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K- bianalytic functions and their related theorems

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Abstract
In this paper, the concept of K- bianalytic functions is presented and the K- bianalytic functions expression is given. Then, we proved the Cauchy theorem and Cauchy integral formula of K- bianalytic function. Also, the power series expansion theorem and the Fourier series of K- bianalytic function were proved in this study.

Keywords: K- bianalytic functions, Cauchy theorem, Cauchy integral formula, power series, Fourier series

1. Introduction

Analytic functions have been applied in many fields, e.g., theoretical physics, astromechanics, and fluid mechanics. Over the past two hundred years, systematic theories in analytic functions have been established through the efforts of many researchers. In 1983, Professor Jianding Wang first proposed the semi-analytic function theory, which leads to establishment of the conjugate analytic function theory [1]. These two theories have been successfully applied to electric fields, magnetic fields, fluid mechanics, elastic mechanics and other fields. Based on these theories, many researchers continued this study and expanded the problem into a series of new fields, bianalytic functions, complex harmonic functions, polyanalytic functions (k analytic functions), and the corresponding boundary value problems, differential equations, integral equations, etc [2-5]. Recently, some researchers have proposed the concept of K-analytic function and studied its analytical properties (see [6-8]). Zhang jianyuan [6] concluded a necessary and sufficient conditions of K-analytic function. In [7], the author deducts the relation between K-analytic function and K-integral, the relation between K-analytic function and K-harmonic function. Zhang jianyuan [8] used the series theory to propose the power series expansion of the K-analytic function and its zero isolation and uniqueness.

In this research, we explained the concept of K- bianalytic analytical function, and we derived the K-bianalytic functions expression. Then, we proved the Cauchy theorem and Cauchy integral formula of K- bianalytic function. In the last part, we discussed the power series expansion and the Fourier series expansion of K-bianalytic functions.

2. Preparatory knowledge

Definition 1 [6] (definition of K- Complex number) Let function $f(z)$ be defined in area D, we say that the form $z(k) = x + iky(k \in R, k \neq 0)$ is k- Complex number of $z = x + iy$.

Definition 2 [6] (definition of K-derivative) Let function $f(z)$ be defined in a neighborhood of z_0, $z(k) = x + iky(k \in R, k \neq 0)$, if the limit $\lim\limits_{\Delta z \to 0} \frac{\Delta f}{\Delta z(k)} = \lim\limits_{\Delta z \to 0} \frac{f(z)-f(z_0)}{z(k)-z_0(k)}$ is exist, then we call function $f(z)$ is k-differentiable at z_0, and this limit is defined as the k-derivative of function $f(z)$ at z_0, denoted as $f'_{(k)}(z_0)$ or $\left.\frac{df(z)}{dz(k)}\right|_{z=z_0}$.

When function $f(z)$ is K-derivable at every point in area D, $f(z)$ is called K-derivative in D. and its k- derivable is denoted as $f'_{(k)}(z), (z \in D)$.

Definition 3 [6] (definition of K-analytic function) If function $f(z)$ is k-differentiable in region D, $f(z)$ is K-analytic in region D. All of functions parsed in k- in area D is denoted as $F\big(D(k)\big)$.

Lemma 1 [6] (Necessary and sufficient conditions of K-analytic) the function $f(z) = u(x, y) + iv(x, y)$ is K-analytic in region D, $\Leftrightarrow u, v$ is differentiable in region D, and the condition of C-R-K holds,
where condition C-R-K is $u_x = v_y/k\, , v_x = -\,u_y/k$.

Let $\frac{\partial}{\partial \bar{z}(\mathrm{k})} = \frac{\partial}{\partial \mathrm{x}} + \frac{1}{\mathrm{k}}\frac{\partial}{\partial \mathrm{y}}$, then the complex form of function C-R-K is $\frac{\partial f}{\partial \bar{z}(k)} = 0$

Lemma 2 [7] The function $f(z)$ is parsed in the single connected region D of z plane, and C is any closed curve in D, then we have the identity
$\oint_C\ f(z)(z-a)^{n-1}(k)dz(k) = 0.$

Lemma 3 [8] (power series expansion of K-analytic) Suppose function $f(z)$ is K-analytic in region D, and $\mathrm{B}(k)\colon |(z-a)(k)| < R \subset D$, we have

$f(z) = \sum c_n (z-a)^n(k), z \in B(k)$

where
$c_n = \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{f(\zeta)}{(\zeta-a)^{n+1}(k)} d\zeta(k) = \frac{f^n(a)}{n!}$

$\big(\Gamma_\rho\colon |(z-a)(k)| = \rho, 0 < \rho < R, n = 0,1,2,\cdots\big)$, and the expansion is unique.

3. K-bianalytic functions and their related theorems

Definition 3 The function $f(z)$ has the two derivative $\frac{\partial^2 f}{\partial \bar{z}^2(k)}$ of $\bar{z}(k)$ in the region D**.** If a given function $f(z)$ satisfies $\frac{\partial^2 f}{\partial \bar{z}^2(k)} = 0$, then $f(z)$ is called K- bianalytic function on D. We use $F(D_2(k))$ to represent the set of all K- Bi-analytic functions on D.

Theorem 1 (expansion of K- bianalytic functions) If function $f(z) \in F(D_2(k))$, then the following is established $f(z) = \bar{z}(k)\varphi(z) + \phi(z)$, where $\phi(z)$ is arbitrary K-analytic functions.

Proof: Let $\varphi(z) = \frac{\partial f}{\partial \bar{z}(k)}$, we can see $\varphi(z)$ is K-analytic in region D. Then $f_1(z) = \bar{z}(k)\varphi(z)$ is K- bianalytic function, and $\frac{\partial f_1}{\partial \bar{z}(k)} = \varphi(z)$. Then, we can show $\frac{\partial (f-f_1)}{\partial \bar{z}(k)} = 0$. Now let $\phi(z) = f - f_1$, then $\phi(z)$ is a K-analytic function. Thus there is $f(z) = \bar{z}(k)\varphi(z) + \phi(z)$.

For convenience, the function $f(z)$ which ignores a K-analytic function term $\varphi(z)$ is also called K-double analytic function.

Theorem 2 (Cauchy theorem of K- bianalytic functions) If $f(z)$ is a K- bianalytic function in region D, and D is a bounded area surrounded by a closed curve C, and $f(z), \frac{\partial f}{\partial \bar{z}(k)}$ are continuous on $\bar{D} = D + C$, then $\oint_C\ \Big(f(z) - \overline{z-a}(k)\frac{\partial f}{\partial \bar{z}(k)}\Big) dz(k) = 0$, or $\oint_C\ f(z)dz(k) = \oint_C\ \overline{z-a}(k)\frac{\partial f}{\partial \bar{z}(k)} dz(k)$, where a is an arbitrary point in complex plane.

Proof : According to the definition of K- bianalytic function $\frac{\partial^2 f}{\partial \bar{z}^2(k)} = 0$, we have $\frac{\partial}{\partial \bar{z}(k)}\Big(f(z) - \overline{z-a}(k)\frac{\partial f}{\partial \bar{z}(k)}\Big) = \frac{\partial f}{\partial \bar{z}(k)} - \Big(\frac{\partial f}{\partial \bar{z}(k)} + \overline{z-a}(k)\frac{\partial^2 f}{\partial \bar{z}^2(k)}\Big) = 0.$

Therefore $\mathrm{f}(z) - \overline{z-a}(k)\frac{\partial f}{\partial \bar{z}(k)}$ is K- analytic in D. By assumption we can know $f(z) - \overline{z-a}(k)\frac{\partial f}{\partial \bar{z}(k)}$ is continuous on $\bar{D}$. From k- integral (see [5]), we can obtain

$$\oint_C \Big(f(z) - \overline{z-a}(k)\frac{\partial f}{\partial \bar{z}(k)}\Big) dz(k) = 0, (\forall a \in D)$$

or

$$\oint_C f(z)dz(k) = \oint_C \overline{z-a}(k)\frac{\partial f}{\partial \bar{z}(k)} dz(k)\, , (\forall a \in D).$$

Corollary Let D be a bounded n+1 connected region surrounded by segmented smooth curve $C = C_0 + C_1^- + C_2^- + \cdots + C_n^-$, if function $f(z)$ is K- bianalytic in D, and $f(z), \frac{\partial f}{\partial \bar{z}(k)}$ are continuous on $\bar{D} = D + C$, then

$$\int_{C}\left(f(z)-\overline{z-a}(k)\frac{\partial f}{\partial\bar{z}(k)}\right)dz(k)=0,$$

Or

$$\int_{c_0}\left(f(z)-\overline{z-a}(k)\frac{\partial f}{\partial\bar{z}(k)}\right)dz(k)=\left(\int_{c_1}+\int_{c_2}+\cdots+\int_{c_n}\right)\left(f(z)-\overline{z-a}(k)\frac{\partial f}{\partial\bar{z}(k)}\right)dz(k)$$

where a is an arbitrary point in complex plane.

Below, we will use the Cauchy theorem of K- bianalytic function to prove the Cauchy integral formula of K- bianalytic function. To this end, we first prove a lemma.

Lemma 4 If function $f(z)\in F(D_2(k))$, and $\phi(z)$ is an arbitrary K- analytic function, then $f(z)\phi(z)$ is K- bianalytic in D.
Proof : Since $\frac{\partial^2 f}{\partial\bar{z}^2(k)}=0$, and $\frac{\partial\phi}{\partial\bar{z}(k)}=0$, we conclude that

$$\frac{\partial}{\partial\bar{z}(k)}\big(f(z)\phi(z)\big)=\frac{\partial f}{\partial\bar{z}(k)}\phi(z)+f(z)\frac{\partial\phi}{\partial\bar{z}(k)}=\frac{\partial f}{\partial\bar{z}(k)}\phi(z).$$

Thus there is

$$\frac{\partial^2}{\partial\bar{z}^2(k)}\big(f(z)\phi(z)\big)=\frac{\partial}{\partial\bar{z}(k)}\left(\frac{\partial f}{\partial\bar{z}(k)}\phi(z)\right)=\frac{\partial^2 f}{\partial\bar{z}^2(k)}\phi(z)+\frac{\partial f}{\partial\bar{z}(k)}\frac{\partial\phi}{\partial\bar{z}(k)}=0.$$

Then we can know that $f(z)\phi(z)$ is K- bianalytic in D.
Theorem 3 (Cauchy integral formula of K- bianalytic functions) Let D be a bounded region surrounded by a closed curve or segmented smooth curve $C=C_0+C_1^-+C_2^-+\cdots+C_n^-$, if function $f(z)$ is K- bianalytic in D, and $f(z),\frac{\partial f}{\partial\bar{z}(k)}$ are continuous on $\overline{D}=D+C$, then

$$f(z)=\frac{1}{2\pi i}\int_{C}\frac{f(\zeta)}{(\zeta-z)(k)}d\zeta(k)-\frac{1}{2\pi i}\int_{C}\frac{\overline{\zeta-z}(k)}{(\zeta-z)(k)}\frac{\partial f}{\partial\bar{z}(k)}d\zeta(k),z\in D.$$

Proof : For every $z\in D$, we make a weekly $C_\rho:(\zeta-z)(k)=\rho e^{i\theta}(0<\theta<2\pi)$, so that its surrounding area

$D_\rho=B(z,\rho)(k):|(\zeta-z)(k)|<\rho\subset D.$

It is easy to prove that $\frac{1}{(\zeta-z)(k)}$ is K-analytic when $\zeta\neq z,\zeta\in D$, then $\frac{f(\zeta)}{(\zeta-z)(k)}$ is bianalytic in $D-D_\rho$ according to hypothesis and lemma 4, and the following can be obtained by theorem 2

$$\int_{C+C_\rho^-}\frac{f(\zeta)}{(\zeta-z)(k)}d\zeta(k)=\int_{C+C_\rho^-}\overline{\zeta-z}(k)\frac{\partial}{\partial\bar{z}(k)}\left(\frac{f(\zeta)}{(\zeta-z)(k)}\right)d\zeta(k),$$

that is

$$\int_C\frac{f(\zeta)}{(\zeta-z)(k)}d\zeta(k)-\int_{C_\rho}\frac{f(\zeta)}{(\zeta-z)(k)}d\zeta(k)=\int_C\frac{\overline{\zeta-z}(k)}{(\zeta-z)(k)}\frac{\partial f}{\partial\bar{z}(k)}d\zeta(k)-\int_{C_\rho}\frac{\overline{\zeta-z}(k)}{(\zeta-z)(k)}\frac{\partial f}{\partial\bar{z}(k)}d\zeta(k). \quad (1)$$

Since $C_\rho:(\zeta-z)(k)=\rho e^{i\theta}$, let $\rho\to 0$, we also conclude that

$$\int_{C_\rho}\frac{f(\zeta)}{(\zeta-z)(k)}d\zeta(k)=2\pi i f(z),\int_{C_\rho}\frac{\overline{\zeta-z}(k)}{(\zeta-z)(k)}\frac{\partial f}{\partial\bar{z}(k)}d\zeta(k)=0. \quad (2)$$

Substituting (2) into (1), we have

$$f(z)=\frac{1}{2\pi i}\int_C\frac{f(\zeta)}{(\zeta-z)(k)}d\zeta(k)-\frac{1}{2\pi i}\int_C\frac{\overline{\zeta-z}(k)}{(\zeta-z)(k)}\frac{\partial f}{\partial\bar{z}(k)}d\zeta(k),\quad \zeta\in \text{D}.$$

By Theorem 3, we can get the power series expansion theorem of K- bianalytic functions.

Theorem 4: (power series of K- bianalytic functions) If $f(z)$ is a K- bianalytic function in region D, and B(k): $|(z-a)(k)| < R \subset D$, for $\forall a \in D$, then $f(z)$ can be expanded into power series in D

$$f(z) = \sum c_n \bar{z}(k) \cdot (z-a)^n(k), z \in B(k), \tag{3}$$

where

$$c_n = \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{\frac{\partial f}{\partial \bar{z}(k)}}{(\zeta-a)^{n+1}(k)} d\zeta(k), \quad (n = 0,1,2\cdots) \tag{4}$$

$\left(\Gamma_\rho: |(z-a)(k)| = \rho, 0 < \rho < R, n = 0,1,2,\cdots\right)$, and the expansion is unique.

Proof: From theorem 1, we know that $f(z) = \bar{z}(k)\frac{\partial f}{\partial \bar{z}(k)}$, and $\frac{\partial f}{\partial \bar{z}(k)}$ is a K-analytic function, which is available from Lemma 3 conclusion established.

Theorem 5 （Fourier series of K- bianalytic function） Suppose function $f(z)$ is K- bianalytic in region D, and B(k): $|(z-a)(k)| < R \subset D, (\forall a \in D)$, then we have

$$f(z) = \frac{-\rho}{2}(\alpha_0 + i\beta_0)(\cos\theta - i\sin\theta) + \sum_{n=1}^{+\infty} \rho(\alpha_n + i\beta_n)\left(\cos(n-1)\theta + i\sin(n-1)\theta\right), \quad \left(z \in B(k)\right)$$

Where

$$\alpha_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\rho,\theta)d\theta,$$

$$\beta_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(\rho,\theta)d\theta,$$

$$\alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} v(\rho,\theta)\sin n\theta\, d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} u(\rho,\theta)\cos n\theta\, d\theta, (n = 1,2,\cdots),$$

$\beta_n = \frac{1}{\pi}\int_{-\pi}^{\pi} v(\rho,\theta)\cos n\theta\, d\theta = -\frac{1}{\pi}\int_{-\pi}^{\pi} u(\rho,\theta)\sin n\theta\, d\theta, (n = 1,2,\cdots)$, and the expansion is unique.

Proof: Because function $f(z)$ is K- bianalytic in region D, form (1) and (2) are established. In $\Gamma_\rho: |(z-a)(k)| = \rho$, suppose $(z-a)(k) = \rho e^{i\theta}(0 < \rho < R)$, then $(z-a)^n(k) = \rho^n e^{in\theta}, dz(k) = i\rho e^{i\theta}d\theta$, thus（4）is

$$c_n = \frac{1}{2\pi i}\int_{\Gamma_\rho} \frac{\frac{\partial f}{\partial \bar{z}(k)}}{(z-a)^{n+1}(k)} dz(k) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} \frac{\frac{\partial f}{\partial \bar{z}(k)}}{\rho^{n+1}e^{i(n+1)\theta}} i\rho e^{i\theta}d\theta = \frac{1}{2\pi\rho^n}\int_{-\pi}^{\pi}\frac{\partial f}{\partial \bar{z}(k)} e^{-in\theta}d\theta \tag{5}$$

When $n \geq 1$, we can see that $f(z)(z-a)^{n-1}(k)$ is K- analytic in region D, and applying Lemma 2, we have

$$\int_{\Gamma_\rho} \frac{\partial f}{\partial \bar{z}(k)} z^{n-1}dz(k) = 0,$$

that is $\int_{-\pi}^{\pi}\frac{\partial f}{\partial \bar{z}(k)} i\rho^n e^{in\theta}d\theta = 0$. Thus there is

$$0 = \frac{1}{2\pi\rho^n}\int_{-\pi}^{\pi}\frac{\partial f}{\partial \bar{z}(k)} ie^{in\theta}d\theta \tag{6}$$

Add form (5) to form (6), we have

$$c_n = \frac{1}{\pi\rho^n}\int_{-\pi}^{\pi}\frac{\partial f}{\partial \bar{z}(k)}\cos n\theta\, d\theta \quad (7)$$

Subtracting (5) from (6) is obtained.

$$c_n = \frac{-i}{\pi\rho^n}\int_{-\pi}^{\pi}\frac{\partial f}{\partial \bar{z}(k)}\sin n\theta\, d\theta \tag{8}$$

Let $\frac{\partial f}{\partial \bar{z}(k)} = u(\rho,\theta) + iv(\rho,\theta)$, by (7), (8)we have

$$c_n\rho^n = \frac{1}{\pi}\int_{-\pi}^{\pi} u(\rho,\theta)\cos n\theta\, d\theta + \frac{i}{\pi}\int_{-\pi}^{\pi} v(\rho,\theta)\cos n\theta\, d\theta$$

$$c_n\rho^n = \frac{1}{\pi}\int_{-\pi}^{\pi} v(\rho,\theta)\sin n\theta\, d\theta + \frac{-i}{\pi}\int_{-\pi}^{\pi} u(\rho,\theta)\sin n\theta\, d\theta$$

Let $c_n\rho^n = \alpha_n + i\beta_n$, then

$$\alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} u(\rho,\theta)\cos n\theta\, d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} v(\rho,\theta)\sin n\theta\, d\theta$$

$$\beta_n = \frac{1}{\pi}\int_{-\pi}^{\pi} v(\rho,\theta)\cos n\theta\, d\theta = -\frac{1}{\pi}\int_{-\pi}^{\pi} u(\rho,\theta)\sin n\theta\, d\theta$$

when $n = 0$, directly derived from (5) formula

$$\alpha_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} u(\rho,\theta)d\theta$$

$$\beta_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(\rho,\theta)d\theta$$

Obviously α_n, β_n are Fourier coefficient of $u(\rho,\theta), v(\rho,\theta)$. Substituting α_n, β_n in (3) formula,

$$f(z) = \sum_{n=0}^{\infty} c_n\bar{z}(k)\cdot(z-a)^n(k)$$

$$= \sum_{n=0}^{\infty} c_n\rho^2(z-a)^{n-1}(k)$$

$$= \sum_{n=1}^{\infty} c_n\rho^{n+1}e^{i(n-1)\theta}$$

$$= \sum_{n=1}^{\infty} \rho(\alpha_n + i\beta_n)(\cos(n-1)\theta + i\sin(n-1)\theta)$$

$$= \frac{-\rho}{2}(\alpha_0 + i\beta_0)(\cos\theta - i\sin\theta) + \sum_{n=1}^{+\infty}\rho(\alpha_n + i\beta_n)(\cos(n-1)\theta + i\sin(n-1)\theta).$$

The uniqueness is obvious, theorem is proved.

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