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Modified Frontini-Sormani iterative method for solving systems of nonlinear equations with singular Jacobian

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Abstract
This paper modifies the family of iterative method in Frontini and Sormani (2004) for solving systems of nonlinear equations with singular Jacobian. The convergence of the methods are established via Taylor series approach. Different numerical examples are given to illustrate the effectiveness of the proposed methods.

Keywords: System of nonlinear equations, singular Jacobian, multi-step iterative methods, convergence

1. Introduction
Several iterative techniques are modified for approximating the solution of systems of nonlinear equations. Newton method and its modifications are being used to approximate solutions of systems of nonlinear equations (SNLE), Weerakoon and Fernando (2000) [8], Frontini and Sormani (2004) [5], Davishi and Barati (2007) [4], Haijun (2007), Noor and Waseem (2009) [7], and Ahmadabadi (2016) [1].
In modifying existing numerical methods for approximation of solution of a problem, the target often is to develop iterative methods with improved convergence rate, computational efficiency or to enable it solve certain class of problem. Consequently, in this paper, we modify the iterative method developed in Frontini and Sormani (2004) [5] to approximating systems of nonlinear equations with singular Jacobian.

2. The Proposed Methods
Consider the system of nonlinear equation

$$F(z) = 0 \tag{1}$$

where $F: \mathbb{D} \subset \mathbb{R}^m \to \mathbb{R}^m$ is a function defined in $\mathbb{D}$ an open domain in $\mathbb{R}^m$. Applying Taylor's series expansion on (1) about β (a neighbourhood of the solution φ of (1)) up to the second term and then approximate $F'(\beta)$ with generic quadrature formula yield

$$F(\beta) + \left[\sum_{i=1}^{q} \eta_i F'\big(\beta + \vartheta_i(z-\beta)\big)\right](z-\beta) + H(z) = 0 \tag{2}$$

where $H(z)$ is higher terms of the Taylor's expansion, $\vartheta_i \in [0,1]$ and $\eta_i, i = 1,2,3, \dots .q$ are knots and weights respectively such that

$$\sum_{i=1}^{q} \eta_i = 1 \tag{3}$$

and

$$\sum_{i=1}^{q} \eta_i \vartheta_i = \frac{1}{2}, \tag{4}$$

Equation (2) is expressed into coupled system of equation as iteration scheme is obtained.
Equations (3) and (4) are quadrature formula consistency conditions (Argyros, 2017) [2].

$$F(\beta) + \left[\sum_{i=1}^{q} \eta_i F'\big(\beta + \vartheta_i (z-\beta)\big)\right](z-\beta) + H(z) = 0 \tag{5}$$

$$H(z) = F(z) - F(\beta) - \left[\sum_{i=1}^{q} \eta_i F'\big(\beta + \vartheta_i (z-\beta)\big)\right](z-\beta) \tag{6}$$

From (5),

$$z = \beta - \left[\sum_{i=1}^{q} \eta_i F'\big(\beta + \vartheta_i (z-\beta)\big)\right]^{-1} F(\beta) - \left[\sum_{i=1}^{q} \eta_i F'\big(\beta + \vartheta_i (z-\beta)\big)\right]^{-1} H(z) \tag{7}$$

Equation (7) can be decomposed into the form
$z = C + \mathfrak{D}(z)$ (8)
Where

$$C = \beta - \left[\sum_{i=1}^{q} \eta_i F'\big(\beta + \vartheta_i (z-\beta)\big)\right]^{-1} F(\beta) \tag{9}$$

And

$$\mathfrak{D}(z) = -\left[\sum_{i=1}^{q} \eta_i F'\big(\beta + \vartheta_i (z-\beta)\big)\right]^{-1} H(z) \tag{10}$$

Applying the decomposition technique due to Daftarda-Gejji (2006) [3] on (10) the following iteration scheme is obtained.

$$\left.\begin{aligned} z_0 &= C \\ z_1 &= \mathfrak{D}(z_0) \\ &\vdots \\ z_{i+1} &= \mathfrak{D}\left(\sum_{j=0}^{i} z_j\right) - \mathfrak{D}\left(\sum_{j=0}^{i-1} z_j\right), i = 1,2,\ldots \end{aligned}\right\} \tag{11}$$

From equation (11) and (9), for
$z \approx z_0$
$= C$

$$= \beta - \left[\sum_{i=1}^{q} \eta_i F'\big(\beta + \vartheta_i (z-\beta)\big)\right]^{-1} F(\beta) \tag{12}$$

Equation (12) enable the proposal of the following iterative method.

Algorithm 1
Suppose z_0 is an initial guess, the solution of $F(z) = 0$ can be approximated using the iterative method:

$$W_k = z_k - [F'(z_k) - F(z_k)\Omega]^{-1} F(z_k)$$

$$z_{k+1} = z_k - \left[\sum_{i=1}^{q} \eta_i F'\big(z_k + \vartheta_i (W_k - z_k)\big)\right]^{-1} F(z_k), k = 1,2,3,\ldots \tag{13}$$

Algorithm 1 is a modified convergence order $\rho = 3$ family of iterative methods developed in Frontini and Sormani (2003) [5]. Where $\Omega = [\Omega_1, \Omega_2, \cdots, \Omega_m]$, $\Omega_i \in \mathbb{R}$. The perturbation term $F(z_k)\Omega$ in the first step of (13), is a dense matrix introduced to circumvent the singularity of the Jacobian $F'(z_k)$ of the target SNLE.
From (11), for

$$z \approx z_0 + z_1$$

$$= C + \mathfrak{D}(z_0)$$

$$= \beta - \left[\sum_{i=1}^{q} \eta_i F'\big(\beta + \vartheta_i(z-\beta)\big)\right]^{-1} F(\beta) - \left[\sum_{i=1}^{q} \eta_i F'\big(\beta + \vartheta_i(z_0-\beta)\big)\right]^{-1} H(z_0) \quad (14)$$

Using (6),

$H(z_0) = F(z_0)$ (15)
Substituting (15) in (14) the yield

$$z = \beta - \left[\sum_{i=1}^{q} \eta_i F'\big(\beta + \vartheta_i(z-\beta)\big)\right]^{-1} F(\beta) - \left[\sum_{i=1}^{q} \eta_i F'\big(\beta + \vartheta_i(z_0-\beta)\big)\right]^{-1} H(z_0) \quad (16)$$

Equation (16) enable the proposal of the following three step iterative method for the approximation of the solution φ of (1).

Algorithm 2
Suppose z_0 is an initial guess, the solution of $F(z) = 0$ can be approximated using the iterative method:

$$W_k = z_k - [F'(z_k) - F(z_k)\Omega]^{-1}F(z_k)$$

$$\mathcal{V}_k = z_k - \left[\sum_{i=1}^{q} \eta_i F'\big(z_k + \vartheta_i(W_k - z_k)\big)\right]^{-1} F(z_k),$$

$$z_{k+1} = \mathcal{V}_k - \left[\sum_{i=1}^{q} \eta_i F'\big(z_k + \vartheta_i(\mathcal{V}_k - z_k)\big)\right]^{-1} F(\mathcal{V}_k), k = 1,2,3, \dots \quad (17)$$

3. Convergence Analysis

This section discusses the convergence analysis of the proposed Algorithm 1 and Algorithm 2.

Theorem 1. Let the function F be continuous and differentiable in $\mathbb{D} \subset \mathbb{R}^m$ containing its solution φ. If z_0 is initial guess close to φ, then the sequence of approximations $\{z_k\}_{k\geq 0}, (z_k \in \mathbb{D})$ generated by Algorithm 1 converges to φ with convergence order $\rho = 3$.

Proof: Let $E_k = \|z_k - \varphi\|$ be error at kth iteration. Applying Taylor's series to expanding $F(z)$ and $F'(z)$ about φ, the following is obtained:

$$F(z) = F(\varphi) + F'(\varphi)(z-\varphi) + \frac{1}{2!}F''(\varphi)(z-\varphi)^2 + \frac{1}{3!}F'''(\varphi)(z-\varphi)^3 + \cdots \quad (18)$$

$$F'(z) = F'(\varphi) + F''(\varphi)(z-\varphi) + \frac{1}{2!}F'''(\varphi)(z-\varphi)^2 + \frac{1}{3!}F^{iv}(\varphi)(z-\varphi)^3 + \cdots \quad (19)$$

Set $z = z_k$ in (18) and (19), implies

$$F(z_k) = F(\varphi + E_k)$$

$$= F'(\varphi)\left[E_k + \sum_{n=2}^{4} C_n E_k^n + O(\| E_k^5\|)\right] k = 0,1,2, \dots \quad (20)$$

$$F'(z_k) = F'(\varphi + E_k)$$

$$= F'(\varphi)\left[I + \sum_{n=2}^{5} nC_n E_k^{n-1} + O(E_k^5)\right], k = 0,1,2, \dots \quad (21)$$

Where I is an identity matrix and $C_n = (1/n!) \left\|\big(F'(\varphi)\big)^{-1}F^{(n)}(\varphi)\right\|, n \geq 2.$

Multiplying (22) by equation (19), yield

$$[F'(z_k) - F(z_k)\Omega]^{-1}F(z_k) = [\, E_k + (\Omega - C_2)\, E_k^2 + (2C_2^2 - 2C_3 - 2C_2\Omega + \Omega^2)\, E_k^3$$

$$+(-4C_2^3 + 7C_2C_3 + 4C_4 - 3C_4 + 5C_2^2\Omega - 4C_3\Omega - 3C_2\Omega^2 + \Omega^3)\, E_k^4 + O(\, E_k^5)] \quad (23)$$

Substitute (23) in (13), yield

$$W_k = \varphi + (\Omega - C_2)\, E_k^2 + (-2C_2^2 + 2C_3 + 2C_2\Omega - \Omega^2)\, E_k^3$$

$$+(4C_2^3 - 7C_2C_3 - 4C_4 + 3C_4 - 5C_2^2\Omega\Omega + 4C_3\Omega + 3C_2\Omega^2 - \Omega^3)\, E_k^4 + O(\, E_k^5) \quad (24)$$

Set $z = W_k$ in equation (18) lead to obtaining the following:

$$F(W_k) = F'(\varphi)\big[(C_2 + \Omega)\, E_k^2 + (-2C_2^2 + 2C_3 + 2C_2\Omega - \Omega^2)\, E_k^3 + O(\|\, E_k^4\|)\big] \quad (25)$$

Similarly, set $z = z_k + \vartheta_i(W_k - z_k)$ in equation (19), then

$$\sum_{i=1}^{q} \eta_i\, F'\big(z_k + \vartheta_i(W_k - z_k)\big) = F'(\varphi)\Bigg[I + 2C_2\left(\sum_{i=1}^{q} \eta_i(1 - \vartheta_i)\right) E_k + \left(\sum_{i=1}^{q} \eta_i(3C_3(\vartheta_i - 1)^2) + 2C_2(C_2 - \Omega)\,\vartheta_i\right) E_k^2$$

$$+\left(\sum_{i=1}^{q} \eta_i\big(4C_4(\vartheta_i - 1)^3 + 2C_2(-2C_2^2 + 2C_3 + 2C_2\Omega - \Omega^2)\big)\,\vartheta_i + 6C_3(C_2 - \Omega)(1 - \vartheta_i)\right) E_k^3$$

$$+\Bigg(\sum_{i=1}^{q} \eta_i(2C_2(4C_2^3 - 7C_2C_3 + 3C_4 - 5C_2^2\Omega + 4C_3\Omega + 3C_2\Omega^2 - \Omega^3)\vartheta_i) + 12C_4(C_2 - \Omega)(\vartheta_i - 1)^2\vartheta_i$$

$$+ 3C_3(2(-2C_2^2 + 2C_3 + 2C_2\Omega - \Omega^2)(1 - \vartheta_i)\vartheta_i + (C_2 - \Omega)^2)\vartheta_i\Bigg) E_k^4 + O(\, E_k^5)\Bigg] \quad (26)$$

Using equations (20) and (26) in the second step of equation (13), the following is obtained:

$$x_{k+1} = z_k - \left[\sum_{i=1}^{q} \eta_i\, F'\big(z_k + \vartheta_i(W_k - z_k)\big)\right]^{-1} F(z_k)$$

$$= \beta + \Bigg(\sum_{i=1}^{q} \left(\eta_i^{\,2}\big(-2C_2^2 + 2C_3 + (8C_2^2 - 6C_3 - 2C_2\Omega)\vartheta_i + (-4C_2^2 + 3C_3)\vartheta_i^{\,2}\big)\right)$$

$$+ \eta_1\eta_2\left(-4C_2^2 + 4C_3 + 3C_3\vartheta_1^{\,2} + (8C_2^2 - 6C_3 - 2C_2\Omega)\vartheta_2 + 3C_3\vartheta_2^2 + (8C_2^2 - 6C_3 - 2C_2\vartheta_2)\right)\Bigg) E_k^3$$

$$+ O(E_k^4) \quad (27)$$

Equation (27) implies that Algorithm 1 has convergence order $\rho = 3$

Theorem 1. Let the function F be continuous and differentiable in $\mathbb{D} \subset \mathbb{R}^m$ containing its solution φ. IF z_0 is initial guess close to φ, then the sequence of approximations $\{z_k\}_{k\geq 0}, (z_k \in \mathbb{D})$ generated by Algorithm 2 converges to φ with convergence order $\rho = 4$.

Proof. Set $z_k = \mathcal{V}_k$ in (18), where $\mathcal{V}_k$ is as in (27), then

$$F(\mathcal{V}_k) = F(\beta)^{-1}\left[\left(\left(\sum_{i=1}^{q}\left(\eta_i{}^2\left(-2C_2^2 + 2C_3 + (8C_2^2 - 6C_3 - 2C_2\Omega)\vartheta_i + (-4C_2^2 + 3C_3)\vartheta_i{}^2\right)\right)\right.\right.\right.$$

$$\left.+ \eta_1\eta_2\left(-4C_2^2 + 4C_3 + 3C_3\vartheta_1{}^2 + (8C_2^2 - 6C_3 - 2C_2\Omega)\vartheta_2 + 3C_3\vartheta_2^2 + (8C_2^2 - 6C_3 - 2C_2\vartheta_2)\right)\right) E_k^3$$

$$\left. + O(E_k^4)\right] \tag{28}$$

Set $z = z_k + \vartheta_i(\mathcal{V}_k - z_k)$ in equation (19) where $\mathcal{V}_k$ equivalent to the relation in equation (27) then is,

$$\left[\sum_{i=1}^{q}\eta_i\, F'\left(z_k + \vartheta_i(\mathcal{V}_k - z_k)\right)\right]^{-1} = F(\beta)^{-1}\left[I + \left(2C_2\sum_{i=1}^{q}\eta_i\,(\vartheta_i - 1)\right)E_k + \left(4C_2^2\sum_{i=1}^{q}\eta_i\,(\vartheta_i - 1)^2\right)E_k^2 + O(E_k^3)\right] \tag{29}$$

By multiplying equation (29) by (28), the following relation is obtained.

$$\left[\sum_{i=1}^{q}\eta_i\, F'\left(z_k + \vartheta_i(\mathcal{V}_k - z_k)\right)\right]^{-1} F(\mathcal{V}_k)$$

$$= \left(2(-2C_2^2 + 2C_3 + 2C_2\Omega - \Omega^2) + 2C_2(C_2 - \Omega)\sum_{i=1}^{q}\eta_i\,(\vartheta_i - 1)\right)E_k^3 + O(E_k^4) \tag{30}$$

Substitute (30) in the third step of (17) yield

$$z_{k+1} = \beta + (-2C_2^2 + 2C_3 + 2C_2\Omega - \Omega^2)(2C_2 - 1) + 2C_2(C_2 - \Omega)\left(\sum_{i=1}^{q}\eta_i\,(\vartheta_i - 1)\right)^2$$

$$\left(\sum_{i=1}^{q}\eta_i\,(\vartheta_i - 1)^2 + (4C_2^3 - 7C_2C_3 + 3C_4 - 5C_2^2\Omega + 4C_3\Omega + 3C_2\Omega^2 - \Omega^3\right.$$

$$-\left((5C_2^3 - 7C_2C_3 + 3C_4 - 7C_2^2\Omega + 4C_3\Omega + 4C_2\Omega^2 - \Omega^3) + 2C_2(-C_2^2 + 2C_3 + C_2\Omega - \Omega^2)\sum_{i=1}^{q}\eta_i\,(\vartheta_i - 1)\right)$$

$$\left.\left.+(C_2 - \Omega)\left(4C_2^2\sum_{i=1}^{q}\eta_i\,(\vartheta_i - 1)^2 - \left(\sum_{i=1}^{q}\eta_i\,(3C_3(\vartheta_i - 1)^2 + 2C_2(C_2 - \Omega)\vartheta_i)\right)\right)\right)\right)E_k^4$$

$$+O\left(E_k^5\right) \tag{31}$$

Therefore, Algorithm 2 is of convergence order $\rho = 4$ ■

4. Particular forms of the proposed methods

In this section, some concrete form of Algorithm 1 and Algorithm 2 are presented. The methods are proposed by assigning values to the parameters η_i and ϑ_i satisfying the conditions given in (3) and (4). It is important to note that with same values of η_i and ϑ_i, corresponding Frontini and Sormani (2004) [5] methods are obtained. The difference between the methods proposed herein and Frontini and Sormani (2004) [5] is the infused perturbation term to the Jacobian of the predictor function.

4.1 Particular forms of Algorithm 1

When $q = 1, \eta_1 = 1,\ \vartheta_1 = \frac{1}{2}$, in Algorithm 1 the following iterative method for approximating the solution φ of $F(z) = 0$ is proposed.

Algorithm 3. Suppose z_0 is an initial guess, the solution of $F(z) = 0$ can be approximated using the iterative method:

$$W_k = z_k - [F'(z_k) - F(z_k)\Omega]^{-1}F(z_k)$$

$$z_{k+1} = W_k - \left[F'\left(\frac{z_k + W_k}{2}\right)\right]^{-1} F(z_k), k = 0,1,2,\ldots \tag{32}$$

Algorith 3 is a proposed modified Frontini and Sormani (2004) [5] iterative method for approximating the solution $\boldsymbol{\varphi}$ of (1) with convergence order $\rho = 3$ and error equation satisfying

$$E_{k+1} = \left(C_2^2 + \frac{C_3}{4} - C_2\Omega\right) E_k^3 + O(E_k^4) \tag{33}$$

When $q = 2, \eta_1 = \frac{1}{4}, \eta_2 = \frac{3}{4},\ \vartheta_1 = 0,\ \vartheta_2 = \frac{2}{3}$, in Algorithm 1 the following iterative method for approximating the solution φ of (1) is proposed.

Algorithm 4. Suppose z_0 is an initial guess, the solution of $F(z) = 0$ can be approximated using the iterative method:

$$W_k = z_k - [F'(z_k) - F(z_k)\Omega]^{-1}F(z_k)$$

$$z_{k+1} = W_k - 4\left[F'(z_k) + 3F'\left(\frac{z_k + 2W_k}{3}\right)\right]^{-1} F(z_k), k = 0,1,2,\ldots \tag{34}$$

The error equation of Algorithm 4 is

$$E_{k+1} = C_2(C_2 - \Omega) E_k^3 + O(E_k^4) \tag{35}$$

4.2 Particular forms of Algorithm 2

for $q = 1, \eta_1 = 1,\ \vartheta_1 = \frac{1}{2}$, in Algorithm 2, enable the proposal of the following iterative method for approximating the solution φ of $F(z) = 0$.

Algorithm 5. Suppose $\mathbf{z}_0$ is an initial guess, the solution of $F(z) = 0$ can be approximated using the iterative method:

$$W_k = z_k - [F'(z_k) - F(z_k)\Omega]^{-1}F(z_k)$$

$$\mathcal{U}_k = W_k - \left[F'\left(\frac{z_k + W_k}{2}\right)\right]^{-1} F(z_k),$$

$$z_{k+1} = \mathcal{U}_k - \left[F'\left(\frac{z_k + \mathcal{U}_k}{2}\right)\right]^{-1} F(\mathcal{U}_k), k = 0,1,2,\ldots \tag{37}$$

Algorithm 5 is a proposed three step iterative method for approximating the solution $\boldsymbol{\varphi}$ of (1) with convergence order $\rho = 4$ and error equation satisfying

$$E_{k+1} = C_2^2(C_2 - \Omega) E_k^4 + O(E_k^5) \tag{38}$$

When $q = 2, \eta_1 = \frac{1}{4}, \eta_2 = \frac{3}{4},\ \vartheta_1 = 0,\ \vartheta_2 = \frac{2}{3}$, in Algorithm 2 the following iterative method for approximating the solution $\boldsymbol{\varphi}$ of (1) is proposed.

Algorithm 6. Suppose $\mathbf{z}_0$ is an initial guess, the solution of $F(z) = 0$ can be approximated using the iterative method:

$$W_k = z_k - [F'(z_k) - F(z_k)\Omega]^{-1}F(z_k)$$

$$\mathcal{U}_k = W_k - 4\left[F'(z_k) + 3F'\left(\frac{z_k + 2W_k}{3}\right)\right]^{-1} F(z_k),$$

$$z_{k+1} = \mathcal{U}_k - 4\left[F'(z_k) + 3F'\left(\frac{z_k + 2\mathcal{U}_k}{3}\right)\right]^{-1} F(\mathcal{U}_k), k = 0,1,2,\ldots \tag{39}$$

Algorithm 6 is a proposed iterative method for approximating the solution φ of (1) with convergence order $\rho = 3$ and error equation satisfying

$$E_{k+1} = C_2^2(C_2 - \Omega)\, E_k^4 + O\left(E_k^5\right) \quad (40)$$

Remark
For efficiency and better convergence in implementation of Algorithm 3-6, the elements Ω_i of Ω are subjectively chosen such that the magnitudes of Ω_i is less 1.

5. Numerical Experimentation
This section discusses numerical results obtained **F**rom the implementation of the proposed iterative methods (Algorithm 3 (A_3^3), Algorithm 4 (A_4^3), Algorithm 5 (A_5^4) and Algorithm 6 (A_6^4)) on some standard numerical problems. The performance of the proposed methods are compared with their corresponding method in Frontini and Sormani (2003) [5]. PYTHON 2.9 software was used for all computational programs with 25 digits and $\|F(z_{k+1})\| < 10^{-15}$ as stopping criteria. Intel Celeron(R) CPU 1.6 GHz with 2 GB of RAM processor is used to execute all programs

Problem 1 (Noor and Waseem, 2009) [7]. Consider the SNLE given as:

$$z_1^2 - 2z_1 - z_2 + 0.5 = 0$$

$$z_1^2 + 4z_2^2 - 4 = 0$$

with the domain $\mathbb{D} = (-1.5, 3.5) \times (-1.5, 3.5)$. The Jacobian matriz of the problem is

$$F'(z) = \begin{bmatrix} 2z_1 - 2 & -1 \\ 2z_1 & 8z_2 \end{bmatrix}$$

The solutions of Problem 1 at $\mathbf{z}_0 = (0.6, 0.12)^T$ in $\mathbb{D}$ approximated up to 25 digits is:

$$\varphi = (-0.2222145550506706125301539, 0.9938084185799846105696574)^T$$

It is shown that the Newton method diverged in approximating the solution of the SNLE in Problem 1 using $\mathbf{z_0} = (0.6, 0.12)^T$ as starting point, Noor and Waseem (2009) [7]. In the work of Noor and Waseem (2009) [7], it was shown using some practical examples that predictor-corrector technique can be useful when a one-step iterative method fails to approximate the solution of a SNLE. However the methods in Frontini and Sormarni (2004) are predictor-corrector methods and they performed poorly in solving Problem 1. The numerical results obtained by each method on Problem 4.7 is given in Table 4.10.

Table 1: Problem 1 computational results

Methods	Ω_i	NIT	CPU Time	$\|\mathbf{z_{k+1}} - \mathbf{z_k}\|_\infty$	$\|\mathbf{F(z_k)}\|_\infty$	ρ_{coc}
N_1^3	-	22	0.374	8.3677E-06	1.5000E-08	1.8597
A_3^3	$\mathbf{\Omega}_1 = \frac{-1}{10}, \mathbf{\Omega}_2 = \frac{1}{5}$	5	0.134	1.3788E-05	1.1767E-15	3.5508
N_2^3	-	22	0.404	8.3677E-06	1.5000E-08	1.8597
A_4^3	$\mathbf{\Omega}_1 = \frac{-1}{10}, \mathbf{\Omega}_2 = \frac{1}{5}$	5	0.144	1.3788E-05	1.1766E-15	3.5508
N_3^4	-	**Fail to converge**				
A_5^4	$\mathbf{\Omega}_1 = \frac{-1}{10}, \mathbf{\Omega}_2 = \frac{1}{5}$	6	0.217	2.8270E-02	7.5959E-04	3.9119
N_4^4	-	**Fail to converge**				
A_6^4	$\mathbf{\Omega}_1 = \frac{-1}{10}, \mathbf{\Omega}_2 = \frac{1}{5}$	6	0.238	7.5959E-04	7.5959E-04	3.9119

Observe that in Table 1, the methods (N_1^3 and N_2^3) converged very slowly while N_3^3 and N_4^3 fail to converge. On the other hand, the proposed methods (A_3^3, A_4^3, A_5^4, and A_6^4) converged within few iterations.

Problem 2 (Haijun, 2009) [6]. Consider the SNLE:

$$e^{z_1} - z_2 - 2 = 0$$

$$\cos(z_1) - z_1 + z_2 - 1 = 0$$

The SNLE in Problem 4.8 has a solution as approximated up to 25 significant figures is
$\varphi = (1.478488896007052054298454, 2.386312496115686598576732\,1)^T$
and lies in $\mathbb{D} = (0, 2) \times (-2, 2)$. The Jacobian of the problem is

$$F'(z) = \begin{bmatrix} e^{z_1} & -1 \\ -\sin(z_1) - 1 & 1 \end{bmatrix}$$

for $z_0 = (0, 0.5)^T$, the computational results are presented in Table 2.

Table 2: Problem 2 computational results

Methods	Ω_i	**NIT**	**CPU Time**	$\|z_{k+1} - z_k\|_\infty$	$\|F(X_k)\|_\infty$	ρ_{coc}
N_1^3	-	**Fail due to singular Jacobian $F'(z)$**				
A_3^3	$\Omega_1 = \frac{1}{3}, \Omega_2 = \frac{1}{2}$	5	0.137	4.8182E-03	1.1088E-08	3.4229
N_2^3		**Fail due to singular Jacobian $F'(z)$**				
A_4^3	$\Omega_1 = \frac{1}{3}, \Omega_2 = \frac{1}{2}$	4	0.120	6.3647E-04	2.2043E-11	3.3786
N_3^4		**Fail due to singular Jacobian $F'(z)$**				
A_5^4	$\Omega_1 = \frac{1}{3}, \Omega_2 = \frac{1}{2}$	3	0.118	29506E-03	5.4442E-12	4.8020
N_4^4		**Fail due to singular Jacobian $F'(z)$**				
A_6^4	$\Omega_1 = \frac{1}{3}, \Omega_2 = \frac{1}{2}$	6	0.228	4.7291E-03	1.8013E-11	4.7658

From Table 4.12, the methods N_1^3, N_2^3, N_3^4 and N_4^4 fail due to singular Jacobian $F'(z)$ of the SNLE, while the proposed modified form converges to the solution of the SNLE in Problem 2 using few number of iterations.

Problem 3 (Haijun, 2009) [6]. Define $F(z) = 0$ such that

$$F(z) = \begin{bmatrix} z_1^2 + 3\,log(z_1) - z_2^2 \\ 2z_1^2 - z_1 z_2 - 5z_1 + 1 \end{bmatrix}$$

The Jacobian matriz of the SNLE is

$$F'(z) = \begin{bmatrix} 2z_1 + \dfrac{3}{z_1} & -z_2 \\ 4z_1 - z_2 - 5 & -z_1 \end{bmatrix}$$

The solution of the above SNLE in $\mathbb{D} = (0, 2) \times (2, 3)$ approximated up to 25 digits is:
$\varphi = (1.4784888960070520542984 54, 2.3863124961156865985767321\,)^T$
If $\det(F'(z)) = 0$ for a vector z, then Jacobian $F'(z)$ is singular. Using $z_0 = (0.5495, -1)^T$ the Jacobian $F'(z_0) = 9.95000000001177E - 005$. This implies that the Jacobian $F'(z)$ is almost or approximately singular at z_0. In Table 3, the performance of the proposed methods are compared.

Table 3: Problem 3 computational results

Methods	Ω_i	**NIT**	**CPU Time**	$\|z_{k+1} - z_k\|_\infty$	$\|F(z_k)\|_\infty$	ρ_{coc}
N_1^3	−	16	0.385	2.9232E-02	8.0557E-06	4.4016
A_3^3	$\Omega_1 = \frac{1}{9}, \Omega_2 = \frac{1}{2}$	6	0.174	4.5912E-05	7.0191E-14	3.3505
N_2^3	−	**Fail**				
A_4^3	$\Omega_1 = \frac{1}{9}, \Omega_2 = \frac{1}{2}$	7	0.216	3.2102E-05	2.2239E-07	3.4182
N_3^4	−	**Fail**				
A_5^4	$\Omega_1 = \frac{1}{9}, \Omega_2 = \frac{1}{2}$	3	0.127	2.0782E-02	1.7693E-08	6.7072
N_4^4	−	Fail				
A_6^4	$\Omega_1 = \frac{1}{9}, \Omega_2 = \frac{1}{2}$	2	0.109	2.5671E-01	5.7077E-04	25.8912

The computational results in Table 3 shows that the methods
N_2^3, N_3^3, and N_4^4 breakdown or fail to approximate the solution of of Problem 3 because the Jacobian $\mathbf{F'(z)}$ is almost singular at iteration point. While the N_1^3 required 16 number of iterations to achieve convergence. On the other hand, the proposed methods A_3^3, $A_{4.}^3$, A_5^4 and A_6^4 does not fail to converge but converged to the solution using few iterations.

6. Conclusion

In this paper, we have suggested a modified Frontini and Sormarni method for approximation of systems of nonlinear equation solution. The proposed methods are developed via decomposition technique. From the numerical computational results, the proposed methods are able to approximate the solution of systems of nonlinear equation with singular Jacobian.

7. Reference

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