)}\right)dpdq\right\}dt$$

Under stated conditions of the above integral, the order of integration can be inter change -ng. Therefore, we have

$$\Gamma^{(\alpha,\beta)}(u)\Gamma^{(\alpha,\beta)}(v)=\frac{1}{B(x+u,y+v)}$$

$$\times\int_0^\infty\int_0^\infty p^{u-1}q^{v-1}\left\{\int_0^1 t^{x-1}(1-t)^{y-1}\ E_{\alpha,\beta}\left(-\frac{p}{t}\right)E_{\alpha,\beta}\left(-\frac{q}{(1-t)}\right)dt\right\}dpdq$$

Finally, using the result (1.12), we obtain the desired result.

Remark 4.
The special case $\alpha=\beta=1$ of (2.5) reduces to a known result of Choi *et al*. [3, p.360 (2.1)].

Theorem 4. The following integral representations holds true:

(i) $B_{p,q}^{(\alpha,\beta)}(x,y)=2\int_0^{\frac{\pi}{2}}\cos^{2x-1}\theta\sin^{2y-1}\theta E_{\alpha,\beta}\left(-\frac{p}{\cos^2\theta}\right)E_{\alpha,\beta}\left(-\frac{q}{\sin^2\theta}\right)d\theta,$ (2.7)

(ii) $B_{p,q}^{(\alpha,\beta)}(x,y)=\int_0^\infty\frac{u^{x-1}}{(1+u)^{x+y}}E_{\alpha,\beta}\left(-\frac{p(1+u)}{u}\right)E_{\alpha,\beta}\big(-q(1+u)\big)\,du$ (2.8)

(iii) $B_{p,q}^{(\alpha,\beta)}(x,y)=(c-a)^{1-x-y}\int_a^c(u-a)^{x-1}(c-u)^{x-1}$

$\times E_{\alpha,\beta}\left(-\frac{p(c-a)}{(u-a)}\right)E_{\alpha,\beta}\left(-\frac{q(c-a)}{(c-u)}\right)du$ (2.9)

Proof. For prove the formula (2.7), putting $t=\cos^2\theta$ in (1.12), we have

$$B_{p,q}^{(\alpha,\beta)}(x,y)=2\int_0^{\frac{\pi}{2}}\cos^{2x-2}\theta\,\sin^{2y-2}\theta$$

$$\times E_{\alpha,\beta}\left(\frac{p}{\cos^2\theta}\right)E_{\alpha,\beta}\left(-\frac{q}{1-\cos^2\theta}\right)\cos\theta\sin\theta d\theta$$

$$B_{p,q}^{(\alpha,\beta)}(x,y)=2\int_0^{\frac{\pi}{2}}\cos^{2x-1}\theta\sin^{2y-1}\theta E_{\alpha,\beta}\left(-\frac{p}{\cos^2\theta}\right)E_{\alpha,\beta}\left(-\frac{q}{\sin^2\theta}\right)d\theta.$$

Similarly, the formulas(2.8) and (2.9) can be proved by taking the transformation $t = \frac{u}{1+u}$ and $t = \frac{u-a}{c-a}$ in (1.12), respectively.

Theorem 5. The following Mellin transform holds true:

$$\mathcal{M}\left\{B_{p,q}^{(\alpha,\beta)}(x,y); p \to r, q \to s\right\} = B(x+r, y+s)\Gamma^{(\alpha,\beta)}(r)\Gamma^{(\alpha,\beta)}(s), (2.10)$$

$\mathrm{Re}(x+r) > 0, \mathrm{Re}(y+s) > 0, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(p), \mathrm{Re}(q) > 0,$

$\mathrm{Re}(r), \mathrm{Re}(s) > 0.$

Proof. By taking the Mellin transform in both sides of (1.12), we have $\mathcal{M}\left\{B_{p,q}^{(\alpha,\beta)}(x,y); p \to r, q \to s\right\}$

$$= \int_0^\infty \int_0^\infty p^{r-1} q^{s-1} \left\{\int_0^1 t^{x-1}(1-t)^{y-1} E_{\alpha,\beta}\left(-\frac{p}{t}\right) E_{\alpha,\beta}\left(-\frac{q}{(1-t)}\right) dt\right\} dpdq.$$

Under stated conditions of the above integral, the order of integration can be inter- changing. Therefore, we have

$$\mathcal{M}\left\{B_{p,q}^{(\alpha,\beta)}(x,y); p \to r, q \to s\right\}$$

$$= \int_0^1 t^{x-1}(1-t)^{y-1} \left\{\int_0^\infty p^{r-1} E_{\alpha,\beta}\left(-\frac{p}{t}\right) dp \int_0^\infty q^{s-1} E_{\alpha,\beta}\left(-\frac{q}{(1-t)}\right) dq\right\} dt.$$

Using the substituting $u = \frac{p}{t}, v = \frac{q}{(1-t)}$, we get

$$\mathcal{M}\left\{B_{p,q}^{(\alpha,\beta)}(x,y); p \to r, q \to s\right\}$$

$$= \left\{\int_0^1 t^{x+r-1}(1-t)^{y+s-1} \left\{\int_0^\infty u^{r-1} E_{\alpha,\beta}(-u) du\right\} \left\{\int_0^\infty v^{s-1} E_{\alpha,\beta}(-v) dv\right\} dt\right\}.$$

Now, by using the definitions (1.2) and (1.13), we obtain the desired result.
By taking the inverse Mellin transform on both sides of (2.15), we get the following Form- ula:

Corollary 1. The following formula of extended Beta function holds true

$$B_{p,q}^{(\alpha,\beta)}(x,y) = \frac{1}{(2\pi i)^2} \int_{\gamma_1 - i\infty}^{\gamma_1 + i\infty} \int_{\gamma_2 - i\infty}^{\gamma_2 + i\infty} B(x+r, y+s)\Gamma^{(\alpha,\beta)}(r)\Gamma^{(\alpha,\beta)}(s)$$

$$\times p^{-r} q^{-s} drds, (2.11)$$

$(\mathrm{Re}(x) > 0, \mathrm{Re}(y) > 0, \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, p, q \geq 0, \gamma_1, \gamma_2 > 0).$

Remark 5.
(i) If $\beta = 1$, then (2.10) reduces to a known result of Rahman *et al*. [6, p.5 (2.2)].
(ii) If$\alpha = \beta = 1$ and $p = q$, then (2.11) reduces to a known result of Chaudhry *et al*. [2, p.28 (5.1)].
(iii) If $\alpha = \beta = 1$, then (2.11) reduces to a known result of Choi *et al*. [3, p.365 (4.2)].

References

1. Chaudhry MA, Zubair SM. Generalized incomplete Gamma functions with App- lications. J Comput. Appl. Math. 1994; 55:99-124.
2. Chaudhry MA, Qadir A, Rafique M, Zubair SM. Extension of Euler's Beta function. J Comput. Appl. Math. 1997; 78:19-32.
3. Choi J, Rathie AK, Parmar RK. Extension of extended Beta, Hypergeom-etric and confluent hypergeometric functions, Honam Mathematical J. 2014; 36(2):357-385.
4. Mittag-Leffler GM. Sur la nouvelle function$E_\alpha(z)$, C. R. Acad. Sci. Paris. 1903; 137:554-558.
5. Pucheta PI. A new extended Beta function, International Journal of Mathematics and its Applications. 2017; 5(3-C):255-260.
6. Rahman G, Kanwal G, Nisar KS, Ghaffar A. A new extension of Beta and hyper-geometric functions. 2018. doi:10.20944/preprints201801.0074.v1.
7. Shadab M, Jabee S, Choi J. An extension of Beta function and its application, Far East Journal of Mathematical Sciences. 2018; 103(1):235-251.
8. Srivastava HM, Manocha HL. A Treatise on Generating Functions, Halsted Press, New York, 1984.
9. Wiman A. Uber den Fundamentals at zin der Teorieder Funktionen $E_\alpha(z)$, Acta Math. 1905; 29(1):191-207.